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AN EFFICIENT METHOD OF ANALYSIS FOR THE DRAINED AND UNDRAINED BEHAVIOUR OF AN ELASTIC SOIL

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Abstract—In this paper a method of analysis for the drained and undrained behaviour of a saturated soil with an elastic anisotropic skeleton is proposed. The method depends upon embedding the problem in a series of problems depending upon a parameter, then expanding the solution in terms of that parameter. The method has the advantage that both the drained and undrained behaviour can be found simultaneously and the problems of a singular stiffness matrix usually associated with undrained behaviour of an elastic soil is avoided.

INTRODUCTION

A saturated soil can be thought of as being made up of two phases; the soil grains making up the soil skeleton constitute the solid phase while the pore fluid which fills the voids constitutes the liquid phase. In many applications it may be assumed that the soil skeleton acts as an elastic body while the pore fluid may be assumed to be incompressible.

The two phase nature of soil causes it to act in two widely different ways. If a soil is loaded suddenly the pressure in the pore fluid will rise and excess pore pressures will develop. The pore fluid will then tend to flow from regions of higher pore pressure to regions of lower pore pressure. This flow can only occur at a finite rate and thus at the instant of loading, since the pore fluid is incompressible relative to the soil skeleton, the soil will deform as an incompressible elastic material, the soil is then said to behave in an undrained manner and is characterised by "undrained" elastic properties. If, however, the soil is loaded gradually no excess pore pressures develop and the soil will behave in a drained manner and will behave as an elastic material with "drained" elastic properties. Similarly, if after an initial sudden loading a long period of time is allowed to elapse, all excess pore pressures dissipate and the final response of the soil is drained.

The finite element technique provides a powerful method for the numerical solution of problems in elasticity [1], This method is readily applicable to the drained behaviour of soil. The application to incompressible materials and thus to the undrained behaviour of soil is not so straightforward. The reason for this is that it is not possible to express the stresses solely in terms of the strains and thus for such materials it is necessary to adopt a special formation [2], thus if the finite element method is applied to an incompressible material it is found that the stiffness matrix is singular [3].

Various attempts [4, 5] have been made to overcome these difficulties, Christian [4] developed a finite element formulation in which the excess pore pressures were treated as independent variables. This has several disadvantages; first, the number of nodal quantities and consequently the number of equations to be solved is increased, so for example under conditions of plane strain the number of equations will be increased by 50% and the band width of those equations will also be increased by 50%. A second disadvantage is that the resultant stiffness matrix is not necessarily positive definite and thus many solution techniques, i.e. Crout-Cholesky factorisation, may breakdown. An alternative approach [5] is to expand the solution as a series in Poisson's ratio about some pivotal value. It is found that this approach is computationally efficient and will always converge, however, it is restricted in its application since it can only be applied to isotropic materials.

In this paper a method of analysis, closely related to that of Ref. [5], but applicable to inhomogeneous anisotropic soils, is developed. The soil behaviour is embedded in a class of material behaviours characterised by a single parameter k. The solution is then expanded as a power series in k about some pivotal value. The drained, undrained behaviours are then obtained by setting $k = 0$, 1 respectively.

DRAINED AND UNDRAINED STRESS-STRAIN BEHAVIOUR

Davis and Poulos [6] have shown that an isotropic soil under undrained conditions behaves as an imcompressible elastic material with a Poisson's ratio of a half and a shear modulus equal to the shear modulus of the soil skeleton. In many situations, especially where deposition has occurred in a succession of layers, soil acts as a anisotropic material. In this section the stress-strain law for a soil with an anisotropic soil skeleton is developed.

The deformation of a saturated soil is governed by the effective stress law [7, 8]. This may be written in the form

$$\boldsymbol{\sigma}' = D\boldsymbol{\epsilon} \tag{1a}$$

where

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + u\mathbf{a} \tag{1b}$$

and

$\boldsymbol{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx})^T$ is the vector of total stress components, tension positive
$\boldsymbol{\epsilon} = (\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})^T$ is the vector of strain components
u = excess pore pressure taken positive when compressive
$\mathbf{a} = (1, 1, 1, 0, 0, 0)^T$
D = the matrix of elastic constants for the soil skeleton.

The quantities in the above definitions all represent the increase above some initial value due to the applied loads.

Equation (1) may be solved for strain and thus

$$\boldsymbol{\epsilon} = D^{-1}\boldsymbol{\sigma} + uD^{-1}\mathbf{a}. \tag{2}$$

Now as was pointed out in the introduction, soil under undrained conditions deforms at constant volume and thus

$$\mathbf{a}^T\boldsymbol{\epsilon} = 0. \tag{3}$$

Equations (2)–(3) imply that under undrained conditions the pore pressure is related to the total stress as follows

$$u = -\frac{\mathbf{a}^T D^{-1}\boldsymbol{\sigma}}{\mathbf{a}^T D^{-1}\mathbf{a}}. \tag{4}$$

Equation (1) may therefore be written

$$\left[1 - \frac{\mathbf{a}\mathbf{a}^T D^{-1}}{\mathbf{a}^T D^{-1}\mathbf{a}}\right]\boldsymbol{\sigma} = D\boldsymbol{\epsilon}. \tag{5}$$

It is not possible to invert eqn (5) since the matrix

$$F = 1 - \frac{\mathbf{a}\mathbf{a}^T D^{-1}}{\mathbf{a}^T D^{-1}\mathbf{a}}$$

is singular. This may be inferred either by physical reasoning or by noticing that

$$F\mathbf{a} = 0.$$

In order to overcome this difficulty, suppose that eqn (5) is embedded in a series of stress-strain laws

$$F(k)\boldsymbol{\sigma} = D\boldsymbol{\epsilon} \tag{6}$$

where

$$F(k) = 1 - \frac{k\mathbf{a}\mathbf{a}^T D^{-1}}{\mathbf{a}^T D^{-1}\mathbf{a}}$$

so that the drained behaviour corresponds to $k = 0$ and the undrained behaviour to $k = 1$.

Equation (6) can now be inverted, as outlined in Appendix A, so that

$$\boldsymbol{\sigma} = D(k)\boldsymbol{\epsilon} \tag{7}$$

where

$$D(k) = D + \left(\frac{k}{1-k}\right)\frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T D^{-1}\mathbf{a}}.$$

FINITE ELEMENT EQUATIONS

If an elastic body has the stress-strain law

$$\boldsymbol{\sigma} = D\boldsymbol{\epsilon}.$$

The finite element method can be used to develop an approximating set of equations

$$K\boldsymbol{\delta} = \mathbf{f} \tag{8}$$

where K is the stiffness matrix of the body
$\boldsymbol{\delta}$ is the vector of nodal deflections
$\mathbf{f}$ is the vector of nodal forces.

The stiffness matrix may be expressed in the form

$$K = \int B^T D B \, \mathrm{d}V \tag{9}$$

where D is the matrix of elastic constants and the matrix B relates the vector of strain components $\boldsymbol{\epsilon}$ to the nodal displacements $\boldsymbol{\delta}$, through the equation

$$\boldsymbol{\epsilon} = B\boldsymbol{\delta}. \tag{10}$$

Similarly if a material has the stress-strain law defined by eqn (7), the finite element method can be used to develop an approximating set of equations

$$K(k)\boldsymbol{\delta} = \mathbf{f} \tag{11}$$

where

$$K(k) = \int (B^T D(k) B) \, \mathrm{d}V$$

Equation (11) can be written in the form

$$[(1-k)A + kC]\boldsymbol{\delta} = (1-k)\mathbf{f}. \tag{12}$$

where $A = \int (B^T D B)\,\mathrm{d}V$ = the stiffness matrix for the soil skeleton
$C = \int (B^T \mathbf{a}\mathbf{a}^T B / \mathbf{a}^T D^{-1}\mathbf{a})\,\mathrm{d}V$

Now suppose that the solution $\boldsymbol{\delta}(k)$ of eqn (11) has the formal expansion

$$\boldsymbol{\delta} = \boldsymbol{\delta}_0 + (k - k_0)\boldsymbol{\delta}_1 + (k - k_0)^2\boldsymbol{\delta}_2 + \cdots +$$ (13)

where k_0 is some pivotal value. Then the coefficients $\boldsymbol{\delta}_i$ may be determined successively from the recurrence relation

$$M\boldsymbol{\delta}_i = \mathbf{g}_i, \qquad i = 0, 1, 2 \ldots$$ (14)

where $M = k_0C + (1 - k_0)A$
$N = C - A$
$\mathbf{g}_0 = (1 - k_0)\mathbf{f}$
$\mathbf{g}_1 = -\mathbf{f} - N\boldsymbol{\delta}_0$
$\mathbf{g}_i = -N\boldsymbol{\delta}_{i-1} \qquad i = 2, 3, \ldots\ldots$

Equation (14) allows the coefficients $\boldsymbol{\delta}_i$ to be determined quite economically. The quantities $\mathbf{g}_i$ can be thought of as representing different "load sets" on a particular structure. Thus if the equations are to be solved by Crout–Cholesky factorisation only one such factorisation need be performed.

Once the coefficients $\boldsymbol{\delta}_i$ have been determined the stresses defined by eqn (7) can be calculated as follows. Suppose $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ have the formal expansions

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + (k - k_0)\boldsymbol{\sigma}_1 + (k - k_0)^2\boldsymbol{\sigma}_2 + \cdots +$$ (15a)

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_0 + (k - k_0)\boldsymbol{\epsilon}_1 + (k - k_0)^2\boldsymbol{\epsilon}_2 + \cdots +$$ (15b)

where $\boldsymbol{\epsilon}_i = B\boldsymbol{\delta}_i$.

If eqns (15a), (15b) are substituted into eqn (7) it is found that the coefficients $\boldsymbol{\sigma}_i$ are given by:

$$\boldsymbol{\sigma}_{i+1} = (\boldsymbol{\sigma}_i + P\boldsymbol{\epsilon}_{i+1} + Q\boldsymbol{\epsilon}_i)/(1 - k_0)$$ (16)

where

$$P = k_0\frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^TD^{-1}\mathbf{a}} + (1 - k_0)D$$

$$Q = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^TD^{-1}\mathbf{a}} - \mathrm{D}$$

and

$$\boldsymbol{\sigma}_0 = P\boldsymbol{\epsilon}_0/(1 - k_0).$$

It would appear that the method outlined above can be used to obtain the solution for both the drained ($k = 0$) and undrained ($k = 1$) behaviour of an elastic soil. However, before the method can be used with confidence it is necessary to establish: (a) the existance of a solution to eqn (11) in the form of the power series, eqn (13), and, (b) the conditions under which such a solution converges. These results are established in the next section.

EXISTENCE AND CONVERGENCE OF THE SOLUTION

The solution of eqn (12) can be written in terms of the eigenvectors of

$$[(1 - k)A + kC]\boldsymbol{\delta} = 0.$$ (17)

A little care needs to be exercised in obtaining this solution, for although the matrix A is positive definite, being the stiffness matrix of the soil skeleton, the matrix C is only positive semi-definite and is in general singular. It can be seen that this implies that $k = k_1 = 1$ is an eigenvalue of eqn (17). In general it will be a multiple eigenvalue and there will be several distinct eigenvectors associated with it.† Suppose that this eigenvalue is of multiplicity p so that there are p distinct eigenvectors which may be denoted $\mathbf{\Delta}_{11}, \mathbf{\Delta}_{12}, \ldots, \mathbf{\Delta}_{1p}$. It may be assumed without loss of generality that these eigenvectors form an orthogonal set with respect to the matrix A, so that

$$\mathbf{\Delta}_{1i} A \mathbf{\Delta}_{1j} = 0 \qquad i \neq j.$$

Suppose that the remaining eigenvalues are denoted $k_2, k_3, \ldots, k_N$ and that these have corresponding eigen-vectors $\mathbf{\Delta}_2, \mathbf{\Delta}_3 \ldots \mathbf{\Delta}_N$. The solution of eqn (12) may now be written

$$\boldsymbol{\delta} = \mathbf{\Delta}_1 + \sum_{n=2}^{N} \frac{k-1}{k-k_n} a_n \mathbf{\Delta}_n \tag{18}$$

where

$$a_n = \frac{\mathbf{f}^T \mathbf{\Delta}_n}{\mathbf{\Delta}_n{}^T A \mathbf{\Delta}_n}$$

$$\mathbf{\Delta}_1 = \left[\sum_{n=1}^{P} \frac{\mathbf{\Delta}_{1n} \mathbf{\Delta}_{1n}^T}{\mathbf{\Delta}_{1n}^T A \mathbf{\Delta}_{1n}} \right] \mathbf{f}.$$

The symmetry of the matrices A, C implies that the eigenvalues of eqn (17) are all real, it also is possible to establish that these eigenvalues are greater than or equal to one and thus that the eigenvalues $k_2, k_3, \ldots, k_N$ are all strictly greater than one. In order to establish this result, let $\mathbf{\Delta}$ be any eigenvector of eqn (17) then the corresponding eigenvalue k must satisfy

$$\frac{k}{1-k} = -\frac{\mathbf{\Delta}^T A \mathbf{\Delta}}{\mathbf{\Delta}^T C \mathbf{\Delta}}.$$

Now in the notation of Appendix A,

$$\frac{k}{1-k} = -\frac{\int (\boldsymbol{\rho}^T \boldsymbol{\rho}) \, \mathrm{d}V}{\int (\boldsymbol{\rho}^T \mathbf{b}\mathbf{b}^T \boldsymbol{\rho}) \, \mathrm{d}V}$$

where $\boldsymbol{\rho} = L^T B \mathbf{\Delta}$. Now recalling that $\mathbf{b}$ is a unit vector

$$\frac{k}{1-k} = -\frac{\int |\mathbf{b}|^2 |\boldsymbol{\rho}|^2 \, \mathrm{d}V}{\int |\mathbf{b}^T \boldsymbol{\rho}|^2 \, \mathrm{d}V}$$

and so using Cauchy's inequality

$$\frac{k}{1-k} \leqslant -1. \tag{19}$$

Now suppose $k < 1$ then multiplying both sides of eqn (19) by the positive quantity $1-k$ implies that

$$k \leqslant k - 1$$

or

$$0 \leqslant -1$$

†Each such eigenvector represents a mode of zero volume change for the body.

and thus

$$k \geqslant 1.$$

To establish the validity of eqn (13) suppose that k_0 is chosen so that $k_0 < 1$, then eqn (18) can be expanded in the form

$$\boldsymbol{\delta} = \boldsymbol{\delta}_0 + (k - k_0)\boldsymbol{\delta}_1 + (k - k_0)^2\boldsymbol{\delta}_2 + \cdots + \tag{20}$$

where

$$\boldsymbol{\delta}_0 = \boldsymbol{\Delta}_1 + \sum_{n=2}^{N} \frac{k_0 - 1}{k_0 - k_n} a_n \boldsymbol{\Delta}_n$$

$$\boldsymbol{\delta}_m = (-1)^m \sum_{n=2}^{N} \frac{(k_n - 1)a_n \boldsymbol{\Delta}_n}{(k_0 - k_n)^{m+1}} \qquad m = 1, 2, \ldots\ldots$$

this expansion being valid whenever

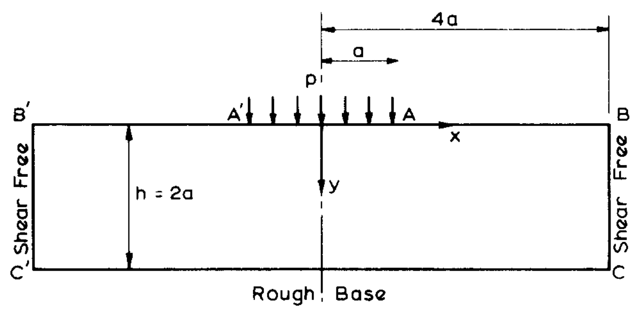
$$|k - k_0| < |k_0 - k_n|.$$

Equation (20) is precisely the form postulated for $\boldsymbol{\delta}$ in eqn (13). If only the undrained solution is required, i.e. $k = 1$, then a value of k_0 less than 1 must be chosen.† In many cases a solution is required for both $k = 0$ and 1, using a common value of k_0. In this case k_0 must be selected so that

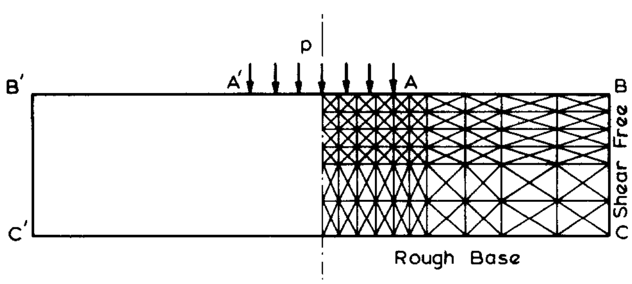
$$k_0 \leqslant \frac{1}{2}.$$

EXAMPLES

To illustrate the foregoing theory, consider the example shown schematically in Fig. 1. The



(a) Problem Definition



(b) Finite Element Mesh

Fig. 1.

†For the undrained case convergence will be more rapid as k_0 approaches a value of 1. The authors have found that rapid convergence without instability is obtained using $k_0 = 0.9$.

rectangular elastic body $ABCC'B'A'$ deforms under conditions of plane strain and is subject to a uniform normal traction of intensity p on AA'. The base of the body is rigidly fixed while the two sides BC, $B'C'$ are laterally restrained. BB', BC, $B'C'$ are all shear free. In the following examples finite element solutions were obtained using a value of $k_0 = 1/2$.

Case (a)

The problem defined above was first solved, for an isotropic material with a skeleton shear modulus G and a Poisson's ratio $\nu' = 0.3$, using the network of constant strain triangles shown in Fig. 1b. The results of this analysis are shown in Figs. 2a–d. In Fig. 2a the drained and undrained

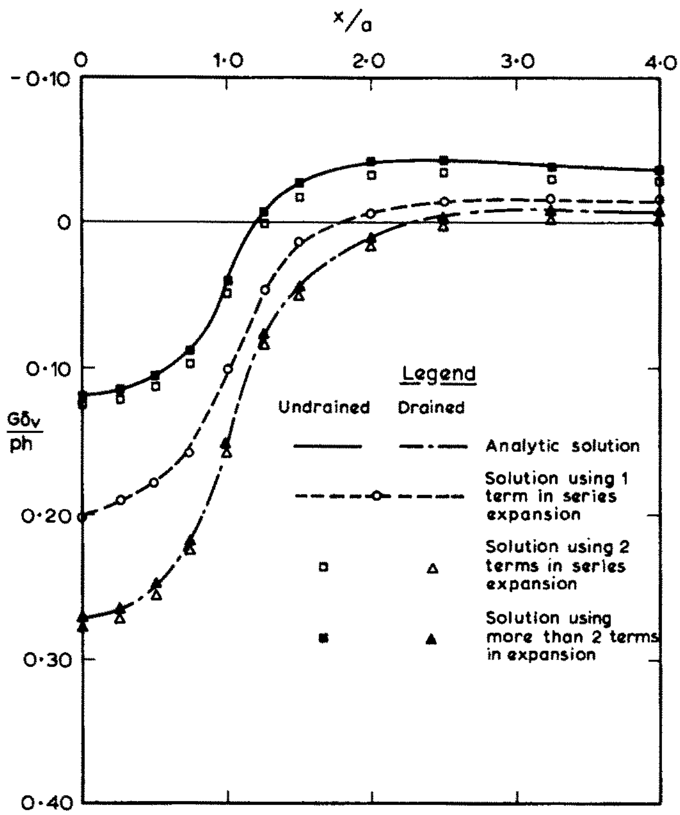


Fig. 2a. Surface profile beneath strip footing: isotropic layer. (δ_v is the vertical component of the surface deflection).

surface profiles are calculated by truncating eqn (13) after $1, 2, \ldots$ terms. These values converge quite rapidly to the solution for an infinite number of terms, viz. the eigenvalue expansion (18), and provide a close numerical approximation to the analytic solution to the problem which was obtained by the authors using a Fourier series analysis. In Fig. 2b, c the drained and undrained horizontal and vertical stress on the centre line are calculated by truncating the expansion (15) after $1, 2 \ldots$ terms. The pore pressure distribution can also be calculated by observing that

$$u\mathbf{a} = D\epsilon - \sigma$$

and its undrained distribution on the centre line is shown in Fig. 2d. As for the deflections, these quantities converge rapidly to the exact solutions of the finite element problem and again the agreement between the finite element solution and the analytic solution is quite close.

Case (b)

Next the problem shown in Fig. 1 was analysed for a cross anisotropic soil. The relationship between stress and strain for conditions of plane strain can be written[1]

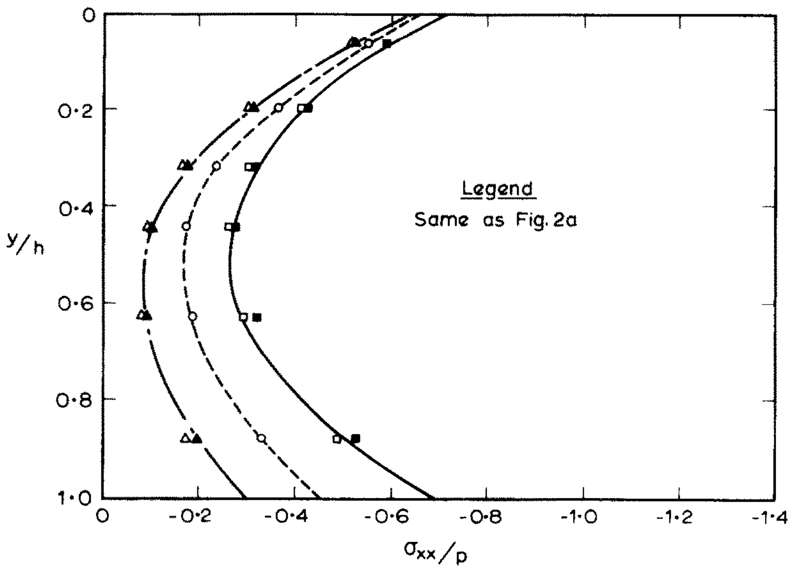


Fig. 2b. Horizontal stress beneath footing centre: isotropic layer.

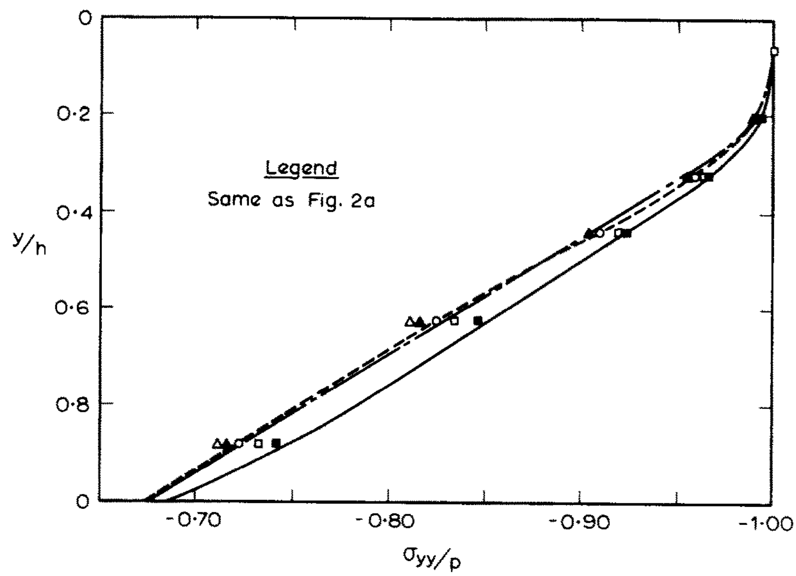


Fig. 2c. Vertical stress beneath footing centre: isotropic layer.

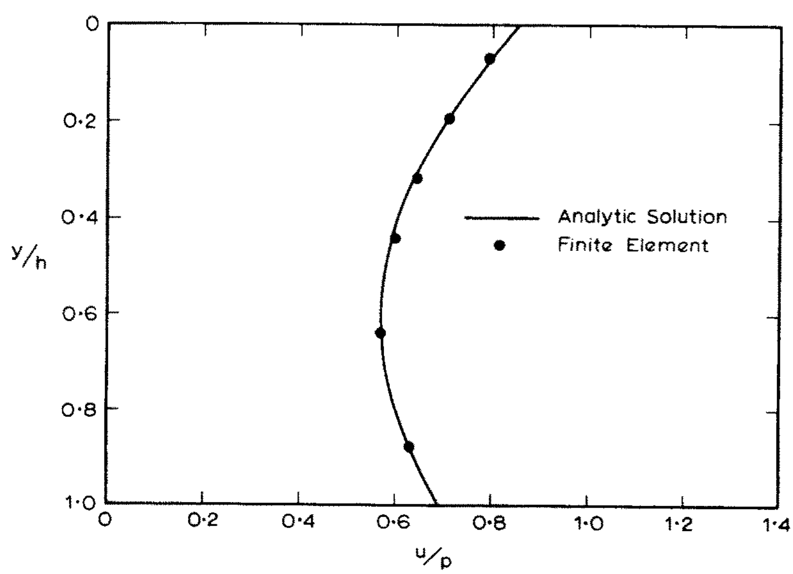


Fig. 2d. Excess pore pressure beneath footing centre: undrained isotropic layer.

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} r & s & 0 \\ s & t & 0 \\ 0 & 0 & u \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

where

$$r = \frac{E_2 n(1 - n\nu_2^2)}{(1+\nu_1)(1-\nu_1-2n\nu_2)}$$

$$s = \frac{E_2 n\nu_2}{1-\nu_1-2n\nu_2^2}$$

$$t = \frac{E_2(1-\nu_1)}{1-\nu_1-2n\nu_2^2}$$

$$u = G_2$$

and

$$n = E_1/E_2$$
$$m = G_2/E_2.$$

This problem was analysed for the case.

$$E_1/E_2 = 2$$
$$G_2/E_2 = 0.5$$
$$\nu_1 = 0.3$$
$$\nu_2 = 0.1.$$

The results of this analysis are shown in Figs. 3a-d. Again the truncated series converges rapidly to the exact solution of the finite element equations which is in turn a close approximation to the analytic solution.

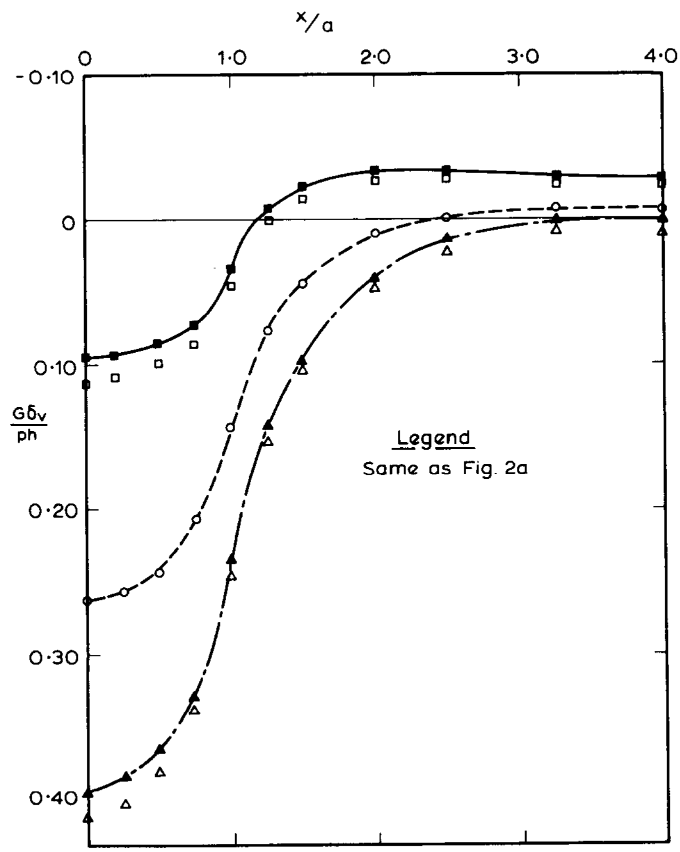


Fig. 3a. Surface profile beneath strip footing: anisotropic layer. (δ_v is the vertical component of the surface deflection).

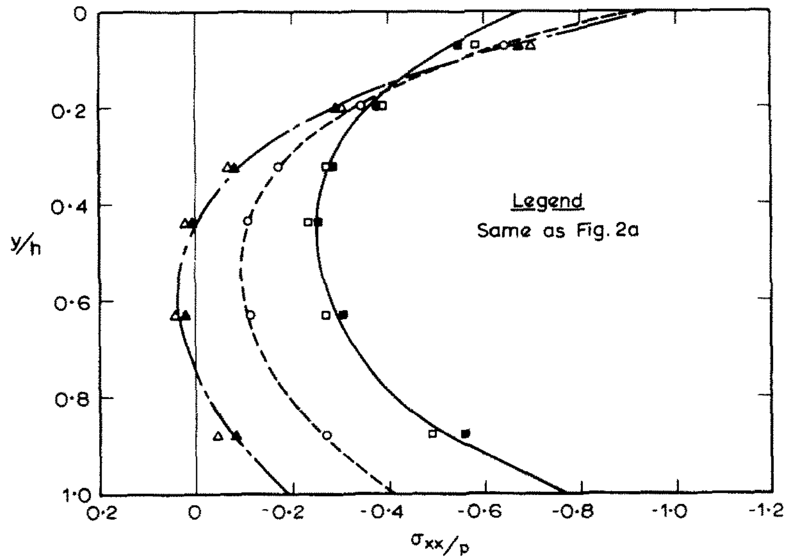


Fig. 3b. Horizontal stress beneath footing centre: anisotropic layer.

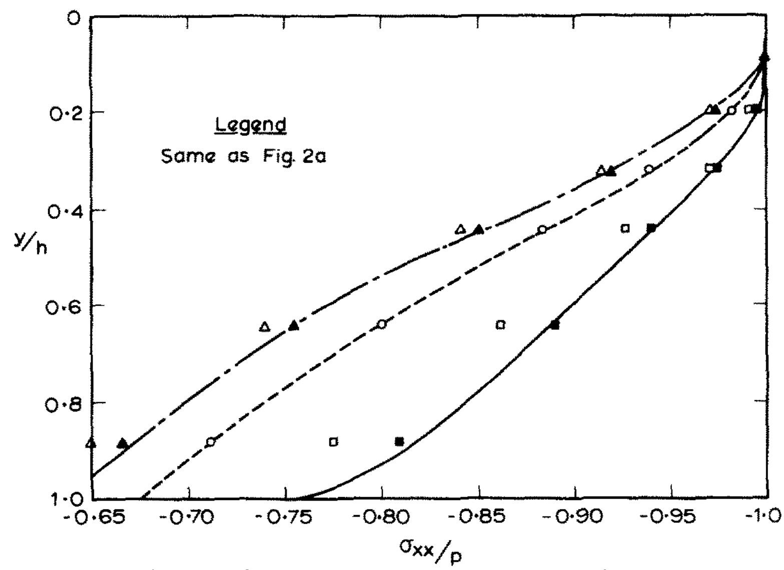


Fig. 3c. Vertical stress beneath footing centre: anisotropic layer.

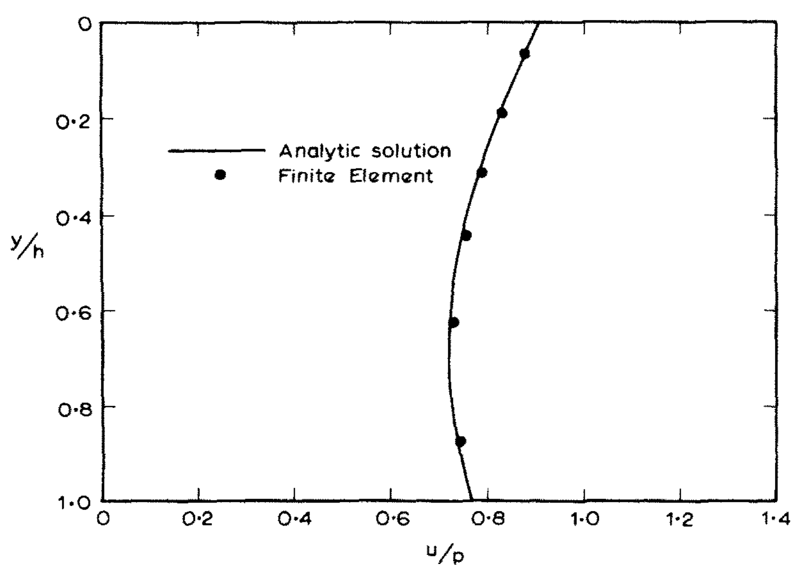


Fig. 3d. Excess pore pressure beneath footing centre: undrained anisotropic layer.

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APPENDIX A

D is a positive definite symmetric matrix and thus may be written as the product of real matrices

$$\begin{aligned} D &= LL^T \\ D^{-1} &= MM^T \\ M &= L^{-1}. \end{aligned} \tag{A1}$$

Now introduce the following quantities

$$\begin{aligned} \mathbf{h} &= M\boldsymbol{\sigma} \\ \mathbf{g} &= L^T\boldsymbol{\epsilon} \\ \mathbf{b} &= M\mathbf{a}/(\mathbf{a}^T D^{-1}\mathbf{a})^{1/2} \end{aligned} \tag{A2}$$

then eqn (6) may be written in the form

$$(1 - k\mathbf{b}\mathbf{b}^T)\mathbf{h} = \mathbf{g} \tag{A3}$$

Now when $|k| < 1$ the solution of eqn (A3) can be expressed as the convergent series

$$\mathbf{h} = [1 + k\beta + k\beta^2 + 1 \cdot \cdots \cdot +]\mathbf{g} \tag{A4}$$

where $\beta = \mathbf{b}\mathbf{b}^T$.

Now β is a projector for it is easily verified that

$$\beta^2 = \beta$$

and thus eqn (A4) may be written

$$\mathbf{h} = [1 + \beta(k + k^2 + \cdots\cdot +)]\mathbf{g} = \left[1 + \frac{k}{1-k}\beta\right]\mathbf{g} \tag{A5}$$

thus

$$\boldsymbol{\sigma} = \left[D + \frac{k}{1-k}\frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T D^{-1}\mathbf{a}}\right]\boldsymbol{\epsilon}. \tag{A6}$$

The above derivation of eqn (A6) depends upon the assumption that $|k| < 1$. However, it can be verified by substitution in eqn (6) that it remains true for all $k \neq 1$.