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FINITE DEFORMATION OF AN ELASTO-PLASTIC SOIL

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SUMMARY

Presented in this paper is a formulation and a numerical solution method for problems which involve finite deformations of an elasto-plastic material. The governing equations are cast in rate form and the constitutive laws are formulated in a frame indifferent manner. Particular reference is made to the finite deformation of soil. Plastic failure is described by a general yield condition and plastic deformation by an arbitrary flow rule. Several examples are treated numerically.

INTRODUCTION

In the formulation of theories in applied mechanics and in particular soil mechanics, it has been a common practice to assume that strains, both elastic and plastic are infinitesimal and, that the initial geometry of a deforming body is not appreciably altered during the deformation process. These assumptions are less justified for soil than for such materials as steel and concrete. Theories of finite strain that relax some of these restrictive assumptions have been developed and there exists a considerable body of literature on what might be called the classical elastic large strain theory[1–5] (e.g. the large deformation of materials like rubber has been studied[6,7]).

In contrast to the methods of these early investigators, many more recent studies have preferred an incremental approach[8,9] to facilitate the analysis of the more general class of inelastic materials whose constitutive laws are expressible in terms of incremental or rate quantities. For such formulations the solution of a given problem is found by following a specified loading path. In most cases the governing equations cannot be solved analytically and it is necessary to adopt an approximate numerical technique.[10]

Much recent work has been devoted to formulating analyses for plate and shell problems involving large displacement but small strains (a survey is given by Marcal[11]). As has been noted[12] these formulations are inappropriate for applications to bulky geometries such as occur in many problems in soil mechanics. Several attempts have been made to formulate a finite deformation theory suitable for use with soils. An example is that of Thoms and Arman[13] who directly applied a technique of Argyris[14] to the problem of an embankment constructed on soft clay. The analysis was restricted to an elastic material and theoretical results were compared with results from photo-elastic model tests. Davidson and Chen[15] have given some solutions for the problem of footings on clay while Fernandez and Christian[16] examined flexible footing and retaining wall problems.

In this paper a formulation is given for the solution of problems of finite elasto-plastic flow without restricting deformation magnitude. Plastic failure is described by a general yield condition and plastic deformation by an arbitrary flow rule. The theory is developed for a general constitutive law which relates an objective stress rate to the strain rate. This theory has applications to such problems as: the penetration of embankments into very soft soil; the behaviour of layers of normally consolidated clay in which both elastic modulus and undrained

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shear strength increase with depth, being virtually zero at the surface; the post-peak load behaviour of very sensitive clays for which large shear strains are required to attain the full softening corresponding to the remoulded strength; bomb crater and underground cavity formation. The theory is illustrated by several examples.

FORMULATION FOR A GENERAL RATE LAW

Preliminary remarks

There are two modes of description of the deformations of a continuous medium, the Lagrangian and the Eulerian. In a fixed reference frame the Lagrangian description employs the co-ordinates a_i of a typical material particle in some initial reference state as independent variables. In the Eulerian description the co-ordinates x_i of a material particle in the current deformed condition are considered independent. When deformations of unrestricted magnitude are described careful distinction must be made between the two systems.

Previous attempts to formulate solutions to finite deformation problems, in a manner suitable for use with the finite element method, have differed both in their choice of the mode of description and in the selection of a suitable constitutive law. Most, however, use an incremental approach in which a specified loading path is followed. For example Hibbitt *et al*[17] use a Lagrangian treatment and stress–strain law given by:

$$\Delta S_{ij} = D_{ijkl}\Delta E_{kl} \tag{1}$$

where

ΔS_{ij}—is the incremental Kirchhoff stress tensor associated with

ΔE_{kl}—the incremental Green's[18] strain tensor, and

D_{ijkl}—is a known function of the current state.

Hofmeister *et al*[19] also use a Lagrangian approach with a constitutive law:

$$\Delta\sigma_{ij} = D_{ijkl}\Delta E_{kl} \tag{2a}$$

where $\Delta\sigma_{ij}$ is an incremental stress tensor given by

$$\Delta\sigma_{ij} = \sigma_{ij} - \sigma_{ij}^0 \tag{2b}$$

with

σ_{ij}^0—as the initial Cartesian stress tensor (before the current increment of loading) referred to a global reference frame, and

σ_{ij}—as the subsequent Cartesian stress tensor (after the current increment of loading) referred also to the global reference frame.

Davidson and Chen[15] adopt the law

$$\Delta\tau_{ij} = D_{ijkl}e_{kl} \tag{3a}$$

where

e_{kl}—is the incremental, infinitesimal strain tensor, and

$\Delta\tau_{ij}$—is the increment stress tensor given by:

$$\Delta\tau_{ij} = \tau_{ij} - \sigma_{ij}^0 \tag{3b}$$

where σ_{ij}^0 is as above in (2), and

τ_{ij}—is a Cartesian stress tensor in the subsequent configuration. These stress components are referred to a locally rotated Cartesian frame which varies from point to point.

Osias and Swedlow[12] have adopted an Eulerian formulation and a material law summarized by:

$$\hat{\sigma}_{ij} = D_{ijkl} d_{kl} \tag{4}$$

where

$\hat{\sigma}_{ij}$—is the Jaumann stress rate,[20] and

d_{kl}—is a deformation rate tensor,

while Fernandez and Christian[16] used one of several laws proposed by Biot.[21] Some idea of the difference in load-deflection response that may arise in a practical problem due to the adoption of different constitutive laws is discussed by Carter.[22] This problem is defined and the essential results are shown in Figure 3. They are discussed further in a later section.

In this paper the following approach is adopted. The solution involves following a specified load path. At any time t_0, the general body of Figure 1 occupies a region in space V_0 bounded by the surface S_0 and is in equilibrium with known body forces and with specified tractions which act on a portion of the surface S_{0T}. On the remainder of the surface, S_{0D}, velocities are specified.† The body has moved to this configuration from some reference state by the combined action of all forces that acted on it from time 0 to t_0. We focus attention on the movement of the body during a subsequent increment in loading in the interval t_0 to t. At time t the body occupies the region V bounded by the surface S and tractions and velocities are specified on S_T and S_D respectively. In such circumstances it seems reasonable to postulate that there exists a relation, which may of course depend on the previous history of the body, between the increments in stress and the increment in strain and that as the time interval becomes infinitesimal this will reduce to a relation between stress rate and strain rate.

A general rate law

For simplicity of presentation it is convenient in this paper to adopt a Cartesian reference frame and deal with problems of plane strain.‡

At a given time t_0 consider a typical particle in the deforming body to occupy a position P_0 in space described by the vector $\mathbf{a} = (a, b)$ relative to some fixed reference frame. At time t this same particle occupies P, described by $\mathbf{x} = (x, y)$. These two vectors are of course related by

$$\mathbf{x} = \mathbf{a} + \mathbf{u} \tag{5}$$

where $\mathbf{u} = (u_x, u_y)$ is the vector of displacement components during the current incremental movement. Adopting an Eulerian description, the instantaneous rate of deformation may be described by the vector of velocity gradients

$$\mathbf{d} = (\boldsymbol{\varepsilon}^{\mathrm{T}}, \omega)^{\mathrm{T}} \tag{6}$$

where

$$\boldsymbol{\varepsilon} = \left(\frac{\partial v_x}{\partial x}, \frac{\partial v_y}{\partial y}, \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^T$$

$$\omega = \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) \quad \text{and}$$

$$\mathbf{v} = (v_x, v_y) = \frac{\partial}{\partial t}\mathbf{u}(\mathbf{a}, t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(\mathbf{x}, t)$$

† More complicated boundary conditions are easily incorporated into the theory.

‡ The extension to more general situations is straightforward, but will not be dealt with here.

is the vector of velocity components at time t. The quantities $\boldsymbol{\varepsilon}$ and ω are related to $\boldsymbol{\varepsilon}'$ and ω', the Lagrangian strain rates and rotation or spin rates respectively, through

$$\mathbf{d}' = \Phi \mathbf{d} \tag{7}$$

where

$$\mathbf{d}' = (\boldsymbol{\varepsilon}'^{\mathrm{T}}, \omega')^{\mathrm{T}},$$

$$\boldsymbol{\varepsilon}' = \left(\frac{\partial v_x}{\partial a}, \frac{\partial v_y}{\partial b}, \frac{\partial v_x}{\partial b} + \frac{\partial v_y}{\partial a}\right)^{\mathrm{T}},$$

$$\omega' = \left(\frac{\partial v_x}{\partial b} - \frac{\partial v_y}{\partial a}\right)$$

and

$$\Phi = \begin{bmatrix} 1 + \dfrac{\partial u_x}{\partial a}, & 0, & \dfrac{1}{2}\dfrac{\partial u_y}{\partial a}, & \dfrac{1}{2}\dfrac{\partial u_y}{\partial a} \\ 0, & 1 + \dfrac{\partial u_y}{\partial b}, & \dfrac{1}{2}\dfrac{\partial u_x}{\partial b}, & -\dfrac{1}{2}\dfrac{\partial u_x}{\partial b} \\ \dfrac{\partial u_x}{\partial b}, & \dfrac{\partial u_y}{\partial a}, & 1 + \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial a} + \dfrac{\partial u_y}{\partial b}\right), & \dfrac{1}{2}\left(\dfrac{\partial u_y}{\partial b} - \dfrac{\partial u_x}{\partial a}\right) \\ \dfrac{\partial u_x}{\partial b}, & -\dfrac{\partial u_y}{\partial a}, & \dfrac{1}{2}\left(\dfrac{\partial u_y}{\partial b} - \dfrac{\partial u_x}{\partial a}\right), & 1 + \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial a} + \dfrac{\partial u_x}{\partial b}\right) \end{bmatrix}$$

An important problem in continuum mechanics has been that of selecting a suitable definition of stress-rate for use in constitutive relations. It has commanded the attention of, amongst others, Jaumann,[20], Truesdell,[3] Green[23] and Cotter and Rivlin,[24] each adopting a slightly different definition. An acceptable stress rate must give an objective measure of the change in stress when viewed from a frame rotating with the material. Its definition must, therefore, contain a rotary term to compensate for the fact that the stress components with respect to a fixed co-ordinate system change, even when there is no change in the stress components with respect to a co-ordinate system that participated in the instantaneous rotation of the neighbourhood of a considered particle. The example of the rigid rotation of a bar in simple tension is often quoted.[25] As Prager[26] and Oldroyd[27] have observed, the difference between acceptable definitions of the stress rate must necessarily consist of a linear combination of strain rates. Thus in any constitutive law which expresses the stress rate as a linear combination of the strain rates, such terms can always be absorbed into this linear relationship, so that the difference between the various definitions is only illusory. As pointed out by Prager[26] there are sound reasons for preferring the definition due to Jaumann[20] when plastic behaviour is involved and this definition is adopted in this paper. It is given by:

$$\hat{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + M\mathbf{d} \tag{8}$$

where

$$M = \begin{bmatrix} 0, & 0, & 0, & -\sigma_{xy} \\ 0, & 0, & 0, & \sigma_{xy} \\ 0, & 0, & 0, & \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy}) \end{bmatrix},$$

$\mathbf{d}$—is the vector defined by equation (6)

$\boldsymbol{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})^{\mathrm{T}}$ is the vector of Cartesian stress components at the instant t.

The superior dot indicates material differentiation with respect to time.

A general linear relationship between the objective stress rate and the strain rate can be written in the form:

$$\hat{\boldsymbol{\sigma}} = D\boldsymbol{\varepsilon} + \mathbf{g} \tag{9a}$$

where

$\hat{\boldsymbol{\sigma}}$ – is the Jaumann stress rate vector,

$\boldsymbol{\varepsilon}$ – is the vector of strain rates defined in equation (6), and

D – is a matrix whose components depend on the current state and perhaps all previous states, in some specified way.

and where the vector $\mathbf{g}$ is included to account for the presence of any features such as pore pressures (i.e. an effective stress law) or thermal and shrinkage effects. If these are absent then $\mathbf{g} = \mathbf{0}$. We may also express equation (9a) as

$$\hat{\boldsymbol{\sigma}} = H\mathbf{d} + \mathbf{g} \tag{9b}$$

where $H = (D, \mathbf{0})$. Alternatively equation (9b) may be written as:

$$\dot{\boldsymbol{\sigma}} = P\mathbf{d} + \mathbf{g} \tag{9c}$$

where

$$P = H - M$$

It should be emphasized here that the displacements $\mathbf{u}$ are measured relative to the position of the body at time t_0 and thus vanish when $t = t_0$, while the stresses $\boldsymbol{\sigma}$ are the current values at time t.

Governing equations

For the body of Figure 1 we assume that at time t the stress field $\boldsymbol{\sigma}(\mathbf{x}, t)$ is in equilibrium with a traction set $\mathbf{T}$ acting over the surface S_T and with bodyforces $\mathbf{F}$ within V. We allow the vector $\mathbf{v}$ to

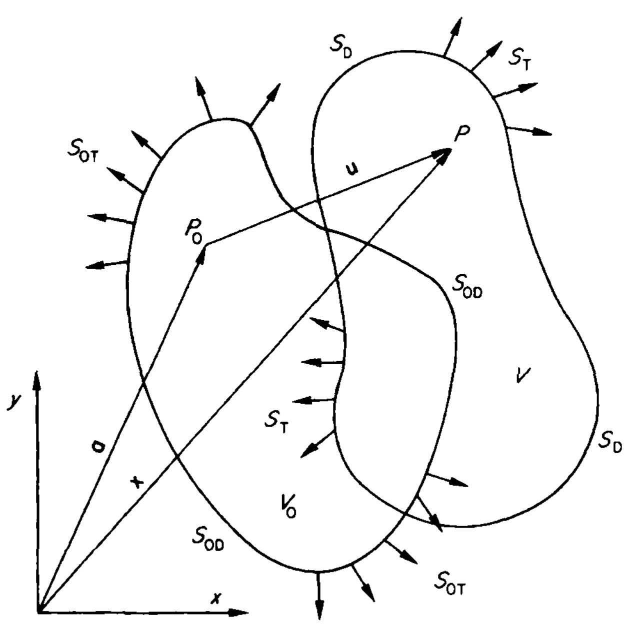

Figure 1

denote the velocity field of all particles within V, which satisfies the velocity boundary conditions. The incremental velocities $\Delta\mathbf{v}$ are compatible with the incremental strain rates $\Delta\boldsymbol{\varepsilon}$. Use may thus be made of the principle of virtual work and hence at time t.

$$\int_V \Delta\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}\,\mathrm{d}V=\int_V \Delta\mathbf{v}^{\mathrm{T}}\mathbf{F}\,\mathrm{d}V+\int_{S_{\mathrm{T}}} \Delta\mathbf{v}^{\mathrm{T}}\mathbf{T}\,\mathrm{d}S \tag{10}$$

Equation (9c) may be integrated to yield

$$\boldsymbol{\sigma}-\boldsymbol{\sigma}_0=\int_{t_0}^{t} (P\mathbf{d}+\mathbf{g})\,\mathrm{d}t \tag{11}$$

where $\boldsymbol{\sigma}_0$ is the Cartesian vector denoting the stress field within V_0 at time t_0.

Substituting equation (11) into (10) gives

$$\int_V \Delta\boldsymbol{\varepsilon}^{\mathrm{T}}\left\{\boldsymbol{\sigma}_0+\int_{t_0}^{t} (P\mathbf{d}+\mathbf{g})\,\mathrm{d}t\right\}\mathrm{d}V=\int_V \Delta\mathbf{v}^{\mathrm{T}}\mathbf{F}\,\mathrm{d}V+\int_{S_{\mathrm{T}}} \Delta\mathbf{v}^{\mathrm{T}}\mathbf{T}\,\mathrm{d}S \tag{12}$$

This is an exact equation governing the behaviour of the body in deforming from V_0 to V.

An approximate solution of equation (12) can be obtained using the finite element method. If the continuous deforming body is divided into a discrete number of conforming elements then suppose that the displacement field can be adequately represented by values at the connecting nodes $1, 2, \ldots, N$ and let

$$\boldsymbol{\delta}^{\mathrm{T}}=(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N)$$

Then suppose that the continuous velocity field can be approximated by

$$\mathbf{v}=A\dot{\boldsymbol{\delta}} \tag{13}$$

where the form of A will depend upon the particular type of element used and will, in general, depend upon its current position. It follows then that the vector $\mathbf{d}$, of velocity gradients, and $\boldsymbol{\varepsilon}$, the strain rates may be related to $\dot{\boldsymbol{\delta}}$ by the approximations

$$\mathbf{d}=B\dot{\boldsymbol{\delta}} \tag{14a}$$

and

$$\boldsymbol{\varepsilon}=C\dot{\boldsymbol{\delta}} \tag{14b}$$

where

$$B=\begin{bmatrix} \dfrac{\partial}{\partial x}, & 0 \\ \dfrac{\partial}{\partial y}, & 0 \\ \dfrac{\partial}{\partial y}, & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y}, & -\dfrac{\partial}{\partial x} \end{bmatrix} A$$

and C is matrix B with the last row deleted. Substituting into equation (12) it is found that for arbitrary variations $\Delta\dot{\boldsymbol{\delta}}$ consistent with the velocity boundary conditions

$$\Delta\dot{\boldsymbol{\delta}}^{\mathrm{T}}\left[\int_V \left\{C^{\mathrm{T}}\int_{t_0}^{t} PB\dot{\boldsymbol{\delta}}\,dt\right\}\mathrm{d}V-\boldsymbol{h}\right]=0 \tag{15}$$

and thus that

$$\int_V \left\{C^{\mathrm{T}} \int_{t_0}^{t} PB\dot{\boldsymbol{\delta}}\,\mathrm{d}t\right\} \mathrm{d}V = \mathbf{h} \tag{16a}$$

where

$$\mathbf{h} = -\int_V C^{\mathrm{T}}\left\{\boldsymbol{\sigma}_0 + \int_{t_0}^{t} \mathbf{g}\,\mathrm{d}t\right\} \mathrm{d}V + \int_V A^{\mathrm{T}}\mathbf{F}\,\mathrm{d}V + \int_{S_{\mathrm{T}}} A^{\mathrm{T}}\mathbf{T}\,\mathrm{d}S \tag{16b}$$

In many cases the conservation of mass can be used to simplify the integral containing the bodyforce. However, in some circumstances it is convenient to consider such forces as coming into being over a period of time. This may be used as a numerical device when material is added to the body or may correspond to an actual physical loading such as in a centrifuge testing apparatus. A numerical integration procedure for the solution of the non-linear, integral equations (16) is discussed in a later section.

APPLICATION TO SOIL PLASTICITY

The analysis of an ideal elasto-plastic soil undergoing infinitesimal deformation has been examined on several occasions.[10,28,29] More recently there has been an attempt to extend this analysis to the problem of finite deformation. However, these investigations have been restricted to materials with an associated flow rule.[12,15,16] In this section the constitutive equations governing a material with an arbitrary yield condition and a general flow law will be developed.

The strain rate $\boldsymbol{\varepsilon}$ can be thought of as consisting of two components, $\boldsymbol{\varepsilon}_{\mathrm{E}}$ the elastic strain rate and $\boldsymbol{\varepsilon}_{\mathrm{P}}$ the plastic strain rate, where

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathrm{E}} + \boldsymbol{\varepsilon}_{\mathrm{P}} \tag{17}$$

The elastic strain rate is related to the stress rate $\hat{\boldsymbol{\sigma}}$ through Hooke's Law

$$\hat{\boldsymbol{\sigma}} = D_{\mathrm{E}}\boldsymbol{\varepsilon}_{\mathrm{E}} + \mathbf{g}_{\mathrm{E}} \tag{18}$$

where D_{E} is the matrix of elastic constants and $\mathbf{g}_{\mathrm{E}}$ serves a similar purpose to that described in equations (9). For an isotropic material under conditions of plane strain

$$D_{\mathrm{E}} = \begin{bmatrix} \Lambda+2G, & \Lambda, & 0 \\ \Lambda, & \Lambda+2G, & 0 \\ 0, & 0, & G \end{bmatrix}$$

where Λ, G are the Lamé parameters..

The plastic strain rate may be written in the form

$$\boldsymbol{\varepsilon}_{\mathrm{P}} = \lambda\mathbf{a} \tag{19}$$

where the vector $\mathbf{a}$ depends upon the current state of the material and may in fact depend upon all previous states,† and λ is a multiplier signifying that there are no viscous effects present. λ has to be positive to fulfil the necessary requirement that the rate of plastic work must always be positive.[30,31] The importance of this in relation to numerical computation is discussed later.

† A correct determination of the properties $\mathbf{a}$, D_{E} occurring in equations (18), (19) would involve the testing of an element whose history (load path) was identical to the given element.

It will be assumed that the yield criterion can be written in the form

$$f(\boldsymbol{\sigma})=0 \tag{20}$$

Thus for a soil obeying the Mohr–Coulomb failure criterion and deforming under conditions of plane strain

$$f(\boldsymbol{\sigma})=(\sigma_{xx}-\sigma_{yy})^2+4\sigma_{xy}^2-\sin^2\phi(\sigma_{xx}+\sigma_{yy}-2c\cot\phi)^2 \tag{21}$$

where c, ϕ are the cohesion and angle of internal friction respectively and tensile stresses are reckoned positive.

Many investigators have assumed that the material has an associated flow rule, that is

$$\mathbf{a}\propto\mathbf{b} \tag{22a}$$

where

$$\mathbf{b}=\left(\frac{\partial f}{\partial\sigma_{xx}},\frac{\partial f}{\partial\sigma_{yy}},\frac{\partial f}{\partial\sigma_{xy}}\right)^{\mathrm{T}} \tag{22b}$$

For a Mohr–Coulomb material under conditions of plane strain this would predict a flow rule of the form

$$\boldsymbol{\varepsilon}_{\mathrm{P}}=4\lambda R(\sin\phi+\cos 2\theta,\ \sin\phi-\cos 2\theta,\ 2\sin 2\theta)^{\mathrm{T}} \tag{23}$$

where

$$R=\left(\frac{(\sigma_{xx}-\sigma_{yy})^2}{2}+\sigma_{xy}^2\right)^{1/2}$$

is the radius of the Mohr circle of stress and θ is the angle between the major principal stress direction and the x axis.

As is well known a flow rule described by equations (22), (23) predicts a rate of dilatancy far in excess of that observed in real soils.† Davis[32] has postulated a class of ideal soils which have a flow rule that may be expressed for conditions of plane strain as:

$$\boldsymbol{\varepsilon}_{\mathrm{P}}=4\lambda R(\sin\psi+\cos 2\theta,\ \sin\psi-\cos 2\theta,\ 2\sin 2\theta)^{\mathrm{T}} \tag{24}$$

where ψ is a measure of the rate of dilatancy of the soil so that $\psi=\phi$ corresponds to a material with an associated flow rule and $\psi=0$ corresponds to material which deforms plastically at constant volume.

Next consider an element in a plastic state, which deforms from an initial position A to an adjacent position B in a time interval dt, as shown schematically in Figure 2.

Thus initially:

$$f\{\sigma_{xx}(t),\ \sigma_{yy}(t),\ \sigma_{xy}(t)\}=0 \tag{25a}$$

The question now arises as to the form of the yield criterion when the element is in position B. One reasonable assumption is that the form of any strength anisotropy is intrinsic to the element so that

$$f\{\sigma_{\xi\xi}(t+\mathrm{d}t),\ \sigma_{\eta\eta}(t+\mathrm{d}t),\ \sigma_{\xi\eta}(t+\mathrm{d}t)\}=0 \tag{25b}$$

† An important exception is the undrained deformation of saturated clays.

Combining equations (25a, b) it is found that

$$\mathbf{b}^{\mathrm{T}}\hat{\boldsymbol{\sigma}}=0 \tag{26}$$

where $\mathbf{b}$ is a vector normal to the yield surface and $\hat{\boldsymbol{\sigma}}$ is the Jaumann stress rate.

It is interesting to note that according to this assumption as an element rotates the form of the yield criterion (referred to the initial set of axes) changes. Thus a material which is anisotropic,

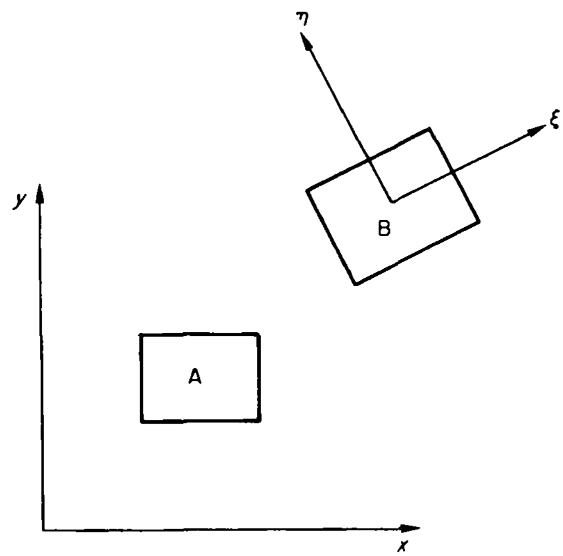

Figure 2

but whose initial anisotropy is homogeneous, gradually develops an inhomogeneity of anisotropy as different elements rotate by different amounts. Of course this does not occur (according to this formulation) if the material is initially isotropic and in such a case

$$\mathbf{b}^{\mathrm{T}}\hat{\boldsymbol{\sigma}}=\mathbf{b}^{\mathrm{T}}\dot{\boldsymbol{\sigma}}=0$$

It is now possible to derive a relationship between strain rate and stress rate. Equations (17)–(19) may be combined to show that

$$D_{\mathrm{E}}\boldsymbol{\varepsilon}=\hat{\boldsymbol{\sigma}}-\mathbf{g}_{\mathrm{E}}+\lambda\boldsymbol{\alpha} \tag{27}$$

where $\boldsymbol{\alpha}=D_{\mathrm{E}}\mathbf{a}$.

Now on premultiplying equation (27) by

$$\chi=1-\frac{\boldsymbol{\alpha}\mathbf{b}^{\mathrm{T}}}{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{b}}$$

it is found that

$$\hat{\boldsymbol{\sigma}}-\chi\mathbf{g}_{\mathrm{E}}=\chi D_{\mathrm{E}}\boldsymbol{\varepsilon} \tag{28}$$

This is precisely the form of equation (9a) considered in the previous section, with

$$D = \chi D_{\mathrm{E}} = \left(1 - \frac{\boldsymbol{\alpha}\mathbf{b}^{\mathrm{T}}}{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{b}}\right) D_{\mathrm{E}}$$

$$\mathbf{g} = \chi \mathbf{g}_{\mathrm{E}} = \left(1 - \frac{\boldsymbol{\alpha}\mathbf{b}^{\mathrm{T}}}{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{b}}\right) \mathbf{g}_{\mathrm{E}} \tag{29}$$

and thus the formulation for this general rate law applies to the elasto-plastic material.

NUMERICAL SOLUTION

The basic finite element equation, (16), is an integral equation and will in general require a numerical solution. The most obvious method will involve following a given load path using a number of finite but small steps. For any such step in time, Δt, we approximate all time dependent quantities by their average values for this particular step. Equation (16) then reduces to the approximate set of equations.

$$\bar{K}\boldsymbol{\delta} = \bar{\mathbf{h}} \tag{30}$$

where

$$\underset{\mathbf{Sd}}{\bar{K}} = \int_{V} \bar{C}^{\mathrm{T}} \bar{P} \bar{B} \, \mathrm{d}V$$

and

$$\bar{\mathbf{h}} = -\int_{\bar{V}} \bar{C}^{\mathrm{T}} (\boldsymbol{\sigma}_0 + \bar{\mathbf{g}}\Delta t) \, \mathrm{d}V + \int_{V} \bar{A}^{\mathrm{T}} \mathbf{F} \, \mathrm{d}V + \int_{\bar{S}_{\mathrm{T}}} \bar{A}^{\mathrm{T}} \mathbf{T} \, \mathrm{d}S$$

The superior bar denotes some average or representative value of the quantity for the current time step and the integrations are carried out over the 'mean' configuration. In general, the set of incremental displacement equations, (30), will be non-symmetric. This lack of symmetry arises from two causes: the inclusion in the analysis of the effect of rotations; and because a non-associated flow rule is used to model soil behaviour. If the material had an associated flow rule and if the effect of rotations in any problem were considered insignificant and neglected, then the matrix $\bar{K}$ would reduce to a symmetric form. Once convergence has been achieved within any one load step the cumulative quantities must be updated. These include the nodal co-ordinates which are adjusted to incorporate the increments of nodal displacement. Thus each new step begins with a new 'initial' geometry.

In approximate solutions, convergence is not the only criterion of satisfactory computation if parts of the body have become plastic. As mentioned in the previous section, it is necessary to check that λ is everywhere positive in order to ensure that the rate of plastic work is positive.

The occurrence of negative values for some elements indicates either numerical inaccuracy or that, in the exact solution, the stress state at the points represented by these elements returns from a plastic to an elastic state. In either case straightforward continuation of the solution is unjustified, since it is clear that the basic constitutive laws of the material are being contravened. This limit to the numerical solution is clearly artificial and may be avoided by ensuring that no negative rate of plastic work occurs in the converged solution. This can be done by incorporating in the iteration procedure of the program the possibility that any plastic element may become elastic as well as the usual possibility that any elastic element may become plastic.[30,31]

EXAMPLES

To illustrate the theory developed in previous sections a number of examples are presented.

Homogeneous deformation—elastic material

The importance of paying close attention to the definition of stress rate is demonstrated by this example. Described in the inset to Figure 3 is an initially square, initially unstressed section of

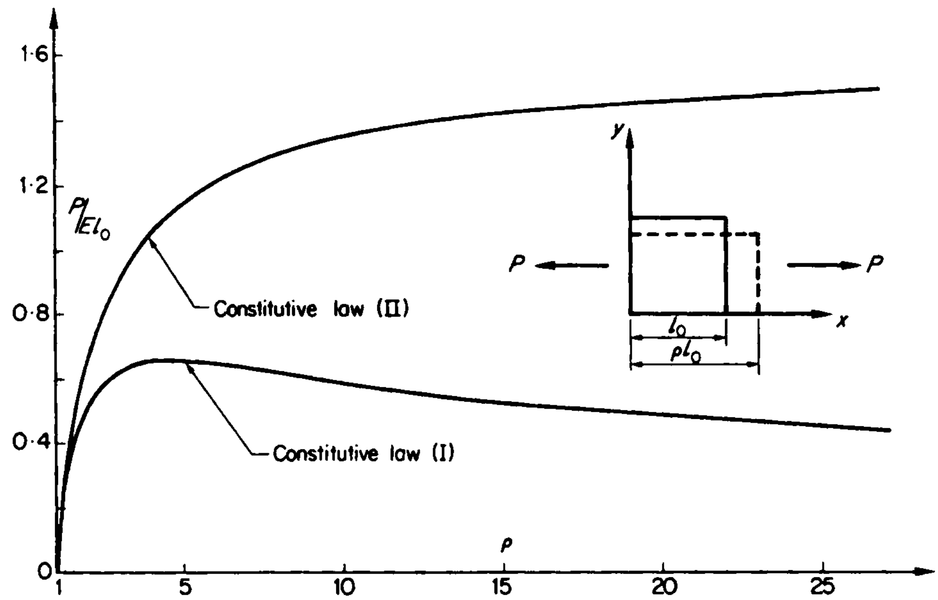

Figure 3. Homogeneous extension—elastic material, $\nu = 0{\cdot}4$

elastic material which is extended under plane strain conditions. Each of the two curves in Figure 3 corresponds to the adoption of a different constitutive law, each relating different stress rates to the same strain rate through identical material constants. The two laws are:

(i) The general rate law given in equation (9a). In this example involving no rotation equation (9a) reduces to

$$\dot{\boldsymbol{\sigma}} = D\boldsymbol{\varepsilon} \tag{9d}$$

(ii) The second constitutive law is expressible as

$$\bar{\hat{\boldsymbol{\sigma}}} = H\mathbf{d} \tag{31}$$

where H and $\mathbf{d}$ are as in subsection 'A general rate law' and $\bar{\hat{\boldsymbol{\sigma}}}$ is a stress rate defined by Biot[21] and is related to $\hat{\boldsymbol{\sigma}}$ by

$$\bar{\hat{\boldsymbol{\sigma}}} = \hat{\boldsymbol{\sigma}} + S\mathbf{d} \tag{32}$$

where

$$S=\begin{bmatrix} 0, & \sigma_{xx}, & -\sigma_{xy}, & 0 \\ \sigma_{yy}, & 0, & -\sigma_{xy}, & 0 \\ \frac{1}{2}\sigma_{xy}, & \frac{1}{2}\sigma_{xy}, & -(\sigma_{xx}+\sigma_{yy}), & 0 \end{bmatrix}$$

Equations (32) may be expressed in the form

$$\hat{\boldsymbol{\sigma}}=(H-S)\mathbf{d} \tag{33a}$$

and for this case involving zero rotation we may write

$$\dot{\boldsymbol{\sigma}}=(D-R)\boldsymbol{\varepsilon} \tag{33b}$$

where R is matrix S with the last column deleted.

Equations (32) and (33) exemplify the point made in subsection 'A general rate law' that the difference between acceptable definitions of stress rate consist merely of a linear combination of strain rates.

The two curves in Figure 3 were computed (both analytically and by finite element method) on the assumption that the matrix D (i.e. Young's modulus, E, and Poisson's ratio, ν) was constant for all states of stress. The two curves are markedly different at larger values of stretch ratio. Of course curve (I) could have been obtained using the stress rate $\bar{\boldsymbol{\sigma}}$ but then D would depend on the stress rate and similarly curve (II) could have been obtained using $\hat{\boldsymbol{\sigma}}$.

For most engineering materials including soil yielding occurs at a sufficiently small strain for distinction between alternative stress rates to be of little consequence in the elastic range. Finite deformation behaviour of massive configurations of such materials will be dominated by finite plastic strains. In the remaining examples all materials obey the general rate law of equation (9).

Homogeneous deformations—elasto-plastic material

The plane strain homogeneous extension and compression of an initially stress free, square section is described in the inset to Figure 4. The body consists of a material with a Mohr–Coulomb failure criterion, with cohesion c and friction angle ϕ. Results are given in Figure 4 for associated ($\psi=\phi$) and non-associated ($\psi\neq\phi$) flow rules as well as the results for a material with infinite strength ($E/c=0$). Young's modulus and Poisson's ratio are considered constant over all ranges of stress. The dilatancy rate is seen to have a significant effect on the load stretch response for those materials considered. The values of E/c used here are not intended to be representative of any real material but were chosen as a severe test of the numerical capabilities of the method.

Surface loading

The problem of normal loading applied to a width $2B$ of the surface of an initially stress free simple elasto-plastic purely cohesive soil, having a strength, c, and an elastic modulus, E, that increase with depth, is defined in the inset to Figure 5. For finite strain problems, the boundary conditions have to be specified more fully than for infinitesimal theory. In this problem it is assumed that the applied traction q remains uniform in intensity and always acting normal to the originally horizontal surface, even when that surface becomes significantly distorted. It is also assumed that this traction is always applied to the same physical part of the surface, even when this part has a length different from its original length $2B$. Figures 6(a)–(d) show the progressive distortion of the surface and the growth of the plastic region (note the extreme distortion of elements close to the loading in Figure 6(d)). Figure 6(d) only gives the arbitrarily selected end to the computation. Even though the plastic zone extends completely from beneath the loading to an appreciable part of the free boundary, the addition of further load, and the consequent further deformation and distortion, could presumably continue indefinitely. This is in contrast to the infinitesimal theory, where unconfined plastic flow signifies final general failure. This point is

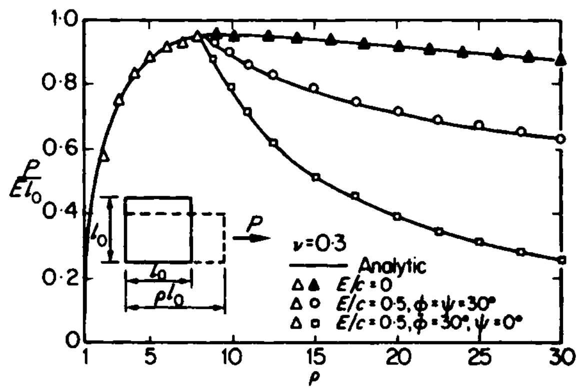

Figure 4(a). Homogeneous extension

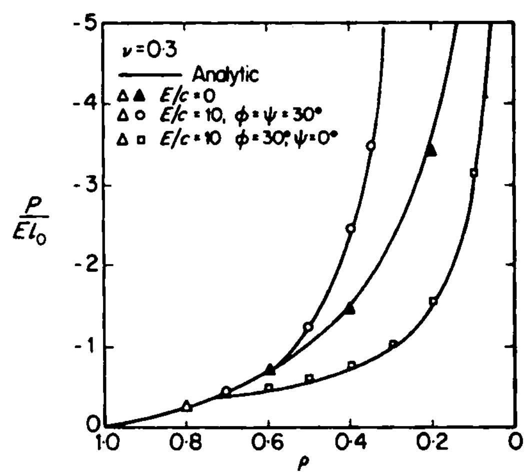

Figure 4(b). Homogeneous compression

brought out by the load deformation curves given in Figure 5. The corresponding curves for homogeneous soil, having a value of ratio $E/c = 200$, which is typical for soil, are given in Figure 7, showing again that no definite value can be assigned to the ultimate bearing capacity when finite strain theory is employed, although the difference between the curves for finite and infinitesimal theory is less than for the inhomogeneous soil case of Figure 5. The full range, from $E/c = 0$ (infinitely strong material showing finite elastic strain effects only) to $E/c = \infty$ (rigid

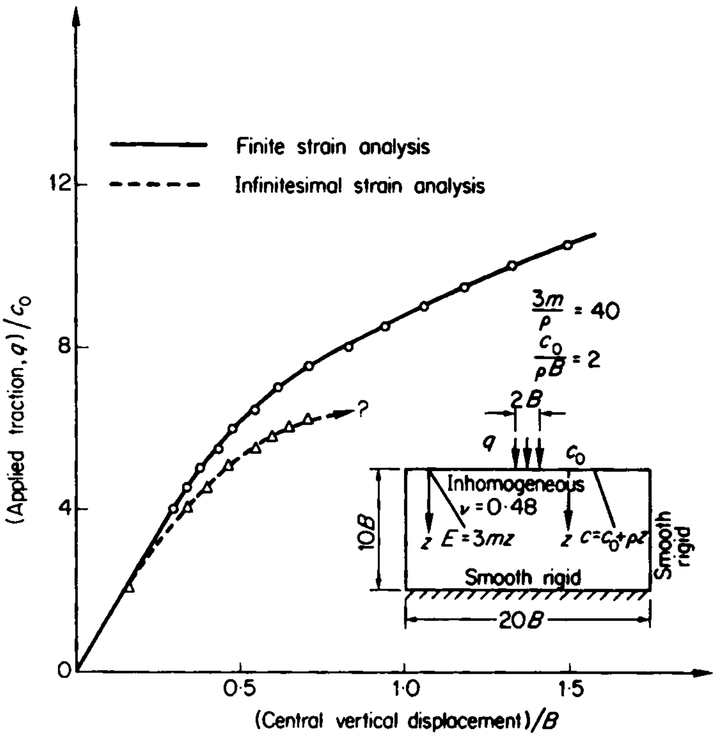

Figure 5. Inhomogeneous case

plastic material) is indicated in Figure 8. Figures 9(a)–(d) show the progressive distortion of the surface and growth of plastic regions for the homogeneous case when $E/c = 10$.

Embankments on soft clay layers

The problem of building up a very long embankment in stages over a soft clay layer is of some practical interest. The present numerical technique allows an examination of the coupled behaviour of the embankment and the clay layer. The example presented here indicates the general trend in behaviour and highlights the importance of performing a finite deformation analysis.

The clay is considered to be an undrained purely cohesive material with $E_c/c_c = 50$ and $\nu_c = 0{\cdot}48$ (to approximate incompressibility behaviour) and of initial depth D. The embankment material is considered to be purely frictional with $\nu_s = 0{\cdot}3$, $\phi_s = \psi_s = 35°$ and density γ_s. The ratio of the moduli of the embankment material to that of the clay layer is $E_s/E_c = 10$.

Construction of the embankment, having a side slope of 30°, takes place in a number of finite lifts. This is modelled numerically, at any stage, by adding a row of elements above the current crest level, then applying the gravity load to these elements. The new row of elements are added so that before gravity is applied they produce a horizontal crest. Figure 10 shows the elevation of the crest centre (above the rigid base of the clay layer) at various stages during construction and for various ratios of density of embankment material to initial density of clay. In all cases the clay was initially in a state of hydrostatic stress (i.e. coefficient of earth pressure at rest, $K_0 = 1$). If an

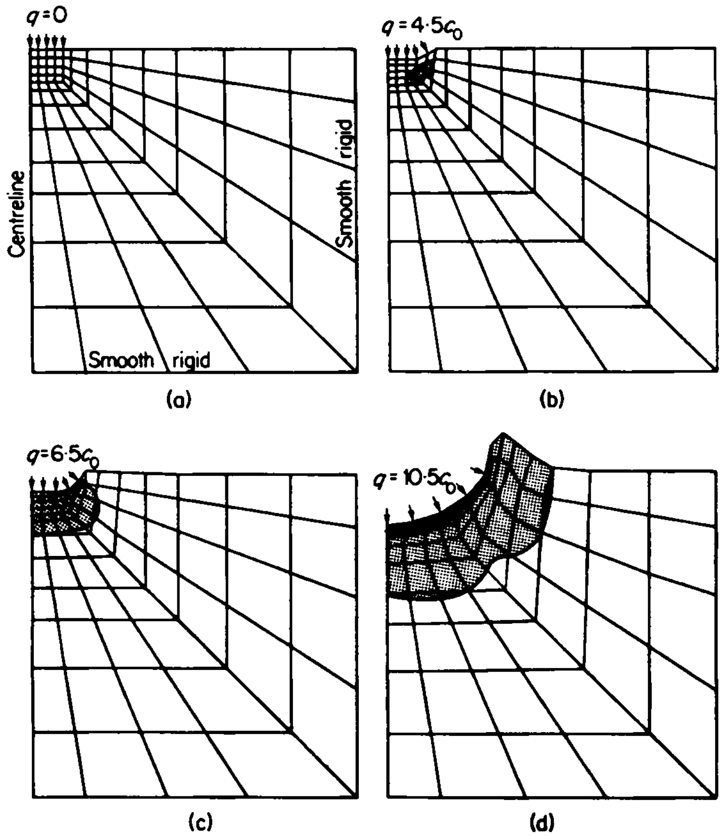

Figure 6. Development of plastic regions, inhomogeneous case

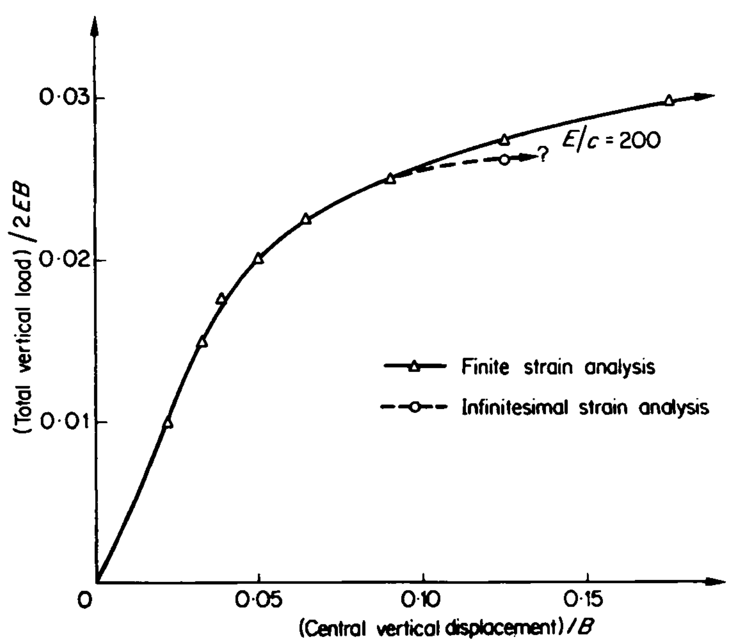

Figure 7. Homogeneous case

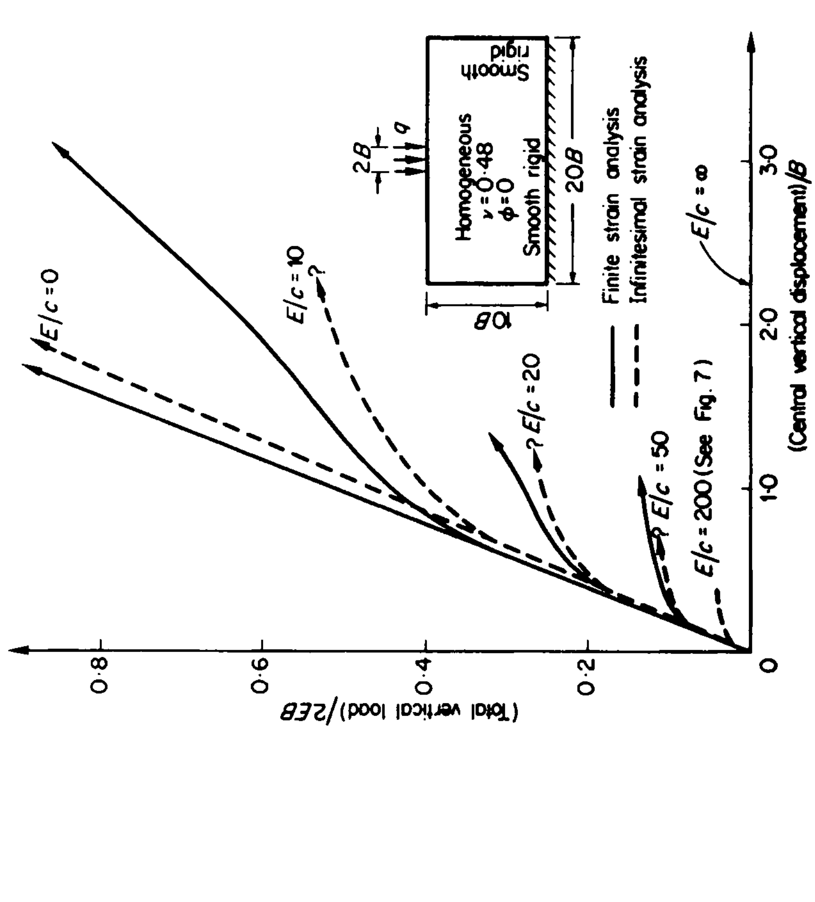

Figure 8. Homogeneous case

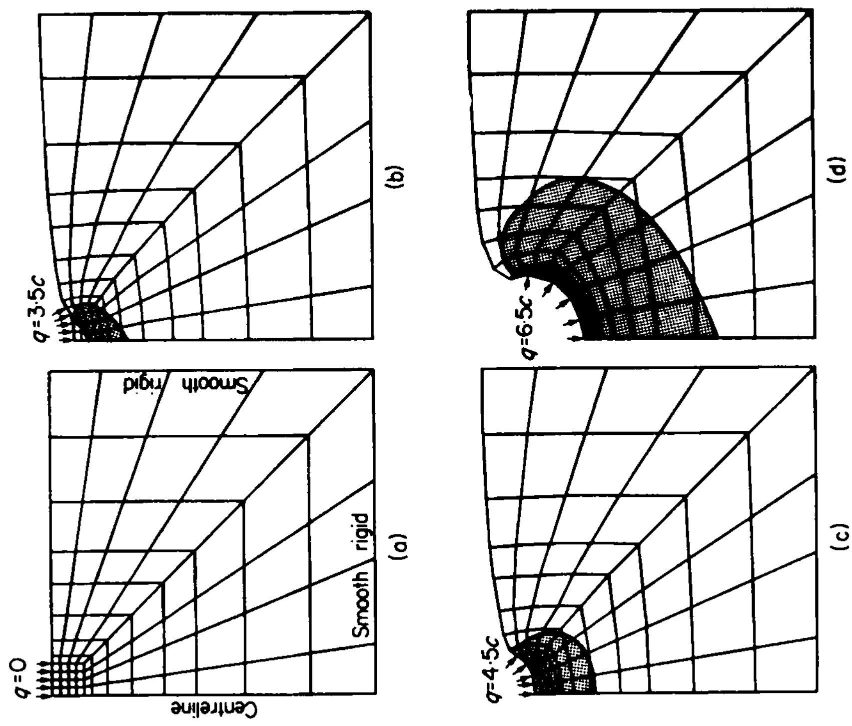

Figure 9. Development of plastic regions, homogeneous case

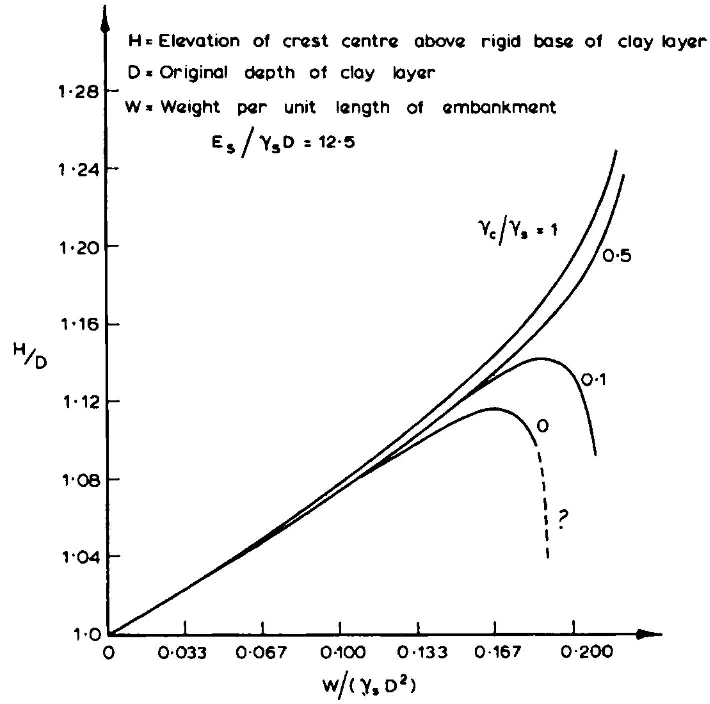

Figure 10. Elevation of crest centre v embankment weight

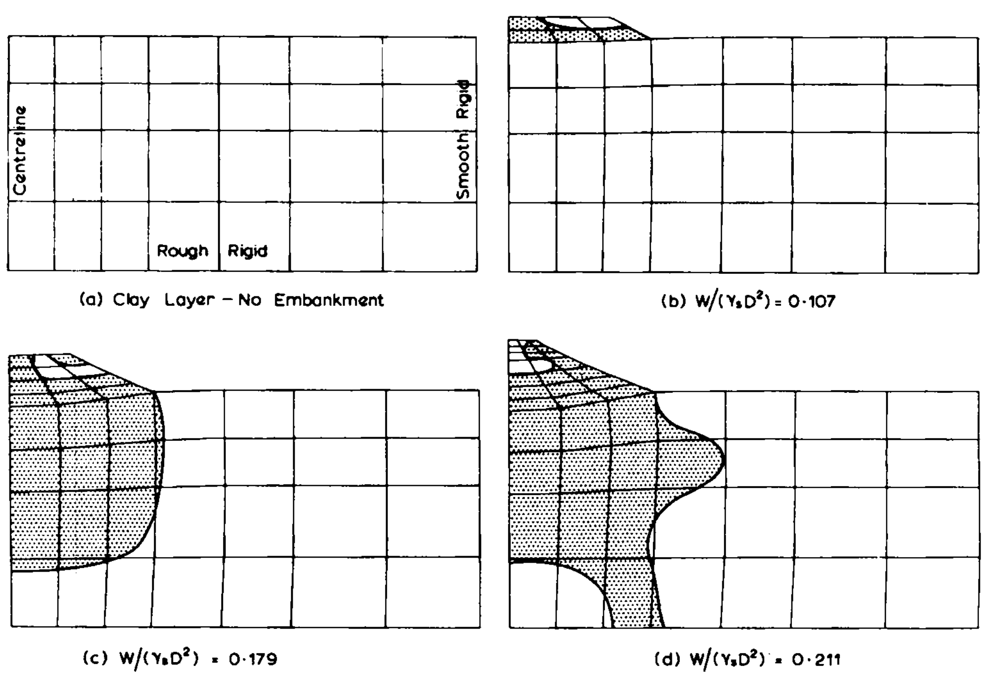

Figure 11. Growth of plastic region for embankment problem ($\gamma_c/\gamma_s = 1$)

infinitesimal analysis or a finite analysis which neglected the importance of the initial stress state was performed for this problem then the four curves of Figure 10 would all coincide. The density of the clay layer would have no influence at all on these results. Moreover there would be no effect on the pattern of yielding in an infinitesimal analysis because initially $K_0 = 1$. Indeed, in the early stages of construction the four cases nearly coincide with each tending to a common solution at zero embankment load.† For low values of γ_c/γ_s there is a critical load (weight of added fill material) beyond which any addition of fill to the crest of the embankment causes a settlement greater than the height of material added. For higher values of γ_c/γ_s crest elevation is limited only by the production of a triangular apex in the embankment cross section. However, this feature is not only a function of γ_c/γ_s but also of the strength and deformation properties of the clay material as well as the geometry of bank section. For a wider bank on similar material the critical load feature will occur at a value of the ratio $\gamma_c/\gamma_s > 0{\cdot}5$.

Figures 11(a)–(d) show the finite element mesh at various stages during construction, for the case when $\gamma_c/\gamma_s = 1$.

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† The non-coincidence at higher bank heights arises essentially because in equation (16b) the first integral no longer exactly cancels that part of the remaining two integrals which arose in the interval $t = 0$ to t_0.

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