Limit Analysis versus Limit Equilibrium for Slope Stability

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Abstract: This paper compares conventional limit-equilibrium results with rigorous upper and lower bound solutions for the stability of simple earth slopes. The bounding solutions presented in this paper are obtained by using two newly developed numerical procedures that are based on finite-element formulations of the bound theorems of limit analysis and linear programming techniques. Although limit-equilibrium analysis is used widely in practice for estimating the stability of slopes, its use may sometimes lead to significant errors as both kinematic and static admissibility are violated in the method. Because there are no exact solutions available against which the results of limit-equilibrium analysis can be checked, the present bounding solutions can be used to benchmark the results of limit-equilibrium analysis. In addition, the lower bound solutions obtained also can be applied directly in practice because they are inherently conservative.

INTRODUCTION

Background and Objectives

Because of its practical importance, the analysis of slope stability has received wide attention in the literature. Limit-equilibrium analysis has been the most popular method for slope stability calculations. A major advantage of this approach is that complex soil profiles, seepage, and a variety of loading conditions can be easily dealt with. Many comparisons of limit-equilibrium methods [see, for example, Whitman and Bailey (1967), Fredlund and Krahn (1977), Duncan and Wright (1980), and Nash (1987)] indicate that techniques that satisfy all conditions of global equilibrium give similar results. Regardless of the different assumptions about the interslice forces, these methods (such as those of Janbu, Spencer, and Morgenstern and Price) give values of the safety factor that differ by no more than 5%. Even though it does not satisfy all conditions of global equilibrium, Bishop's simplified method also gives very similar results. Partly because of this and partly because of its simplicity, the slice method of limit-equilibrium analysis proposed by Bishop (1955) has been used widely for predicting slope stability under both drained and undrained loading conditions. Because of the approximate and somewhat arbitrary nature of limit-equilibrium analysis, concern is often voiced about how accurate these types of solutions really are. The answer to this question is particularly important in cases where designs are based on slim margins of safety.

The following are objectives of the present paper: (1) To present rigorous lower and upper bound solutions for the stability of simple slopes in both homogeneous and inhomogeneous soils; and (2) to check the accuracy of the method of Bishop (1955) by comparing its solutions against those derived from limit analysis. To overcome the difficulties of manually constructing both statically admissible stress fields and kinematically admissible velocity fields, two newly developed numerical procedures are used to calculate both upper and lower bound solutions for the stability of simple earth slopes in both purely cohesive and cohesive-frictional soils. These numerical procedures are based on finite-element formulations of the bound theorems of limit analysis and the best lower and upper bound solutions are obtained, respectively, by optimizing statically admissible stress fields and kinematically admissible velocity fields using linear programming techniques.

Over the years, many limit-equilibrium computer codes have been developed to locate the most critical failure surface by using various search strategies. In this study, the limit-equilibrium results are obtained from the well-known slope stability computer code STABL with the option of Bishop's simplified method (Kim et al. 1997).

Limit Analysis Method

The limit analysis method models the soil as a perfectly plastic material obeying an associated flow rule. With this idealization of the soil behavior, two plastic bounding theorems (lower and upper bounds) can be proved (Drucker et al. 1952; Chen 1975).

According to the upper bound theorem, if a set of external loads acts on a failure mechanism and the work done by the external loads in an increment of displacement equals the work done by the internal stresses, the external loads obtained are not lower than the true collapse loads. It is noted that the external loads are not necessarily in equilibrium with the internal stresses and the mechanism of failure is not necessarily the actual failure mechanism. By examining different mechanisms, the best (least) upper bound value may be found. The lower bound theorem states if an equilibrium distribution of stress covering the whole body can be found that balances a set of external loads on the stress boundary and is nowhere above the failure criterion of the material, the external loads are not higher than the true collapse loads. It is noted that in the lower bound theorem, the strain and displacements are not considered and that the state of stress is not necessarily the actual state of stress at collapse. By examining different admissible states of stress, the best (highest) lower bound value may be found.

The bound theorems of limit analysis are particularly useful if both upper and lower bound solutions can be calculated, because the true collapse load can then be bracketed from above and below. This feature is invaluable in cases for which an exact solution cannot be determined (such as slope stability problems), because it provides a built-in error check on the accuracy of the approximate collapse load.

Limit-Equilibrium Method

The limit-equilibrium method has been used to analyze slope stability problems in soil mechanics for many years by

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assuming that soil at failure obeys the perfectly plastic Mohr-Coulomb criterion [e.g., Fellenius (1926) and Terzaghi (1943)]. However, over the last 30 years the analysis of slopes using the limit-equilibrium method has been significantly refined by using various methods of vertical slices [e.g., Bishop (1955), Janbu (1954), Morgenstern and Price (1965), and Spencer (1967)]. An excellent review of popular limit-equilibrium techniques for predicting slope stability can be found in Nash (1987) or Graham (1984).

Using a global equilibrium condition, the limit-equilibrium approach is purely static as it neglects altogether the plastic flow rule for the soil (i.e., constitutive relation). If the soil at failure is assumed to be a rigid perfectly plastic material obeying an associated flow rule, then collapse mechanisms selected by the limit-equilibrium method are usually kinematically inadmissible. In addition, the static admissibility of the stress field also is not satisfied, because some arbitrary assumptions are made to remove statical indeterminacy and, more importantly, only a global equilibrium condition (rather than equilibrium conditions at every point in the soil) is satisfied. Michalowski (1989) showed that upper bound solutions based on kinematically admissible rigid-block velocity fields (associated with the linear Mohr-Coulomb criterion) also satisfy global force equilibrium equations. Hence an upper bound limit analysis solution also may be regarded as a special limit-equilibrium solution but not vice versa. However, by no means can these two methods be regarded as equivalent (Collins 1974; Chen 1975). Based on the bound theorems of limit analysis, it can be concluded that, in general, the limit-equilibrium method is of an approximate and arbitrary nature and the results obtained from this method are neither upper bounds nor lower bounds on the true collapse loads. Any attempt to validate the limit-equilibrium approach by comparing different limit-equilibrium solutions, without reference to a more rigorous analysis, is considered to be inconclusive.

PROBLEM OF STABILITY OF SIMPLE SLOPES

The slope geometry analyzed in this paper is shown in Fig. 1. Two types of analysis are considered: undrained stability analysis of purely cohesive slopes and drained stability analysis of cohesive-frictional slopes. The purely cohesive soil under undrained loading conditions is modeled by a rigid perfectly plastic Tresca yield criterion with an associated flow rule. The strength of the cohesive soil is determined by the undrained shear strength S_u, which may increase linearly with depth as is the case in normally consolidated clays (Gibson and Morgenstern 1962; Hunter and Schuster 1968). Under drained loading conditions, a perfectly plastic Mohr-Coulomb model is used to describe the soil behavior. For this case, the strength parameters are the effective cohesion c' and the effective friction angle ϕ'. Both of these quantities are assumed to be constant throughout the slope. For simplicity, the effect of seepage (or pore pressures) on the stability of cohesive-frictional slopes has not been included in this study. The solutions obtained are therefore only relevant for fully drained loading conditions where the effect of pore pressures can be neglected. Recent work by Miller and Hamilton (1989) and Michalowski (1994) suggests that it is also possible to incorporate the effect of pore pressures in limit analysis, but this extension will not be covered here.

The solutions for the simple slopes considered in this paper are relevant to excavations and man-made fills built on soil or rock (Taylor 1948; Gibson and Morgenstern 1962; Hunter and Schuster 1968; Duncan et al. 1987). Duncan et al. (1987) have showed that stability charts for simple slopes also can be used to obtain reasonably accurate answers for more complex problems if irregular slopes are approximated by simple slopes and

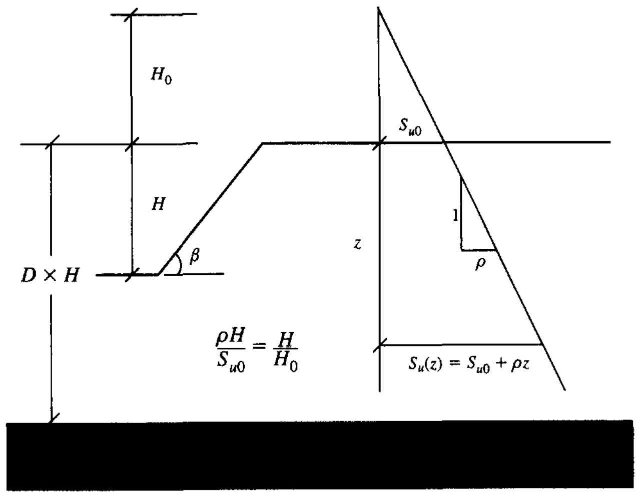

FIG. 1. Geometry of Simple Slopes and Parameters for Describing Increasing Strength with Depth in Slope Stability Analysis

carefully determined averaged values of unit weight, cohesion, and friction angles are used.

FINITE-ELEMENT LIMIT ANALYSIS

Because of the difficulties of constructing statically admissible stress fields manually, the application of limit analysis has in the past almost exclusively concentrated on the upper bound method (Chen 1975; Chen and Liu 1990). In fact, the authors are not aware of any rigorous lower bound solutions for the stability of slopes in cohesive-frictional soils. Although the upper bound solutions may be used as an estimate for the true collapse load, it is the lower bound solutions that are generally more useful in practice, because they are inherently conservative.

Finite-Element Lower Bound Limit Analysis

The use of finite elements and linear programming to compute rigorous lower bounds for soil mechanics problems appears to have been first proposed by Lysmer (1970). Although Lysmer's procedure was potentially very powerful, its usefulness was limited initially by the slowness of the algorithms that were available for solving large linear programming problems. In recent years, significant progress has been made in developing more efficient algorithms for solving large linear programming problems (Sloan 1988a). Detailed discussions of the recent developments in finite-element formulations of the lower bound theorem may be found in Anderheggen and Knopfel (1972), Bottero et al. (1980), Sloan (1988b), and Yu and Sloan (1991a,b). In this study, the formulation of Sloan (1988b) has been used because it has proven to be very efficient and robust when applied to large practical problems.

The lower bound formulation under conditions of plane strain uses the three types of elements shown in Fig. 2. The stress field for each of these elements is assumed to vary linearly. The extension elements may be used to extend the solution over a semiinfinite domain and therefore provide a complete statically admissible stress field for infinite half-space problems. Because this paper is concerned mainly with the stability of finite depth slopes resting on a firm base, extension elements are needed only at the left and right boundaries of the problems (shown in Fig. 4). In fact, the extension elements shown in Fig. 2 can be used readily to extend the stress fields into a semiinfinite domain if a slope in an infinitely deep layer needs to be analyzed (i.e., the depth factor $D = \infty$). Examples

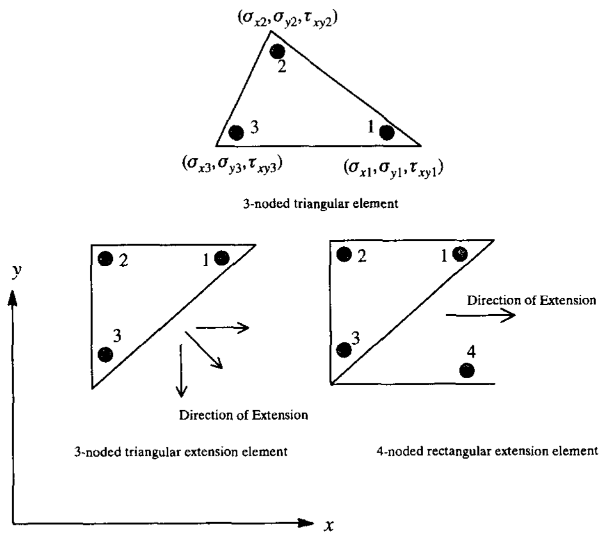

FIG. 2. Elements Used for Lower Bound Limit Analysis

of how this can be done can be found in Yu and Sloan (1994). As will be shown later in this paper, the depth factor has a very small effect on the stability of slopes provided the value of depth factor D is greater than approximately 4 [in agreement with the slope stability charts presented by Taylor (1948)]. It may therefore be reasonable to assume that the stability solutions for a slope in a finite layer with a large D value (say $D \geq 4$) would be very close to those of a slope in a layer of infinite depth.

A lower bound solution is obtained by insisting that the stresses obey equilibrium and satisfy both the stress boundary conditions and the yield criterion. Each of these requirements imposes a separate set of constraints on the nodal stresses. In the lower bound finite-element analysis, statically admissible stress discontinuities are permitted at edges shared by adjacent triangles and also along borders between adjacent rectangular extension elements. The Mohr-Coulomb and Tresca yield functions can be shown to plot as a circle in the $2\tau_{xy}$ versus $(\sigma_x - \sigma_y)$ stress space, where τ_{xy} is shear stress and σ_x and σ_y are normal stresses. To avoid nonlinear constraints occurring in the constraint matrix, a key idea behind the numerical lower bound technique is to use an internal linear approximation of the Mohr-Coulomb or the Tresca yield surface in the $2\tau_{xy}$ versus $(\sigma_x - \sigma_y)$ stress space.

In a typical lower bound analysis, a statically admissible stress field is sought that maximizes either a collapse load over some part of the boundary or the magnitude of body forces acting within a region. Both the collapse load and soil unit weight can be used to define an objective function for linear programming calculations. In the simple slope problem considered in this paper, it is convenient to find a static stress field that maximizes the unit weight. As a result, we will treat the unit weight as the unknown and optimize it directly. A typical lower bound mesh, together with the applied boundary conditions, is shown in Fig. 4(a). Note that the rectangular extension elements are arranged to ensure that the computed stress field is statically admissible throughout the slope, so that the solution is a rigorous lower bound solution.

Finite-Element Upper Bound Limit Analysis

The first formulation of the upper bound theorem, which used constant-strain triangular finite elements and linear programming, appears to have been developed by Anderheggen and Knopfel (1972) who analyzed plate problems. This formulation was later generalized by Bottero et al. (1981) and Sloan (1989) to include velocity discontinuities in plane strain limit analysis. When these constant-strain finite-element formulations are used, the grid must be arranged so that four triangles form a quadrilateral, with the central node lying at the intersection of the diagonals. If this pattern is not used, the elements cannot provide a sufficient number of degrees of freedom to satisfy the incompressibility condition that accompanies undrained failure (Nagtegaal et al. 1974). To overcome this limitation, Yu et al. (1992) developed a six-noded quadratic element for upper bound limit analysis. This formulation can be used to model an incompressible velocity field without resorting to special grid arrangements and also is more efficient than an equivalent three-noded formulation with the same number of nodes. However, it does suffer from the same shortcomings as the formulation of Bottero et al. (1980) and Sloan (1989), in that the direction of shearing for each velocity discontinuity must be specified a priori. Very recently, a new upper bound formulation that permits a large number of discontinuities in the velocity field has been derived by Sloan and Kleeman (1995). This method is again based on the three-noded triangle, but a velocity discontinuity may occur at any edge that is shared by a pair of adjacent triangles, and the sign of shearing is chosen automatically during the optimization process to give the least-dissipated energy rate.

In this study, the upper bound formulation developed by Sloan and Kleeman (1995) is used to calculate upper bound solutions for slope stability. A typical constant-strain triangular element used in the upper bound analysis is shown in Fig. 3.

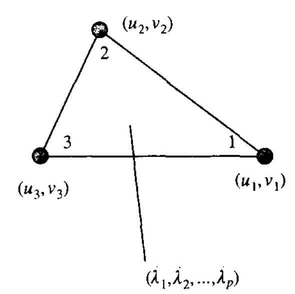

FIG. 3. Triangular Element Used for Upper Bound Limit Analysis

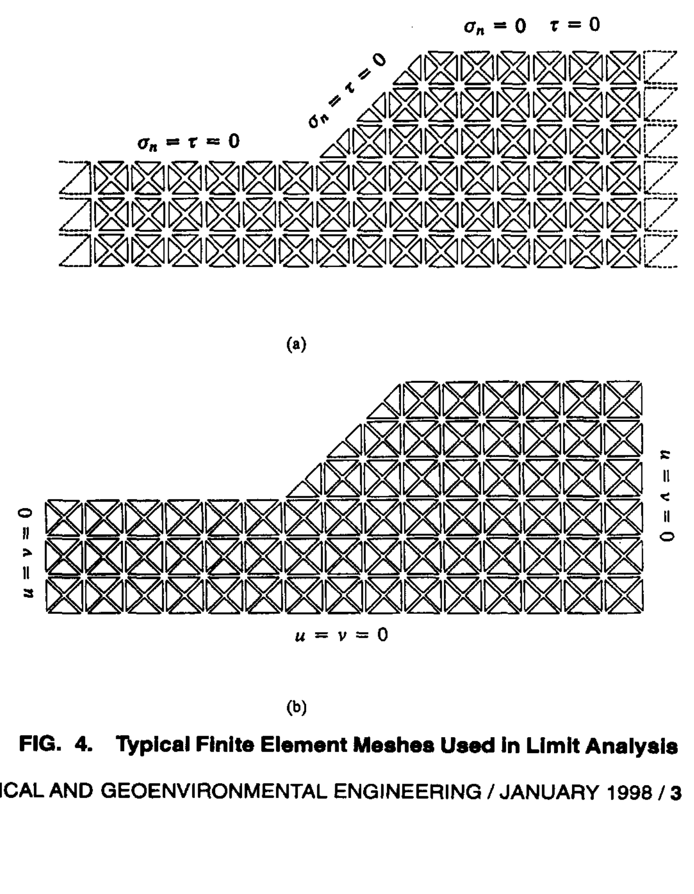

FIG. 4. Typical Finite Element Meshes Used in Limit Analysis

Within each element, the velocities are assumed to vary linearly. Each node has two velocity components and each element has p plastic multiplier rates $\dot{\lambda}_1, \dot{\lambda}_2, \ldots \dot{\lambda}_p$, where p is the number of planes used to linearize the yield criterion.

An upper bound solution is obtained by requiring that the velocity fields obey an associated plastic flow rule and satisfy the velocity boundary conditions. Each of these requirements imposes a separate set of constraints on the nodal velocities. To remove the stress terms from the flow rule equations, and thus provide a linear relationship between the unknown nodal velocities and plastic multiplier rates, an external linear approximation to the yield surface in the stress space of $2\tau_{xy}$ against $(\sigma_x - \sigma_y)$ is employed to ensure that the solution obtained is truly an upper bound.

In a typical upper bound analysis, a kinematically admissible velocity field is sought that minimizes the amount of dissipated power. The dissipated power can be expressed in terms of the unknown plastic multiplier rates and the discontinuity parameters used to define an objective function [see Sloan and Kleeman (1995)]. For the slope stability problem considered in this paper, we will minimize the unit weight directly and this can be achieved by equating the power expended by the external loads to the internal power dissipation [see Sloan (1995) for details]. A typical upper bound mesh is shown in Fig. 4(b). This is very similar to the equivalent lower bound grid of Fig. 4(a) except that the extension elements are no longer necessary because a rigid velocity boundary condition used in the upper bound limit analysis will ensure that the solution obtained is a rigorous upper bound.

RESULTS FOR UNDRAINED STABILITY OF SLOPES

The results of undrained slope stability calculations using the upper and lower bound methods of limit analysis and Bishop's limit-equilibrium method are presented in Figs. 5–9 for slope angles of β = 75, 60, 45, 30, and 15°, respectively. Note that the top and bottom solid lines are used to represent numerical upper and lower bound solutions. The solutions from the limit-equilibrium computer code STABL are plotted as the dashed lines. For each slope angle, the results for four different depth factors of D = 1, 1.5, 2, and 4 are given. To account for the effect of increasing strength with depth, the results are presented in terms of the stability number $N_F = \gamma H/s_{u0}^m = \gamma HF/s_{u0}$ against a dimensionless parameter $\lambda_{cp} = \rho H/s_{u0}^m = \rho HF/s_{u0}$, where γ is the unit weight of the soil, ρ denotes the rate of increase of the undrained shear strength with depth (Fig. 1), and s_{u0} and s_{u0}^m denote, respectively, the actual (or available) shear strength and the mobilized strength of the soil. Note that for undrained cases, the factor of safety is defined as $F = s_{u0}/s_{u0}^m$.

In the undrained limit analyses, for given ρ and slope geometry parameters of H, D, β, and mobilized shear strength s_{u0}^m, the upper and lower bound programs are used to obtain the best upper and lower bounds on the unit weight γ. Once this is known, the stability number $N_F = \gamma H/s_{u0}^m = \gamma HF/s_{u0}$ and the dimensionless parameter $\lambda_{cp} = \rho H/s_{u0}^m = (\rho HF)/s_{u0}$ can be calculated. On the other hand, when the limit-equilibrium code is used, we first set the values of H, D, γ, β, s_{u0}, and ρ and then determine the safety factor F. As a result, the stability number $N_F = (\gamma HF)/s_{u0}$ and the dimensionless parameter $\lambda_{cp} = (\rho HF)/s_{u0}$ can be calculated.

For almost all the cases considered, Figs. 5–9 show that the exact solutions are bracketed within 5–10% by the upper and lower bound solutions. The comparison of the bounding solutions with the limit-equilibrium results can best be considered separately for soils with a constant undrained shear-strength profile and those with increasing strength with depth. If the undrained shear strength of the soil is constant (λ_{cp} = 0), Bishop's limit-equilibrium solutions are found to be in good agreement with the rigorous upper and lower bounds. On the other hand, if the undrained shear strength of the soil increases linearly with depth ($\lambda_{cp} > 0$), the limit-equilibrium results are generally close to the upper bound solutions for steep slopes. When the slope angle is less than 30°, the limit-equilibrium analysis tends to underestimate the true stability number. This underestimation is particularly significant when β = 15° and $\lambda_{cp} > 0.5$. For example, Fig. 9 shows that when D = 4, β = 15° and λ_{cp} = 1.0, the limit-equilibrium method underestimates the true stability number by as much as 35%.

In summary, the Bishop limit-equilibrium method produces reasonably accurate solutions for the stability of homogeneous slopes. For slopes in soils whose strength increases with depth, significant underestimation of the stability number can be obtained from the limit-equilibrium analysis for slopes with a low slope angle and a high λ_{cp} value.

As far as the effect of increasing strength with depth is concerned, it is most interesting to note that the bounding solutions suggest that the stability number increases approximately linearly with the value of λ_{cp}. Fig. 10 presents the effect of slope angle on the stability number for slopes with two different λ_{cp} values and two different depth factors. As expected, for all the cases considered the stability number decreases with increasing value of the slope angle. The effect of slope angle on the stability number is found to be more significant for the slopes with a low value of depth factor and a high λ_{cp} value.

Presented in Fig. 11 are the bounding solutions, which show the effect of the depth factor on the stability number for slopes with two different λ_{cp} values and two different slope angles. In general, these results indicate that the depth factor has a small effect on the obtained stability numbers as long as the depth factor is greater than 2. This is why slopes with a depth factor greater than 4 (with infinite slopes as a special case when $D = \infty$) have not been included in this study. For the cases with low slope angles, the stability number decreases slightly with increasing depth factor and then remains unchanged when the depth factor is greater than 2. It is also noted that the effect of the depth factor for slopes with a constant undrained shear-strength profile is more significant than for cases with increasing strength with depth. This is because a slope is more likely to fail along a slip surface that passes through or above the toe when the strength increases with depth.

Example of Application

We now illustrate how the results presented in Figs. 5–9 can be used to determine the factor of safety for a clay slope.

Problem. A cut slope is to be excavated in a normally consolidated clay. The slope has the following parameters: the slope angle β = 60°, the height of the slope is H = 12 m, the depth factor is D = 1.5, and the soil unit weight is γ = 18.5 kN/m^3. The undrained shear strength of the soil on the top of the slope surface is s_{u0} = 40 kN/m^2 and the rate of increase of the undrained shear strength with depth is estimated as ρ = 1.5 kN/m^3. What is the factor of safety of this soil slope against undrained failure?

A procedure for using the results of the present study to solve the foregoing slope stability problem can be summarized as follows:

1. From the values of γ, H, s_{u0}, and ρ, we can calculate the dimensionless parameters $\gamma H/s_{u0} = (18.5 \times 12)/40 = 5.55$ and determine the ratio of $N_F/\lambda_{cp} = [(\gamma H/s_{u0})F]/[(\rho H/s_{u0})F] = \gamma/\rho = 18.5/1.5 = 12.33$.
2. With β = 60° and D = 1.5, it follows that the results presented in Fig. 6(b) should be used to determine the safety factor.

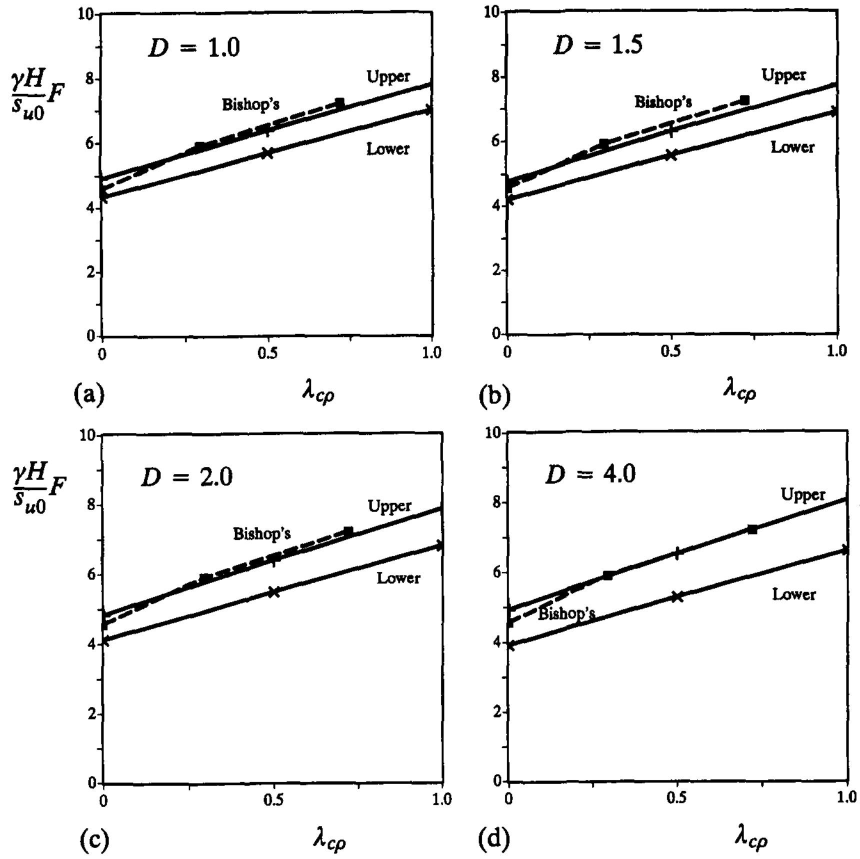

FIG. 5. Effect of Increasing Strength with Depth on Stability Number for Slope Angle of 75°

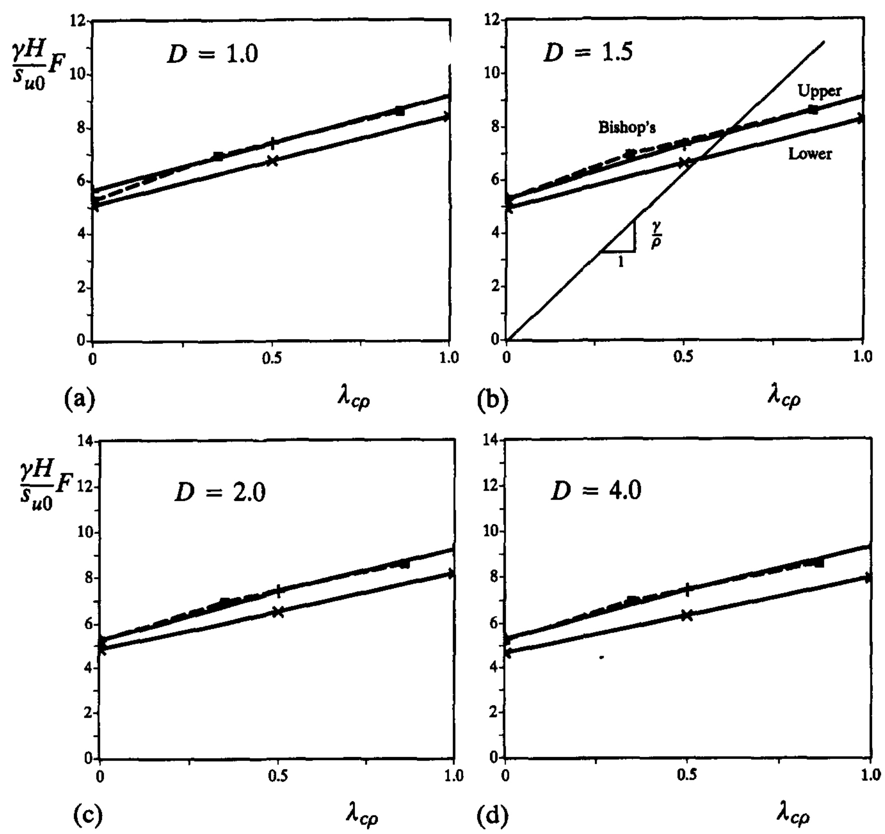

FIG. 6. Effect of Increasing Strength with Depth on Stability Number for Slope Angle of 60°

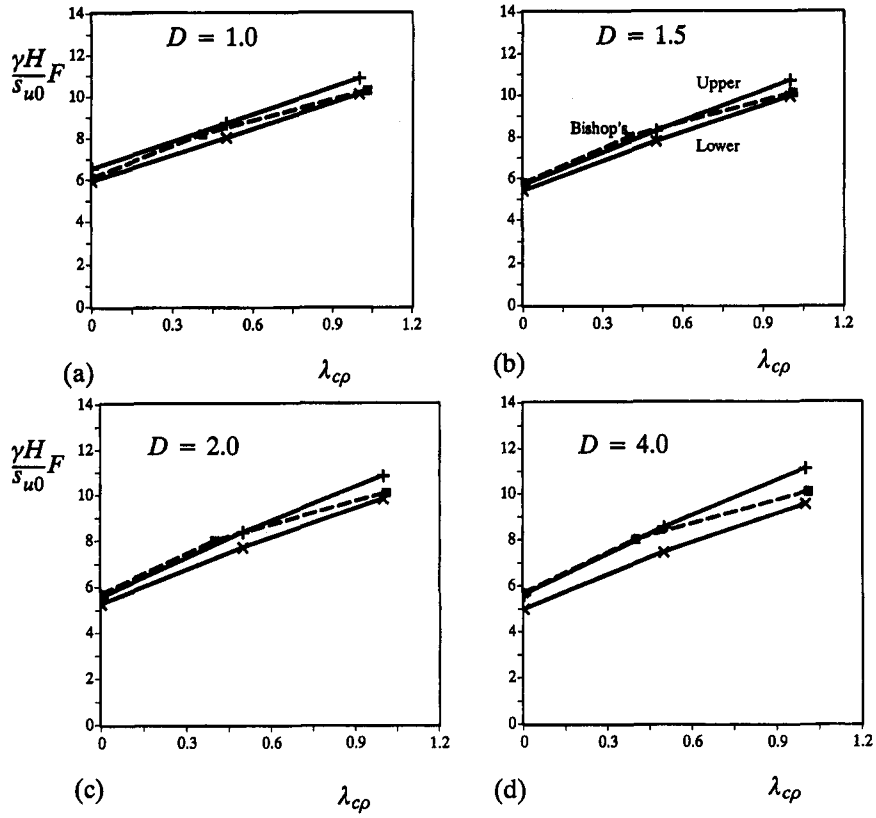

FIG. 7. Effect of Increasing Strength with Depth on Stability Number for Slope Angle of 45°

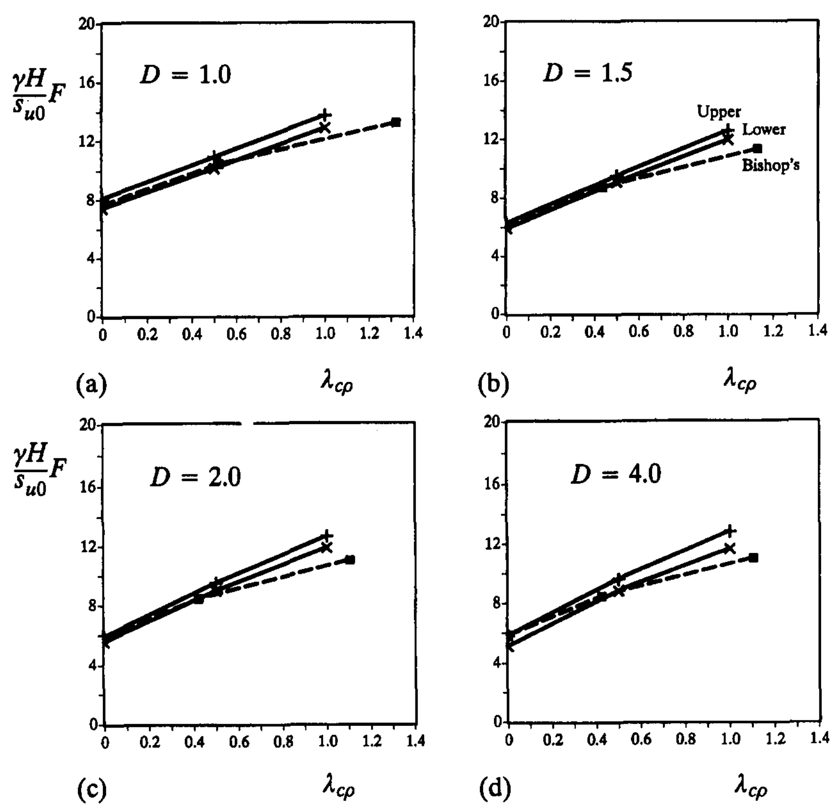

FIG. 8. Effect of Increasing Strength with Depth on Stability Number for Slope Angle of 30°

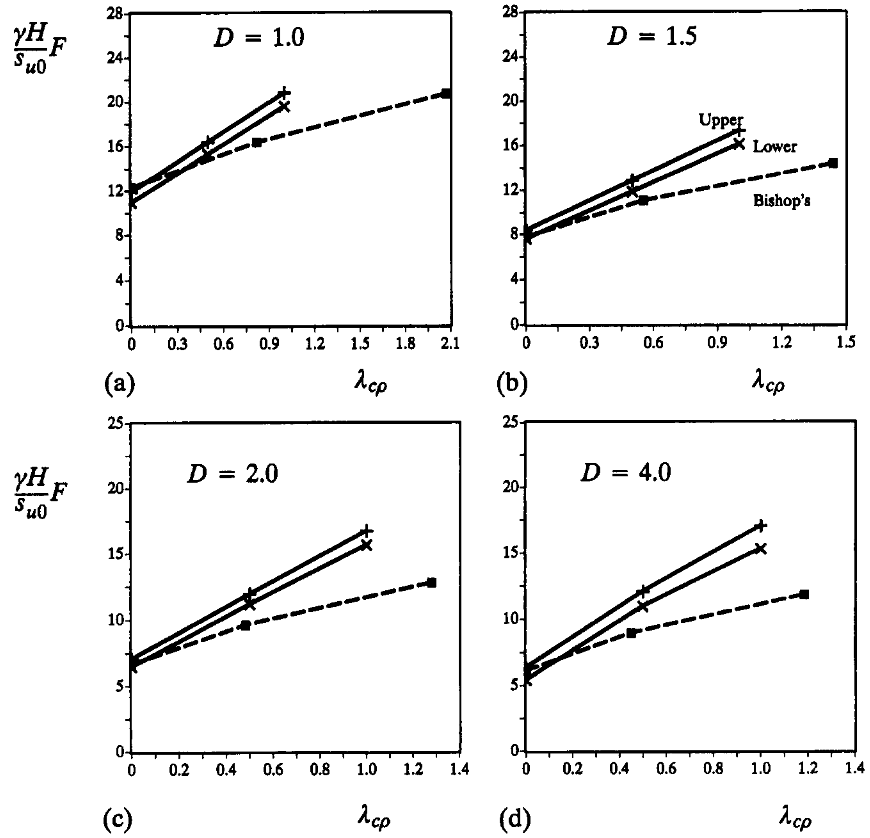

FIG. 9. Effect of Increasing Strength with Depth on Stability Number for Slope Angle of 15°

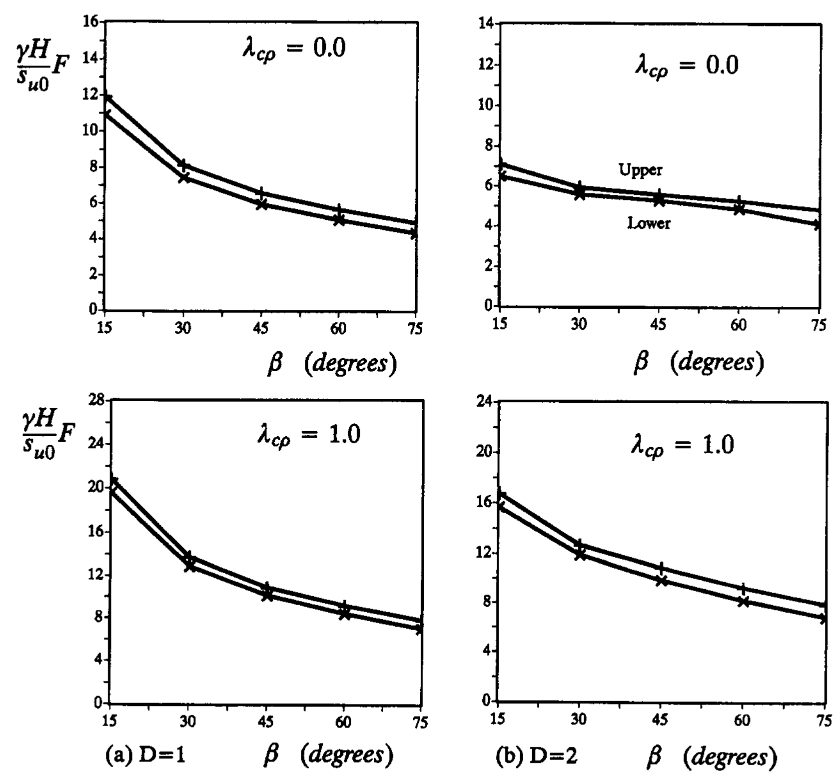

FIG. 10. Effect of Slope Angle on Stability Number

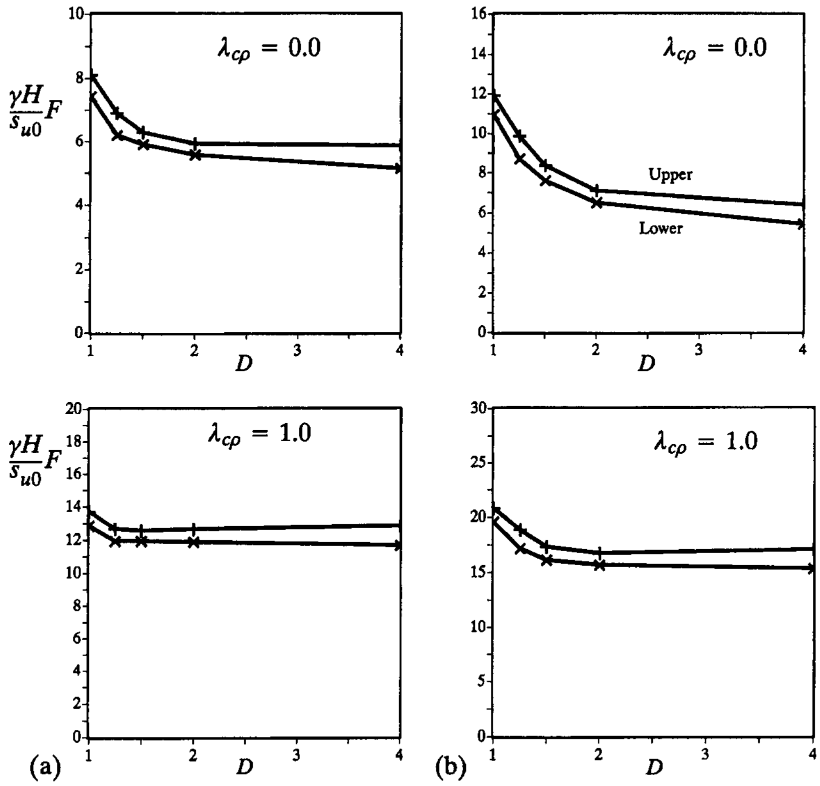

FIG. 11. Effect of Depth Factor on Stability Number: (a) β = 30°; (b) β = 15°

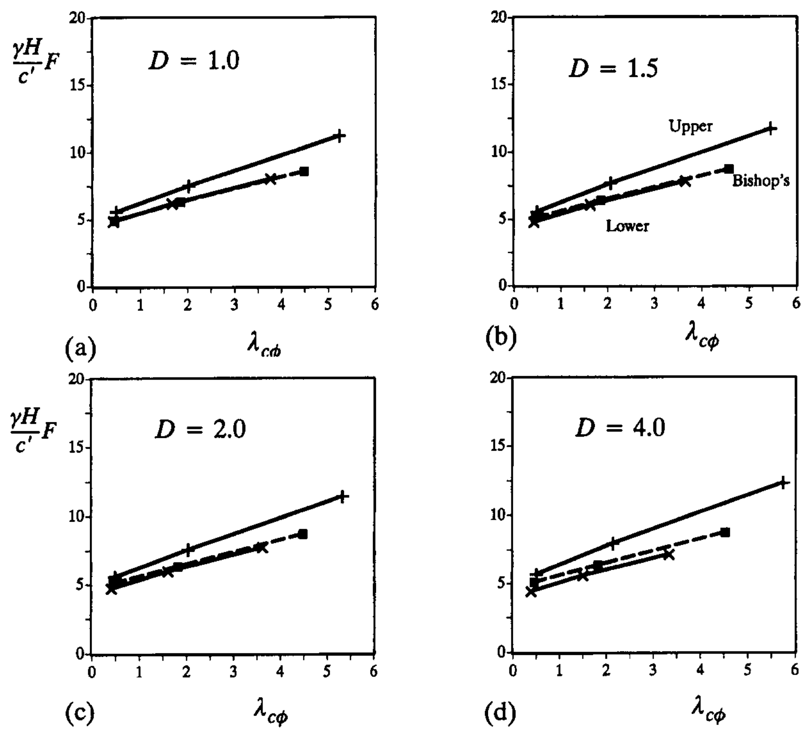

FIG. 12. Stability Number against $\lambda_{c\phi}$ for Cohesive-Frictional Slopes with Slope Angle of 75°

3. In Fig. 6(b), draw a straight line passing through the origin with a gradient of γ/ρ = 12.33. This straight line will intersect with the three curves representing the lower bound, upper bound, and Bishop's limit-equilibrium solutions.
4. From these three intersection points, we can back-figure the following stability numbers N_F = 6.8, 7.8, and 8.0, from which the lower bound, upper bound, and Bishop's limit-equilibrium solutions of the factor of safety can be calculated as $F = N_F/(\gamma H/s_{u0})$ = 1.23, 1.41, and 1.44.

RESULTS FOR DRAINED STABILITY OF SLOPES

Drained slope stability calculations using the upper and lower bound methods of limit analysis and Bishop's limit-equilibrium method have been carried out with four values of

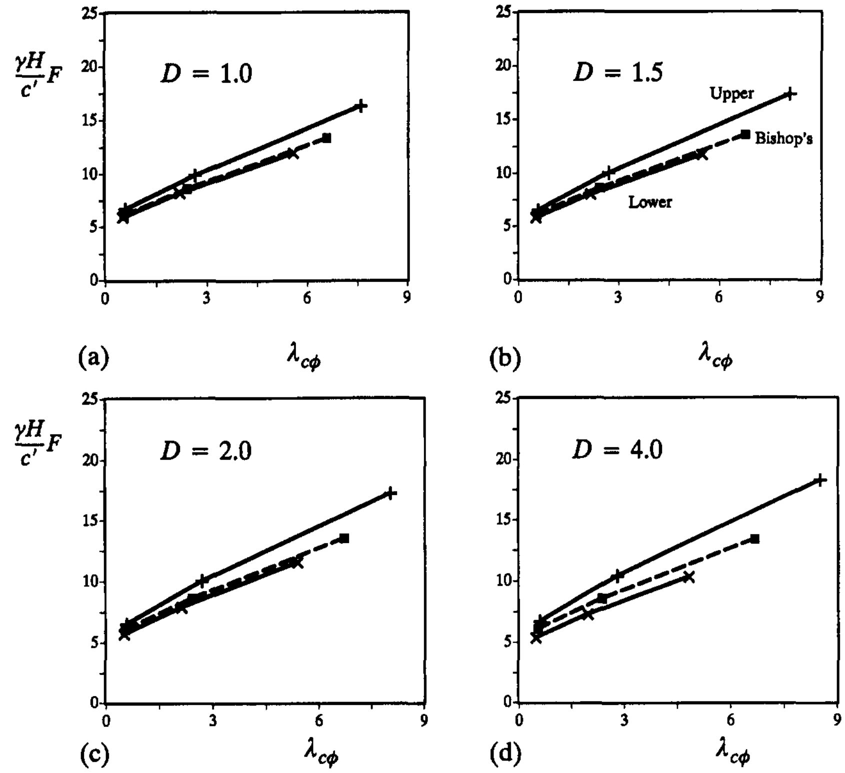

FIG. 13. Stability Number against $\lambda_{c\phi}$ for Cohesive-Frictional Slopes with Slope Angle of 60°

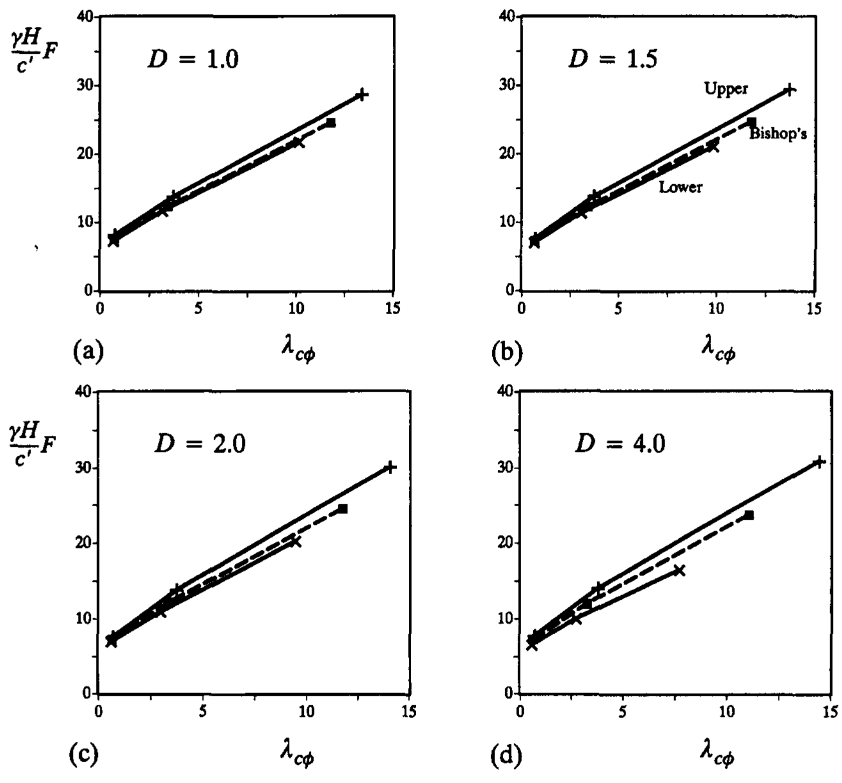

FIG. 14. Stability Number against $\lambda_{c\phi}$ for Cohesive-Frictional Slopes with Slope Angle of 45°

the depth factor D = 1, 1.5, 2, and 4. As in the undrained cases, the drained results suggest that the effect of the depth factor on the stability of cohesive-frictional slopes is very small once the depth factor exceeds approximately 2. This is because for most cases (except for slopes with a very low slope angle and an unrealistically low friction angle) the critical failure surface tends to pass through the toe for cohesive-frictional slopes (Taylor 1948; Chen 1975).

To compare the present bounding solutions with the limit-equilibrium results, the solutions for the drained stability of slopes are presented in terms of the stability number $N_F = \gamma H/c'_m = (\gamma HF)/c'$ against a dimensionless parameter $\lambda_{c\phi} = \gamma H \tan \phi'_m/c'_m = (\gamma H \tan \phi')/c'$, where c', ϕ', and c'_m, ϕ'_m denote, respectively, the actual soil cohesion and friction angle and mobilized cohesion and friction angle. For drained stability, the factor of safety is defined as $F = c'/c'_m = \tan \phi'/\tan \phi'_m$.

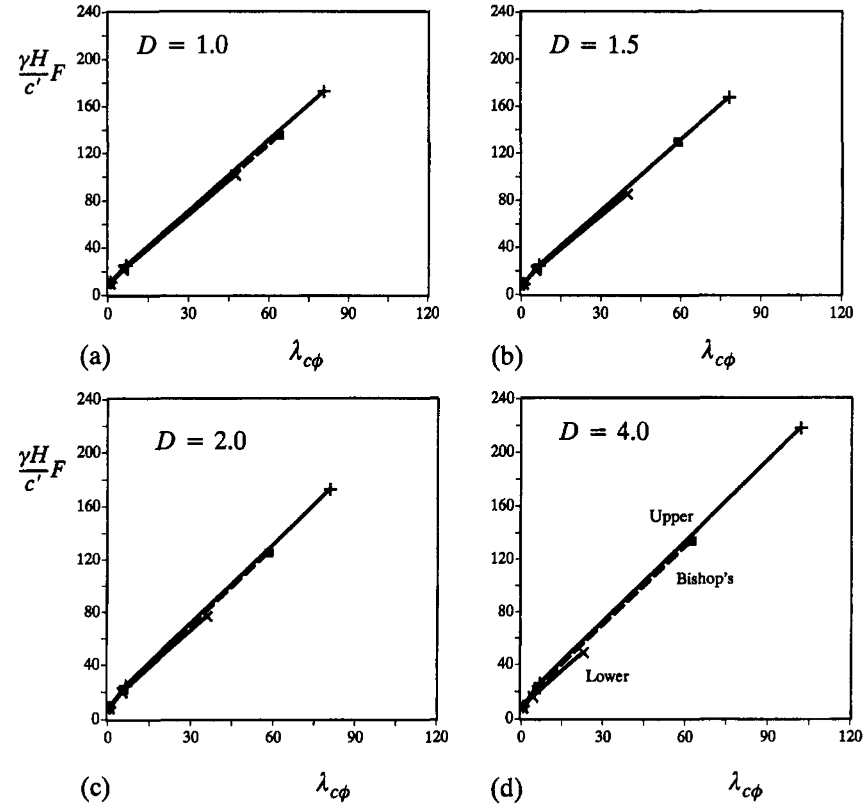

FIG. 15. Stability Number against $\lambda_{c\phi}$ for Cohesive-Frictional Slopes with Slope Angle of 30°

The dimensionless parameter $\lambda_{c\phi}$ was introduced by Janbu (1954) and later used by Cousins (1978) among many others to develop stability charts for cohesive-frictional slopes. A major advantage of using Janbu's parameter to develop a stability chart is that for a given slope with known soil-strength parameters, slope angle, unit weight, slope height, and depth factor, the safety factor can be obtained without resorting to an iterative procedure. This is so because, with the assumption of the safety factors for both cohesion and friction angle being equal, the parameter $\lambda_{c\phi}$ is no longer a function of the safety factor F.

In the present drained limit analyses, for given values of H, D, β, c'_m, and ϕ'_m, the upper and lower bound programs are used to determine the best upper and lower bound solutions of the unit weight γ from which the stability number $N_F = \gamma H/c'_m = (\gamma HF)/c'$ and Janbu's dimensionless parameter $\lambda_{c\phi} = \gamma H \tan \phi'_m/c'_m = (\gamma H \tan \phi')/c'$ can be calculated. In the limit-equilibrium analysis, we first set the values of H, D, γ, β, c', and ϕ' and therefore the dimensionless parameter $\lambda_{c\phi} = (\gamma H \tan \phi')/c'$ and then determine the safety factor F from which the stability number $N_F = (\gamma HF)/c'$ is calculated.

The results from the upper and lower bound calculations and the limit-equilibrium analyses are presented in terms of the stability number $N_F = (\gamma HF)/c'$ against $\lambda_{c\phi}$ in Figs. 12–15 for four slope angles and four depth factors. Figs. 12–15 indicate that the upper and lower bound solutions generally close together and converge rapidly when the slope angle decreases. Again, it is observed that the difference between the upper and lower bound solutions increases slightly as the depth factor is increased. This is caused by the effect of the mesh densities on the results of the numerical limit analyses. Figs. 14 and 15 show that when the slope angle $\beta \leq 45°$ a remarkably good agreement is observed between the upper and lower bound solutions with a maximum difference being less than 5%. It is evident from these figures that the limit-equilibrium analysis produces accurate stability numbers for homogeneous cohesive-frictional slopes, although they are generally closer to the lower bound solutions.

Example of Application

We now demonstrate how the results presented in Figs. 12–15 can be used to determine the factor of safety for a given soil slope with known geometry and actual soil strength.

Problem. A simple soil slope has the following properties: the slope angle $\beta = 45°$, the height of the slope is $H = 9$ m, the depth factor is $D = 2$, the soil unit weight is $\gamma = 19$ kN/m^3, the soil strength is defined by $c' = 20$ kN/m^2, and $\phi' = 35°$. What is the factor of safety of this soil slope against failure?

A simple procedure for using the results of the present study to solve the foregoing slope stability problem may be summarized as follows:

1. From the values of γ, H, c', and ϕ', we can calculate Janbu's dimensionless parameter: $\lambda_{c\phi} = (\gamma H \tan \phi')/c' = (19 \times 9 \times \tan 35°)/20 = 5.99$.
2. With $\beta = 45°$ and $D = 2$, it follows that the results presented in Fig. 14(c) should be used to determine the safety factor.
3. From $\lambda_{c\phi} = 5.99$ as calculated in 1, Fig. 14(c) can be used to give the stability numbers of $N_F = \gamma HF/c' = 15.4$, 16.3, and 17.5 corresponding to lower bound, limit-equilibrium, and upper bound solutions, respectively.
4. With $\gamma = 19$ kN/m^3, $H = 9$ m, $c' = 20$ kN/m^2, and the derived values of Janbu's parameter $N_F = 15.4$, 16.3, and 17.5 from 3, the lower bound, limit-equilibrium, and upper bound solutions of the factor of safety are calculated as $F = cN_F/(\gamma H) = 1.8$, 1.91, and 2.05.

CONCLUSIONS

The following main conclusions can be drawn from the results presented in this paper:

1. For most cases considered in this study, it is found that the exact stability solution for both drained and undrained slopes can be predicted to within 5–10% by the present numerical upper and lower bound solutions.
2. For the special case of homogeneous slopes, the numerical upper bound solutions obtained for slopes with a large depth factor are slightly higher (i.e., worse) than Chen's upper bounds for slopes in an infinitely deep layer with a failure surface passing below the toe (Chen 1975). The lower bound solutions obtained in this paper are the most valuable results for two reasons: few rigorous lower bound solutions exist for slope stability problems in the literature and the lower bound solutions can be used in practice to give a safe design.
3. A detailed comparison of the present bounding solutions with those from the Bishop limit-equilibrium method suggests that although the limit-equilibrium analysis gives reasonable solutions for homogeneous slopes, it tends to underestimate the true stability solution significantly for inhomogeneous slopes with a low slope angle.
4. For undrained slopes, the increasing strength with depth has a significant effect on the stability number. It is interesting to note that the stability number increases approximately linearly with the values of the dimensionless parameter λ_{cp}.

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