CANTILEVER SHEET-PILE WALL MODELLED BY FRICTIONAL CONTACT

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ABSTRACT

This paper demonstrates the application of computational contact mechanics in geotechnical engineering. It first presents a general formulation of kinematic constraints for frictional contact and a general description of the associated numerical algorithms. These numerical algorithms are then used to analyse the stability of cantilever sheet-pile walls. It is shown that the finite element method incorporated with the modelling capacity for frictional contact is very useful in geotechnical analysis and can provide solutions to problems that are otherwise difficult to solve.

KEY WORDS: Finite element method, frictional contact, sheet-pile, soil-structure interaction (IGC: E0/E2/E5)

INTRODUCTION

The soil-structure interaction is traditionally simplified to prescribed boundary conditions or modelled by joint elements. Prescribed boundary conditions, which include prescribed displacements for perfectly rigid structures and prescribed loads for perfectly flexible structures, simply neglect the interface and are crude simplifications of the realities. Interface or joint elements have also been used to model the soil-structure interaction (Goodman et al, 1968; Desai et al, 1984; Wilson, 1992). These elements, initially developed for rock joints, typically use normal and tangential stiffness to model the pressure transfer and friction at the interface. They can have a small or even zero thickness, but otherwise are not much different to other continuum elements. Because they are predefined and their topology remains unchanged during the solution procedure, they are only suitable for predefined interfaces with small and continuous interfacial deformation. A more recent alternative for modelling interface interaction is the use of kinematic constraints for frictional contact between solid bodies. This approach is particularly attractive for modelling interfaces with large relative and/or discontinuous displacement, and will be used to model soil-structure interaction in this paper.

This paper first presents a general finite element formulation of contact kinematics and contact constraints for geotechnical problems. We first present a brief review of contact kinematics and constitutive modelling of contact based on the penalty method. Such descriptions can also be found in the literature (e.g. Wriggers, 2002), but, for continuity and completeness, they are repeated here. A node-to-segment discretisation of contact surfaces is then proposed. A Newton-Raphson solution scheme is presented to solve the discretised contact problem. The numerical method is then used to analyse cantilever sheet-pile walls used for temporary support in geotechnical engineering. The numerical solutions are compared with the solutions based on the classic earth pressure theories.

FORMULATION OF FRICTIONAL CONTACT

Kinematics at the Interface

Consider a system of solid bodies in contact. Contact kinematics state that for any admissible displacement, there is no inter-penetration between any two bodies, and the contact normal stress $t_N$ can only be zero or compressive. The normal contact constraints can be represented as

$$g_N = 0, \quad \text{when } t_N > 0$$

$$g_N > 0, \quad \text{when } t_N = 0$$

$$g_N t_N = 0$$

where $g_N$ is the relative displacement in the normal direction (or the normal gap). When the displacement fields describing the motion of the bodies are known, the normal gap $g_N$ can be computed. For this we consider a point $\mathbf{x}$ in body $B^2$ and its closest projection point $\mathbf{x}'$ in body $B^1$. Here the superscripts 1 and 2 stand for the body 1 and body 2, respectively. The gap function between them is then (see Fig.1):

$$g_N = (\mathbf{x} - \mathbf{x}') \cdot \mathbf{n} = (\mathbf{u} - \mathbf{u}') \cdot \mathbf{n} + g_{N0}$$

where $\mathbf{u}$ is the displacement at $\mathbf{x}$, $\mathbf{u}'$ is the displacement at $\mathbf{x}'$, and $g_{N0}$ is the initial gap between $\mathbf{x}$ and $\mathbf{x}'$. Once the gap function is known, the constraints in Eq. (1) can be used to detect actual contacts along predefined pairs of contact surfaces.

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For frictional contact with a Coulomb-type law, the tangential constraints can be expressed as

\[ g_T = 0, \quad \text{when} \quad \mu t_N - |t_T| > 0 \]
\[ |g_T| > 0, \quad \text{when} \quad \mu t_N - |t_T| = 0 \]
\[ g_T (\mu t_N - |t_T|) = 0 \]  

where \( g_T \) is the relative displacement in the tangential direction (the tangential gap), \( t_T \) is the tangential stress at contact, and \( \mu \) is the coefficient of friction. The state when \( \mu t_N - |t_T| > 0 \) refers to the stick state, while the state when \( \mu t_N - |t_T| = 0 \) refers to the slip state. Again, the tangential gap can be computed once the displacement fields are known. For this, we consider the tangential relative movement at the contact interface during a small time interval between \( t_0 \) and \( t \). For 2D contact as depicted in Fig.1, we have:

\[ g_T = (\vec{u}^2 - \vec{u}^1) \cdot \vec{a}^t \quad \text{or} \quad g_T = \int_{t_0}^{t} (\vec{u}^2 - \vec{u}^1) \cdot \vec{a}^t \, dt \]

(3)

Note, in 3D problems, there are two tangential directions and hence \( g_T \) and \( f_T \) are vectors.

**Constitutive Behaviour at Interface**

The constitutive behaviour at the interface can be split into constitutive relations for the normal and the tangential direction. There are different possibilities to formulate the constitutive equations at the contact interface and in this paper we use the penalty method, due to its simplicity and wide application in engineering software. Extension to other methods such as the augmented Lagrangian method is straightforward (Wriggers, 2002).

In the normal direction of the contact interface, the normal component of the traction vector is given by

\[ t_N = \varepsilon_N g_N \]  

(4)

where \( \varepsilon_N \) is a penalty parameter for the normal contact. In the case of frictionless contact, this is the only stress at the contact interface. In this case, the traction vector at the interface is then given by \( t = t_N \vec{a}^N \). Note that normal contact force becomes zero when \( g_N \geq 0 \) (open gap), see Eq. (1). In the penalty method, contact occurs when \( g_N \leq 0 \) (gap close or a small penetration). Eq. (4) is applied independently of the tangential contact state.

In the case of frictional contact, the tangential component of the traction vector should be differentiated between stick and slip states, following the Coulomb friction law defined by Eq. 2,

\[ t_{T_n} = \begin{cases} \varepsilon_T (g_{T_n} - g_{T_n}^{est-1}) & \text{for stick} \\ \mu \cdot t_{N_n} & \text{for slip} \end{cases} \]

(5)

where \( t_{T_n} \) is the tangential component of the traction at time level \( t_n \), \( \varepsilon_T \) is a penalty parameter for the tangential contact, \( g_{T_n} \) is the total tangential gap at \( t_n \), \( g_{T_n}^{est-1} \) is the slip part of the tangential gap at \( t_{n-1} \), \( t_{N_n} \) is the normal component of the traction at \( t_n \). Note that Eq. (5) can also be used to detect the modes of the tangential movement. The constitutive equation for the normal contact, i.e. Eq. (4), applies irrespective of tangential contact states.

**Boundary Value Problem and Weak Forms**

The basis of the finite element method is the principle of virtual work. This states that if \( \delta u \) are virtual displacement fields satisfying the displacement boundary conditions, then equilibrium is satisfied provided:

\[ \sum_{\alpha} \left[ \int_{V_{\alpha}} \delta \sigma^T \mathbf{d}V - \int_{V_{\alpha}} \mathbf{b} \delta \mathbf{d}V - \int_{S_{\alpha}^c} \mathbf{t} \delta \mathbf{d}S \right] = 0 \]

(6)

where \( \delta \mathbf{e} \) denotes the variation of the strain tensor, \( \sigma \) is the stress tensor, \( \rho \) is the density, \( \mathbf{b} \) is the body force vector, \( \mathbf{t} \) is the distributed force acting on the boundary \( S_{\alpha} \) of the volume \( V \), and the summation is over the number of bodies. The interface contributions are the virtual work done by the traction at the contact interface for a virtual normal and tangential gap and must be added whenever the contact takes place

\[ \int \left( t_N \delta g_N + t_T \delta g_T \right) \, dS \]

(7)

When the penalty method is used, the normal and tangential tractions can be replaced by the normal and tangential gap functions using Eqs. (4) and (5), respectively.

Combining Eqs. (6) and (7) for the penalty formulation leads to

\[ \sum_{\alpha} \left[ \int_{V_{\alpha}} \delta \sigma^T \mathbf{d}V - \int_{V_{\alpha}} \mathbf{b} \delta \mathbf{d}V - \int_{S_{\alpha}^c} \mathbf{t} \delta \mathbf{d}S \right] + \int \left( t_N \delta g_N + t_T \delta g_T \right) \, dS = 0 \]

(8)

To solve the weak form (8), we must first discretise the domain as well the contact interfaces.

**Discretisation of the Contact Surfaces**

The discretisation for the domain contributions of the bodies in contact is not the objective of this work.
We concentrate here on the contact part of the discretisation. One approach that is widely used in nonlinear finite element simulation of contact problem is the so-called node-to-segment contact element as depicted in Fig. 2.

Assume that the discrete slave node \( x_1 \) comes into contact with the master segment \( x_1 - x_2 \), then the kinematical relations can be directly computed using the equations stated in Section 2.1. With the interpolation for the current position of master segment: \( x_a(\xi) = x_1 + (x_2 - x_1)\xi \), the tangent vector of the segment follows as: \( \mathbf{a} = x_a(\xi)/\xi = x_2 - x_1 \). It is connected to an orthonormal base vector \( \mathbf{n} = \mathbf{a}/\|\mathbf{a}\| = (x_2 - x_1) \), with \( l \) being the current length of the master segment. Now the unit normal to the segment can be defined as \( \mathbf{n} = \mathbf{a} \times \mathbf{a} \). The location of the slave node and the gap \( g_N \) are given by the projection \( \tilde{\xi} \) of the slave node \( x_1 \) onto the master segment:

\[
\tilde{\xi} = \frac{1}{l}(x_2 - x_1) \cdot \mathbf{a} \quad \quad (9)
\]

\[
g_{N,s} = \left[ x_1 - (1 - \xi) x_1 - \tilde{\xi} x_2 \right] \cdot \mathbf{n} \quad \quad (10)
\]

From these equations and the local formulation we compute directly the variation of the gap function \( \delta g_N \) on the straight master segment:

\[
\delta g_{N,s} = \left[ \delta x_1 - (1 - \tilde{\xi}) \delta x_1 - \tilde{\xi} \mathbf{n} \right] \cdot \mathbf{n} \quad \quad (11)
\]

In the case of frictional contact, we have to distinguish between two different states in the tangential direction, which are discussed below. For either state, we introduce the relative tangential movement by

\[
g_{T,s} = \tilde{\xi} l d\xi = (\tilde{\xi} - \xi_0) l \quad \quad (12)
\]

where \( \xi_0 \) is the stick point on the master segment and \( \tilde{\xi} \) is the current projection defined in Eq. (9). Note that the equation above has to be modified if the stick point and the current projection are not on the same master segment. The variation of the tangential gap for stick and slip must be distinguished and is given in Wriggers (2002):

\[
\delta g_{T,s}^u = \left[ \delta u_s - (1 - \tilde{\xi}) \delta u_1 - \tilde{\xi} \delta u_2 \right] \cdot \mathbf{a}
\]

\[
+ \frac{g_{N,s}}{l} \left( \delta u_2 - \delta u_1 \right) \cdot \mathbf{n} + \frac{\delta g_{T,s}}{l} \left( \delta u_2 - \delta u_1 \right) \cdot \mathbf{n} \quad \quad (13)
\]

\[
\delta g_{T,s}^a = \left[ \delta u_s - (1 - \tilde{\xi}) \delta u_1 - \tilde{\xi} \delta u_2 \right] \cdot \mathbf{a}
\]

\[
+ \frac{g_{N,s}}{l} \left( \delta u_2 - \delta u_1 \right) \cdot \mathbf{n} \quad \quad (14)
\]

For small deformations, only the first terms in (13) and (14) are needed, as the variations are small compared to the segment length. Eqs. (10)-(14) characterise the main kinematic relations of the contact element.

We can conveniently treat the slave node \( x_1 \), and the master segment \( x_2 - x_1 \) as one contact element. In what follows, we compute the contribution of the node-to-segment element to the weak form (8). We assume that we know the normal force \( T_{N,s} - t_{N,s} A_s \), and the tangential force \( T_{T,s} = t_{T,s} A_s \) at the discrete contact point \( x_1 \) of the contact element under consideration, with \( A_s \) being the area of the contact element. Both forces \( T_{N,s} \) and \( T_{T,s} \) can be obtained from the constitutive relations discussed above. This leads to

\[
\int_{c} \left( t_s \delta g_{N,s} + t_{T,s} \delta g_{T,s} \right) dS \approx \sum_{i=1}^{n} \left( T_{N,i} \delta g_{N,i} + T_{T,i} \delta g_{T,i} \right) \quad (15)
\]

where \( t_s \) and hence \( T_{N,i} \) follow directly Eq. (4), and \( t_{T,s} \) and hence \( T_{T,i} \) follow Eq. (5), and \( n_s \) is the total number of slave nodes in the system.

The contributions of one contact element in Eq. (15) can now be cast into a matrix formulation. For the variations of the normal and tangential gap functions we have

\[
\delta g_{N,s} = \left( \delta u_s, \delta u_1, \delta u_2 \right) \left[ \begin{array}{c} n \\ -(1 - \tilde{\xi}) n \\ -\tilde{\xi} n \end{array} \right] = \delta u^T N_s \quad (16)
\]

\[
\delta g_{T,s}^u = \left( \delta u_s, \delta u_1, \delta u_2 \right) \left[ \begin{array}{c} a \\ (\tilde{\xi} - 1) a \\ -\tilde{\xi} a \end{array} \right] + \frac{g_{N,s}}{l} \left[ \begin{array}{c} 0 \\ -n \end{array} \right] + \frac{\delta g_{T,s}}{l} \left[ \begin{array}{c} 0 \\ -a \end{array} \right] = \delta u^T T_s^u \quad (17)
\]

\[
\delta g_{T,s}^a = \left( \delta u_s, \delta u_1, \delta u_2 \right) \left[ \begin{array}{c} a \\ (\tilde{\xi} - 1) a \\ -\tilde{\xi} a \end{array} \right] + \frac{g_{N,s}}{l} \left[ \begin{array}{c} 0 \\ -n \end{array} \right] + \frac{\delta g_{T,s}}{l} \left[ \begin{array}{c} 0 \\ -a \end{array} \right] = \delta u^T T_s^a \quad (18)
\]

Thus the virtual work of the contact element can be written as \( \delta u^T G \) in the matrix formulation, with the contact element residual vector
1. Initialise algorithm: set $U=0$
2. Locate the master segment for each slave node $s$ and compute the normal gap $g^0_N$
3. Check for contact: $g^0_N \leq 0 \rightarrow$ active slave node
4. Set all active slave nodes to state stick
5. Update stiffness matrix $K(U^s)$ and the residual $R(U^s)$
6. LOOP over load increments: $n=1$, ...
   6.1. LOOP over iteration: $i=1$, ..., convergence
      6.1.1. Solve: $\Delta U = (K(U^i))^{-1} R(U^i)$
      6.1.2. Update $\Delta U^i = \Delta U^i + \delta U^i$ and $U^i = U_{\text{old}} + \Delta U^i$
      6.1.3. Check for convergence $\| R(U^i) \| \leq ITOL \Rightarrow$ exit iteration
      6.1.4. Locate the master segment for each slave node $s$ and compute normal gap $g^0_N$
      6.1.5. Check for contact: $g^0_s \leq 0 \rightarrow$ active slave node
      6.1.6. Compute tangential gap and check tangential contact state
      6.1.7. Stress integration and update internal force vector
      6.1.8. Update stiffness matrix $K(U^i)$ and the residual $R(U^i)$
6.2. END LOOP
7. END LOOP

Box 1. Contact algorithm using the Newton method.

$G^s_i = T_{Ns}N_s + T_{Ts}T_s$ \hspace{1cm} (19)

where $T_s$ stands for $T_{ss}$ for stick and $T_{sl}$ for slip. The tangent matrix for the contact element is derived by linearising the terms in Eq. (15). For the normal contact, we have

$$\Delta T_{Ns} \delta g_{Ns} + T_{Ns} \Delta g_{Ns} = \varepsilon_N \delta g_{Ns} + \varepsilon_N^* \delta g_{Ns} = \delta u^T (K_{Ns} \Delta u)$$ \hspace{1cm} (20)

where

$$K_{Ns} = \varepsilon_N^* N_sN_s^T - \varepsilon_N \frac{g_{Ns}}{l} \left( N_{0s}T_s + T_{Ns}N_{0s}^T + \frac{g_{Ns}}{l} N_{0s}N_{0s}^T \right)$$ \hspace{1cm} (21)

For tangential contact in stick state, we have

$$K^s_{Ts} = \varepsilon_N \left[ T_{Ts}^T (T_{Ts}^s)^T + \frac{g_{Ts}}{l} (N_{0s}N_{0s}^T + N_{0s}T_{Ts}^s) - \frac{g_{Ns}}{l} (T_{0s}N_{0s}^T + N_{0s}T_{Ts}^s) \right]$$ \hspace{1cm} (22)

For tangential contact in slip state, we have

$$K^d_{Ts} = \frac{T_{Ts}}{l} \left[ N_sN_s^T + N_sN_s^T T_{Ts}^s - T_{Ts}^s N_sN_s^T \right] - \frac{g_{Ns}}{l} \left( N_{0s}T_{Ts}^s + 2T_{Ts}N_{0s}^T \right) + \mu \varepsilon_N \text{sign}(T_{Ts}^s)T_{Ts}^sN_s^T$$ \hspace{1cm} (23)

The matrices in Eqs. (21) to (23) are given as

$N_s = \begin{bmatrix} n \\ -(1-\xi) n \\ -\xi n \end{bmatrix}$ \hspace{1cm} $T_s = \begin{bmatrix} a \\ -(1-\xi) a \\ -\xi a \end{bmatrix}$

Note that the matrices $K_{Ns}$ and $K^s_{Ts}$ are symmetric, but not the matrix $K^d_{Ts}$. Also note that the computing of these stiffness matrices for a slave node $s$ is only necessary when it comes to in contact with a master segment.

**SOLUTION ALGORITHMS FOR CONTACT PROBLEMS**

The algorithms which are applied in many standard finite element programs is related to the active set strategy in which the problem is solved for a chosen active set of constraints. Before we state the algorithms we combine (8) and (19) to the global set of equations

$$\sum_a \left( \int_{S_a} \delta e^T \sigma \ dV - \int_{S_a} \delta u^T b \ dV - \int_{S_a} \delta u^T t \ dS \right) + \int_{\bar{S}_i} (t_s \delta g_N + t_r \delta g_r) \ dS = -\delta U^T G_s(U)$$ \hspace{1cm} (24)

where $U$ is used in the place of $u$ to indicate the discretised global displacement field, $G(U)$ denotes the domain contributions to the residual vector and its expression can be found in standard finite element texts (e.g. Zienkiewicz and Taylor, 1991). $G_s(U)$ denotes the contact contributions and is given by (19), and $R(U)$ is the global residual vector. The negative sign is introduced in front of the residual vector so that it follows the standard expression in the finite element procedure, i.e.
the difference between the external force and internal force. The global tangent matrix is obtained through the linearisation of (24) at a given U

$$\begin{align*}
\partial \left( G(U) + \sum_{i=1}^{n} G_i(U) \right) \quad & \mathbf{K}(U) = - \frac{\partial U}{\partial G(U)} \\
& = \left( \mathbf{K}_{ep}(U) + \sum_{i=1}^{n} \left( \mathbf{K}_{N_i}(U) + \mathbf{K}_{F_i}(U) \right) \right) \\
& \text{where } \mathbf{K}_{ep} \text{ is the tangent stiffness matrix due to the} \quad & \text{elastoplastic domains, } \mathbf{K}_{N_i} \text{ and } \mathbf{K}_{F_i} \text{ are the tangent matrices due to the normal and tangential contact and are} \quad & \text{given by Eqs. (21)-(23).}
\end{align*}$$

Newton-Raphson Scheme

The equation system (24) is nonlinear due to the interface friction and possibly also due to material and geometrical nonlinearity, and can be solved by a range of iterative and incremental schemes (see e.g. Sheng and Sloan, 2001). Here we use one of the most common solution scheme for nonlinear systems of equations, i.e. the Newton-Raphson scheme. The Newton-Raphson scheme solves the system (24) by iteration as follows

$$\begin{align*}
\delta \mathbf{U}^i &= \left( \frac{\partial \mathbf{R}}{\partial \mathbf{U}} \right)^{-1} \mathbf{R}(\mathbf{U}^{i-1}) = \left( \mathbf{K}(\mathbf{U}^{i-1}) \right)^{-1} \mathbf{R}(\mathbf{U}^{i-1}) \\
\mathbf{U}^i &= \mathbf{U}^{i-1} + \delta \mathbf{U}^i
\end{align*}$$

In practice, the Newton-Raphson scheme is often applied incrementally over the load path. This scheme combined with the active set strategy is summarised in Box 1. Note that the stick point $\xi$ for each slave node is updated only when the tangential contact is in slip state during a load increment and is updated after the iteration loop is converged. The convergence of the iteration is determined by comparing the residual vector with the current external force vector $\mathbf{F}^{ext}$ (step 4.1.3 in Box 1). The contact mode is always set to state stick when a slave node first time gets in contact with a master surface, or re-gains contact with a master surface after losing contact in a previous step. In the standard Newton-Raphson method, the tangent matrix is updated in each iteration loop, while in the modified Newton-Raphson method, the tangent matrix is updated once at the start of the load increment and is then kept constant during the iteration. The residual vector $\mathbf{R}(\mathbf{U})$ and hence the contact force vector $\mathbf{G}^{i}(\mathbf{U})$ are updated in each iteration.

Augmented Lagrangian Method

We see that the penalty method uses only displacements to approximate the contact constraints and can be easily implemented into the displacement finite element method. However, good approximations of the contact constraints require high penalty parameters which may, in turn, lead to ill-conditioned global equations. A newer method that overcomes the shortcomings of the penalty method is the so-called augmented Lagrangian method. The augmented Lagrangian method is implemented using a double loop algorithm with an additional loop outside the Newton iteration loop in Box 1 to update the Lagrange multipliers. The Lagrange multipliers (usually the normal contact stresses) are first held constant during the Newton iteration loop to solve for the weak form and are then updated by adding to them the current penalty forces. Within this structure, the updates of the Lagrange multipliers are performed in such a way that the new equilibrated solution results in a reduction of the penalty forces, and hence a reduction of the constraint violation. Since the Lagrange multipliers are fixed during the Newton inner loop for the displacements, the augmented Lagrangian method can be considered as a displacement method, but the Lagrange multiplier ensures that its accuracy is not significantly affected by the penalty parameters.

APPLICATIONS

For simplicity, we use the Coulomb friction law for constitutive behaviour at contact. If not indicated otherwise, we use the simple penalty method for contact treatment and the penalty parameter is set to $10^6$. Test runs indicate that variation of this parameter by an order of one or two does not significantly affect the results (with a variation less than 1% in the maximum horizontal displacement of the sheet-pile wall). The Newton-Raphson method in Box 1 is used to solve the quasistatic problems, with an iteration tolerance set to $10^{-4}$ if not indicated otherwise. The explicit stress integration scheme with automatic substepping and error control is used to solve the stresses at integration points (Sloan et al, 2001), due to its efficiency and robustness. The stress integration error is set to $10^{-5}$ if not stated otherwise.
As an example, we look at a cantilever sheet-pile wall problem. Such a wall is commonly used for temporary support in excavation. The problem is difficult to model by the traditional methods that use prescribed boundary conditions, because neither the displacements nor the forces at the soil-wall interface are known. The classic earth pressure theories such as the Rankine and the Coulomb theories are often used to analyse the stability of such walls. The validity of these theories depends very much on the assumptions in them, such as the assumption about the stress state in the Rankine theory and about the slip surface in the Coulomb theory. The validity of these assumptions in turn depends very much on the deformations of the retaining wall. Here, we demonstrate the use of contact constraints in the analysis of sheet-pile walls and compare the finite element analysis with the classic earth theories.

The geometry and the mesh of the sheet-pile wall problem are shown in Fig. 3. Quadrilateral elements with 4 nodes are used for both the soil and the wall. The soil is model by the associated Mohr-Coulomb model, while the wall is treated as an elastic material with a Young’s modulus of 200 GPa for a stiff wall (such as concrete and steel). The unit weight of the wall is set to the same value as the soil, to minimising differential settlement during body force loading. Other soil and wall properties are given in Fig. 3. The contact between the wall and soil is assumed to be frictional, with an interfacial frictional coefficient of 0.1.

The analysis is carried out in the following steps. In the first step, body forces corresponding to the given unit weights are applied to the wall and the soil below the level of the excavation (5m from the bottom). The soil above this level (the backfill) is assumed to be weightless during the first step. In the second step, the unit weight of one meter soil above the level of the excavation increases from zero to 20 kN/m² in 100 coarse increments, while the unit weight of the soil elsewhere is kept constant. The change of unit weight is realised through body force application. This procedure is equivalent to backfilling one meter of soil. In each step afterwards, one meter of soil is backfilled by activating its unit weight. In order to reduce the effects of stiffness of the weightless soil above the backfill, we set the top, left and right boundaries free. Just before the activating the unit weight, we restrain the horizontal movement of the right boundary for the one meter backfill. While this arrangement does not completely remove the stiffness effect (because the weightless soil is still attached to the deform ing soil below), it does reduces the stiffness effect to a possible minimum. The deformed mesh shown in Figure 8 demonstrates that the deformation in the weightless soil above the backfill is not very significant.

In Fig. 4, the horizontal displacements at the head of the wall and the active earth thrust on the wall are plotted against the backfill height for two embedment depths of the wall. The active earth thrust is taken as the sum of the horizontal nodal forces along the back face of the wall. For comparison, the active earth thrust calculated according to the Coulomb earth pressure theory is also plotted in the figure. We see that the finite element model in general predicts a smaller active earth pressure than the Coulomb theory. At the backfill of 5m, the active earth thrusts predicted by the finite element model are about 70–80% of those calculated according to the Coulomb theory.

For both embedment depths $D=2m$ and $D=1.5m$, it is possible to backfill to a total height of 5.0m without loosening the stability of the wall. However, we see that the wall has suffered significant displacements. The total horizontal displacement at the wall head is 111.7cm for $D=2m$ and 179.6cm for $D=1.5m$. Such large displacements are of course not allowed in reality. Therefore, we need to introduce a displacement control mechanism in the wall design. If we limit the horizontal
displacement at the wall head to for example 10cm, the wall with $D=2m$ can retain 3.66m height of backfill and the wall with $D=1.5m$ only 2.95m backfill. The critical backfill height naturally depends on the allowable wall head movement used in practical design, of which the authors are unfortunately not aware. The choice of the value of 10cm in this paper is purely based on the fact that it seems to represent a turning point of the curvature of the displacement curve in Fig. 4.

Fig. 5 shows the predicted horizontal stresses on the wall compared with the Coulomb active earth pressure. The horizontal stresses can also interpreted as the frictional stresses on the wall, as the latter equals the former multiplied by 0.1. We see that the predicted horizontal stresses are smaller than the Coulomb earth pressure, except near the lower end of the wall. The difference between the two is largely due to the failure mechanism assumed in the Coulomb theory (forward sliding) and the predicted failure mechanism (rotation, see Fig. 8). Rotation of the wall also causes a huge frictional stresses at the lower end of the wall.

Using the classic earth pressure theory, we can also approximately estimate the critical backfill height $H$. First assume the sheet pile wall rotates about its lower end. Moment equilibrium about its lower end gives,

$$ P_p \frac{1}{3} D = P_a \frac{1}{3} (H + D)$$

$$ \Rightarrow \frac{1}{6} \gamma D^3 K_p = \frac{1}{6} \gamma (H + D)^3 K_a $$

where $D$ is the effective embedment depth of the wall, $P_a$ and $P_p$ are the active and passive earth thrust acting on the wall, respectively, $K_a$ and $K_p$ are the active and passive earth pressure coefficients, respectively, and $\gamma$ is the unit weight of the soil. Sometimes, the depth $D$ so found is further increased by 20% to allow a rotation point above the lower end of the wall.

The earth pressure coefficients can be found in soil mechanics textbooks (e.g. Craig, 1992)

$$\begin{cases}
K_a = 0.56 \\
K_p = 3.47
\end{cases} \quad \text{for } \phi=30^\circ \text{ and } \tan \phi_w = 0.1$$

The effective embedment depth $D$ is the actual depth of the wall when the wall rotates about its lower end (see Fig. 8). The critical depth so estimated is about 1.67m for $D=2m$ and 1.26m for $D=1.5m$. These values are...
where soil is assumed to be linearly elastic and perfectly plastic. The soil and the wall play a significant role in designing the height of the soil that the wall can retain. For example, for an embedment depth of 1m and a soil friction angle of 35°, the critical height of the wall increases from 2.4m to 2.9m when the Young’s modulus increases from $2.0 \times 10^4$ kPa to $2.0 \times 10^5$ kPa. In addition, for the same soil, the critical height of the wall increases approximately linearly as the embedment increases between 0.5m and 2.5m, which is consistent with Eq. (28). However, this linearity no longer exists when the stiffness of the wall reduces to $2.0 \times 10^6$ kPa, a value equivalent to that of timber. In Fig. 6, the predicted critical heights of the wall are plotted against the wall embedment depths for a wall stiffness of $2.0 \times 10^6$ kPa. In this case, we see clearly that the critical depth $H$ approaches a constant as the embedment depth increases to 2.5m. The reason for this constant $H$ is that the wall with a lower stiffness tends to bend about the dredge line whereas the wall with a higher stiffness tends to rotate about its lower end. Such deformation patterns are further confirmed in Fig. 8. We also note that the stiff wall (Fig. 8a) experiences very little bending. The soil behind the wall (backfill zone) is basically under the active failure mode, whereas the soil in front of the wall is basically under the passive failure mode. Therefore, this type of deformation pattern is closer to the classic earth pressure theories. Because the wall is rotating instead of sliding, the principal stresses in the soil are no longer horizontal and vertical, which is different from the earth pressure theories. On the other hand, the flexible wall shown in Fig. 8b experiences significant bending as well as rotation. The bending of the wall also creates a passive zone in the soil behind the wall and an active zone in the soil in front of the wall. Therefore, the deformation pattern in the soil is more complex than those assumed in the classic earth pressure theories. In both cases, we see small relative sliding and large opening at the wall-soil interface. In the case of the flexible wall shown in Fig. 8b, we also see some penetration at the wall-soil interface. The penetration of soil into the wall is a computational inaccuracy caused by the penalty method. A larger penalty number would decrease this penetration, and theoretically only an infinite penalty number can avoid this penetration.

The example demonstrated here shows that the numerical method with contact modelling capacity is indeed very useful for analysing the deformation and stability of cantilever sheet-pile walls. The classic earth pressure theories, though useful for simple problems, are not very accurate, mainly due to the assumed deformation pattern and the principal stress directions. The numerical method can not only provide detailed information about the deformation and stability of the wall, but can also produce design charts that relate soil and wall properties to design parameters like the embedment of the wall and the backfill height. The problem analysed here is relatively simple, with a homogeneous soil modelled by a simple soil model under plane-strain condition. The true advantage of the finite

![Image](image-url)
element method with contact modelling capacity lies at dealing with practical engineering problems that may well be three-dimensional, with complex soil conditions and complex soil behaviour, and that can not be easily analysed otherwise.

CONCLUSIONS

This paper demonstrates the application of computational contact mechanics in geotechnical engineering. It presents a general formulation for problems involving frictional contact and a general description of the associated numerical algorithms. The cantilever sheet-pile wall problems are chosen as an example and are analysed using contact constraints. Some key conclusions drawn from this study are:

1. Incorporating contact kinematics into the finite element method indeed proves to be useful for solving geotechnical problems that are otherwise difficult to solve, in particular for those problems where the deformation of the super-structure is important and where the soil-structure interfacial displacement is large and not continuous. But the method does become more complex, due to the added difficulties caused by the non-smooth frictional contact. Special solution schemes are required and these include the solution methods for the weak forms of the global equilibrium equation including contact contributions and the contact search algorithms which are not discussed in this paper.

2. The classic Coulomb friction law used to represent the interface behaviour can be incorporated into the finite element method with relative ease. More advanced interface constitutive laws can be incorporated, but may require special treatment such as numerical integration of the constitutive relations at contacting points.

3. In the example of the cantilever sheet-pile wall, the predicted earth thrusts acting on the wall are compared with those obtained from the Coulomb earth pressure theory. The predicted active earth thrusts are significantly smaller than those values according to the Coulomb earth pressure theory. The difference between the numerical results and the earth pressure theories is attributed to the difference between the actual and assumed deformation patterns and principal stress orientation.

4. According to the numerical analysis, the critical height of the soil that a cantilever sheet-pile wall can retain increases with increasing soil shear strength and soil stiffness. The critical height also increases with increasing embedment depth of a stiff wall, but can approach a constant a value with increasing embedment depth of a flexible wall.

5. The problem analysed in this paper is relatively simple, with a homogeneous soil modelled by a simple soil model under plane-strain condition. The true advantage of the finite element method with contact modelling capacity lies at dealing with practical engineering problems that may well be three-dimensional, with complex soil conditions and complex soil behaviour, and that can not be easily analysed otherwise.

6. Other geotechnical problems that can be solved by contact constraint include installation of displacement piles and testing devices pushed into the ground (see e.g. Sheng et al, 2004). These problems involve large interfacial displacements and can not be solved easily otherwise. They are not considered in this paper because they contain other complexities such as large deformation that must be handled by adaptive meshing and fluid flow in the soil voids that the continuity equation must be coupled to solve the pore pressures. However, the general strategies presented in this paper should apply.

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1. Craig, RF (1992), Soil Mechanics, Chapman & Hall.