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LETTERS TO THE EDITOR

A LARGE STRAIN THEORY AND ITS APPLICATION IN THE ANALYSIS OF THE CONE PENETRATION MECHANISM

(*P. D. Kiousis, G. Z. Voyiadjis and M. T. Tumay, Int. j. numer. anal. methods geomech.* **12,** 45–60 (1988))

This paper has addressed a very significant problem in geotechnical engineering, namely the analysis of the cone penetration test. As the authors note, this is a very difficult problem to solve, even numerically, since it involves large deformations, moving boundaries and non-linear material behaviour.

The authors have used an axi-symmetric finite element analysis and presented numerical results for the displacement, stress and pore water pressure fields around the cone tip at failure for penetration under undrained conditions. Failure is defined as the condition where penetration proceeds indefinitely at a constant cone pressure. Results are also given for the relationships between penetration resistance and penetration depth.

In the analysis reported in the paper it was assumed that the initial effective stress state in the soil was hydrostatic with a magnitude of $p'_o = 100\,\mathrm{kPa}$ (where compression is considered positive). The stress–strain behaviour of the material was formulated in terms of effective stress and the elasto-plastic soil model included a work hardening yield cap and a fixed yield surface. The model parameters were selected so that the initial undrained shear strength of the soil under triaxial test conditions, c_u, was approximately 50 kPa.

Furthermore, the elastic behaviour of the soil skeleton was defined by Young's modulus $E = 30{,}000\,\mathrm{kPa}$ and Poisson's ratio, $v = 0.28$. Thus the elastic shear modulus $G = E/2(1+v)$ can be computed simply as $G = 11{,}719\,\mathrm{kPa}$. The choice of soil model and initial stress state implies that the soil will work harden prior to reaching the shear failure surface.

For the ideal case presented in the paper the authors have computed an ultimate cone pressure, $q_c = 525\,\mathrm{kPa}$. it is interesting to compare this value of ultimate cone resistance with the limit pressure predicted by the theory for spherical cavity expansion. For the cavity expansion problem, the writers have adopted a total stress approach and assumed a simple-elastic–perfectly-plastic soil model with deformations proceeding under constant volume (undrained) conditions. The elastic behaviour of the ideal material is characterized by an elastic shear modulus $G = 11{,}719\,\mathrm{kPa}$ and an undrained shear strength, $c_u = 50\,\mathrm{kPa}$. This material will yield plastically whenever the following criterion is satisfied

$$\sigma_1 - \sigma_3 = 2c_u \tag{1}$$

where σ_1 and σ_3 are major and minor principal total stress components, respectively.

The solution for the limit pressure, at which indefinite spherical cavity expansion occurs in this type of material, has been given in closed form by Bishop *et al.*[1] The expression for the limit pressure p_L can be written:

$$p_L = p_o + \left(\frac{4}{3}\right) c_u \left[1 + \ln\left(\frac{G}{c_u}\right)\right] \tag{2}$$

where p_o is the initial hydrostatic total stress in the material. It would appear that in the cone penetration problem solved by the authors, the ambient (initial) pore water pressure was set to zero, so that the total and effective stress components are identical, in which case $p_o = p_o = 100\,\mathrm{kPa}$. Substitution of the appropriate values of p_o, G and c_u into equation (2) gives the limit pressure for cavity expression as $p_L = 530\,\mathrm{kPa}$. This compares most favourably with the computed cone resistance of 525 kPa.

The cavity expansion solution may also be used to predict the magnitude of excess pore water pressure in the soil at the cavity boundary at the limit condition. The simple-elastic–perfectly-plastic model predicts no change in mean effective stress prior to and during yielding, and hence the maximum excess pore pressure generated during the cavity expansion is predicted to be $u_i = p_L - p'_o = 530 - 100 = 430\,\mathrm{kPa}$. This value is significantly below the maximum excess pore pressure predicted by the finite element analysis (see Figure 9 of the original paper), but is in reasonable agreement with the values predicted along about the middle third of the cone tip. In the cavity expansion prediction the simple model does not allow for the generation of excess pore pressure prior to shear failure (due to work hardening in the cap model), but it is expected that this would only be a small proportion of the overall increase in excess pore pressure.

An acknowledgement in the paper of a 'Supercomputer Utilization Grant' is noted with interest. It would be of further general interest if the authors could also indicate what type and amount of computing effort was required to obtain their finite element solutions. Of course, the evaluation of the cavity expansion solutions (equation (2)) required the use of only a hand calculator for the present case.

While it can be regarded as satisfying to find the close agreement between the cavity expansion limit pressures and the more rigorous numerical solution of cone penetration in this particular case, the writers wish to make it clear that they would encourage the pursuit of further numerical solutions to this important problem. The cavity expansion model makes a gross simplification of the kinematic field near the tip of a cone; many workers have made that point in the past and this paper demonstrates well what the correct pattern may be for a constant volume penetration (Figure 5). It would be of great benefit to geotechnical engineering if further numerical solutions to the cone penetration problem were available. In particular, it would be of interest to know if the more economical cavity expansion solution also can provide reasonable predictions of the cone resistance at other values of the ratio of stiffness to strength, G/c_u, in the undrained problem and for other types of soil behaviour, notably those involving either dilation or contraction of a granular material. For the latter type of material the more general, closed form cavity expansion solutions presented by Carter *et al.*[2] may be of some use in making these comparisons.

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1. R. F. Bishop, R. Hill and N. F. Mott, 'Theory of indentation and hardness tests', *Proc. Phys. Soc.*, **57**, 147 (1945).
2. J. P. Carter, J. R. Booker and S. K. Yeung, 'Cavity expansion in cohesive frictional soils', *Géotechnique*, **36**, (3), 349–358 (1986).

AUTHORS' REPLY

Carter and Yeung present interesting comparisons of the finite element method (FEM) results, presented in the original paper, with the spherical cavity expansion theory. The similarities in results obtained by the two methods are impressive, especially with regard to the tip resistance. This agreement becomes interesting if one considers that the FEM solution was obtained on a CYBER 205 with the use of approximately 10 CPU hours, while the cavity expansion solution was obtained within a few minutes of calculator level effort. As it is stated in the last paragraph of the introduction of the original paper, the FEM approach is not well suited for a routine analysis of the problem since it is very expensive. However, this type of analysis can provide a useful insight to the problem as the authors have stated in the paper and Carter and Yeung agreed in their letter to the editor.

Discussion on the comparisons of Carter and Yeung

The agreement between the FEM prediction of tip resistance and the spherical cavity expansion prediction is not very surprising. A close examination of the deformation field around the penetrometer (Figure 5 of the paper) shows that the soil around the tip displaces in a spherical cavity expansion fashion, i.e. almost radially downwards, for penetrations of 5 to 20 mm. In addition, the constitutive model formulation did not allow for damage effects due to large strains (in excess of 100 per cent), which would lead to softening and thus to a possibly different prediction of the tip resistance.

Also, it is not very surprising that the agreement of excess water pressure predictions is not as good. The excess water pressures measured in the soil depend on the tendency of the soil skeleton to compress. One can create different elastic and plastic (cap model) soil parameters that result in the same values of shear modulus G and undrained shear strength c_u. The excess water pressure prediction of the cavity expansion does not change in this case, while the FEM solution predicts larger water pressures, or smaller water pressures, depending on the compressibility of the soil. Consider, for example, the following material properties for the cap model: $\alpha=\gamma=350$ kPa and $\beta=0.000594$, $R=1.2$.

Also consider the following elastic parameters: $E=25{,}782$ kPa, and $\nu=0{\cdot}1$. This material has a larger tendency for volumetric compression compared to the one used in the paper. However it yields exactly the same values of G and c_u as the original material. FEM undrained analysis should predict larger excess water pressure than the one found in the paper. The cavity expansion solution, on the other hand, does not change. We thus conclude that one can use the cavity expansion solution for certain problems, but should be cautious of the limitations of the method.

General Discussion on the Cavity Expansion Approach, and Conclusions.

We recall here that the cavity expansion approach suggested by Carter and Yeung is based on the assumptions of undrained deformations and homogeneous, perfectly plastic material behavior. Such an approach, of course, cannot be used in the analysis of sand deposits or layered soil strata, especially if some of the layers are clean sands, i.e. they deform under drained dilatant or contractive behaviour. However, it is probable that an analysis similar to the one performed by the authors, will show that the deformation field of the soil around the tip is similar to that of a cavity expansion. In this case one can simulate the cone penetration by performing a FEM analysis of cavity expansion using a more sophisticated and accurate constitutive model. Such an approach, of course, would be significantly more economical and, if accurate enough, certainly preferable for the routine interpretation of cone penetration results.

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MODELLING THE ELASTIC BEHAVIOUR OF GRANULAR MATERIALS

(*P. V. Lade and R. B. Nelson, Int. j. numer. anal. methods in geomech.* **11**, 521–542 (1987))

In reviewing various models available for the elastic response of soils, the authors mentioned that the simple formulation

$$E_i = K_i p_a(\sigma'_3/p_a)^n \qquad (1)$$

has been widely used because it appears to capture the elastic behaviour of soils observed in triaxial and isotropic compression.

Similar formulations have also been used to characterize the behaviour of granular materials under repeated loading,[1,2] in which the deviatoric stress $\sigma_1-\sigma_3$ is repeatedly applied from zero at a relatively low frequency ($\leqslant 1$ Hz):

$$M_r = K_1\sigma_3^{K_2} \qquad (2)$$

or

$$M_r = K'_1 I_1^{K'_2} \qquad (3)$$

where M_r is the resilient modulus of the material, I_1 is the first stress invariant and K_1, K_2, K'_1 and K'_2 are material parameters. Under repeated loading, the material response is essentially elastic after

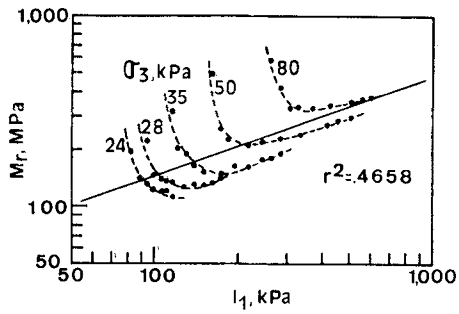

Figure 1. Resilient modulus *v.* first stress invariant for test with constant confining pressure

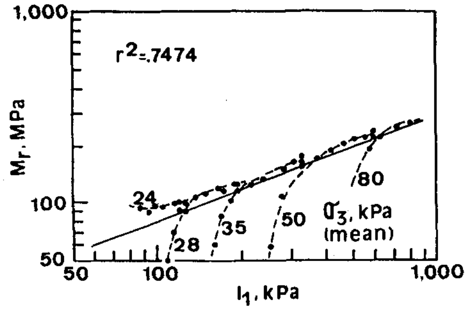

Figure 2. Resilient modulus v. first stress invariant for test with cyclic confining pressure

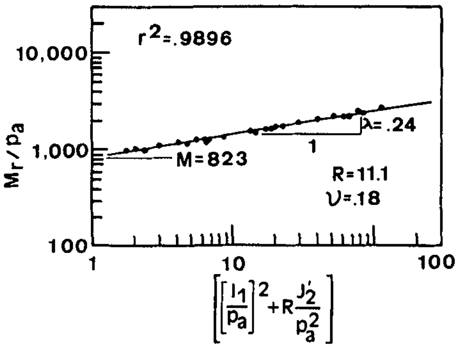

Figure 4. Author's proposed model

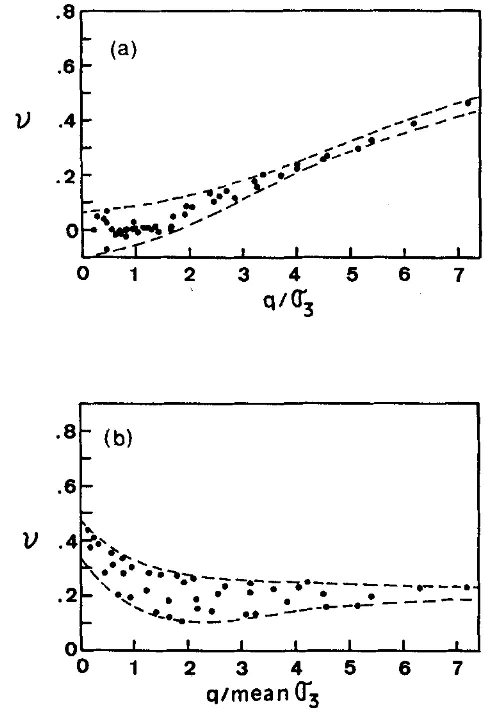

Figure 3. Variation of Poisson's ratio for tests with (a) static and (b) cyclic confining pressure

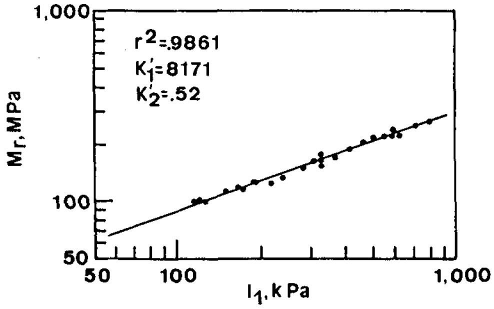

Figure 5. Resilient modulus v. first stress invariant for test with cyclic confining pressure ($\sigma_1/\sigma_3 \geq 2$)

applying a sufficient number of conditioning cycles, so the resilient modulus and Poisson's ratio can be calculated by using the generalized form of Hooke's law.

Equation (3) has been used extensively in many pavement analyses, mainly because it is simple and can be used to predict the resilient modulus of a granular material under the limited stress regime normally encountered in a base course layer. However, it is known that this formulation does not properly model the stress–strain relations of granular materials under repeated loading, especially when a low stress ratio (<2 to 3) is used.[3]

The limitation of equation (3) is demonstrated in Figure 1, which shows a typical result obtained from a repeated load triaxial test with constant confining pressure, and in Figure 2, which presents a typical result from a test with cyclic confining pressure. The material was a well-graded, dry crushed rock with a maximum particle size of 19 mm and a coefficient of uniformity of 32. Tests were carried out on 200 mm diameter specimens with vertical and radial strains measured on-sample with devices similar to those used by Brown and Snaith.[4] It is seen that equation (3) cannot properly model the modulus for low stress ratio. A more accurate model, as the authors have pointed out, must be a function of the mean normal stress as well as the deviatoric stress.

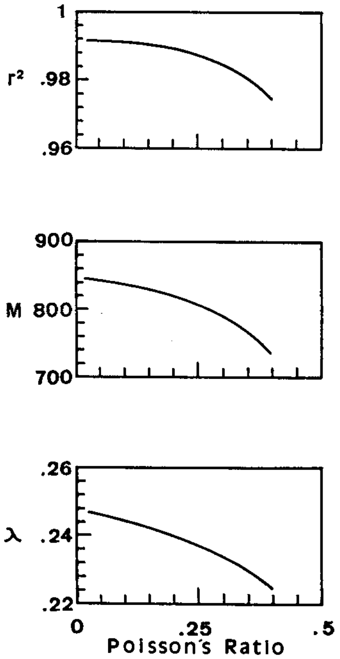

Figure 6. Variations of r^2, M and λ with Poisson's ratio

For design purposes, a test with either a constant confining pressure or a cyclic confining pressure may be used to give a similar modulus, provided the constant confining pressure equals the mean cyclic confining pressure; however, Poisson's ratio is quite different.[5] Figure 3 shows the variation of Poisson's ratio with the ratio of deviatoric stress to mean confining pressure. It is seen that, in both types of test, Poisson's ratio is not constant. However, in the test with cyclic confining pressure Poisson's ratio does not vary as much as in the test with static confining pressure, and for $\sigma_1/\sigma_3 \geqslant 2$ (which corresponds to a vertical strain of greater than about 0·01 per cent), the variation is 0·18 $\pm$0·08.

Taking Poisson's ratio as 0·18 and using the authors' proposed model, a coefficient of determination (not to be confused with coefficient of correlation), r^2, of 0·9896 is obtained (Figure 4), which is statistically significant. However, for the same stress range equation (3) gives a coefficient of determination of 0·9861 (Figure 5), which is not much different from that of the proposed model. Moreover, it is seen that r^2, M and λ are very sensitive to the value of Poisson's ratio, and the maximum r^2 according to the model is 0·9916 for a Poisson's ratio of zero (Figure 6). During plotting, it was also noticed that by inputting data of different σ_1/σ_3, different variations of r^2, M and λ could be obtained, indicating the sensitivity of the parameters to stress variation. Thus, regrettably, from the above comparison it seems that we still have to find a model which adequately describes the unloading–reloading of soils and is also conservative.

REFERENCES

1. R. G. Hicks and C. L. Monismith, 'Factors influencing the resilient response of granular materials', *Highway Res. Record*, **345**, 15–31 (1971).
2. J. J. Allen and M. R. Thompson, 'Resilient response of granular materials subjected to time-dependent lateral stresses', *Transport. Res. Record*, **510**, 1–13 (1974).
3. J. Uzan, 'Characterization of granular material', *Transport. Res. Record*, **1022**, 52–59 (1985).
4. S. F. Brown and M. S. Snaith, 'The measurement of recoverable and irrecoverable deformations in the repeated load triaxial test', *Geotechnique*, **24**, 255–259 (1974).
5. S. F. Brown and A. F. L. Hyde, 'Significance of cyclic confining stress in repeated-load triaxial testing of granular material', *Transport. Res. Record*, **537**, 49–58 (1975).

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AUTHOR'S REPLY

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(*P. V. Lade and R. B. Nelson, Int. j. numer. anal. methods geomech.* **11**, 521–542 (1987))

Mr. Nataatmadja comments on the proposed model for the elastic behaviour of granular materials relative to discrepancies between model predictions and behaviour observed in certain types of test. It is pointed out that results of repeated loading tests on granular materials indicate varying Poisson's ratio, and that the resilient modulus found from such tests may be characterized equally well by a simpler model involving only the mean normal stress.

Results of repeated loading tests often indicate values of Poisson's ratio greater than 0.5 (see e.g. References 1–3), which are incompatible with an isotropic formulation of the elastic response. It is, in fact, questionable whether any portion of the soil response can be characterized as being elastic: resonant column tests on soils show hysteresis loops (resulting in damping) even at very low strain magnitudes (10^{-6}), thus indicating dissipation of energy and casting doubt on the elastic characterization of even the smallest strains observed in such tests. Furthermore, the modulus values obtained from resonant column tests are higher than those obtained from tests with larger strains such as conventional static tests (see e.g. Reference 4).

Nevertheless, most constitutive models for the static behaviour of soils are based on elasticity and plasticity theories. Even though it may be small, the elastic component is required in order to invert the material matrix. Thus a mathematical expression for the elastic behaviour is required.

The formulation presented in the paper is based on experimental observations from static tests which indicate that the elastic response is isotropic and that Poisson's ratio is practically constant for a given soil (see references therein). Furthermore, characterization of true elastic behaviour should be based on the existence of an elastic potential, thus guaranteeing neither creation nor dissipation of energy in a closed stress or strain loop. Although a specific elastic potential function is not given, the existence of such a function is assumed. The resulting expression for the non-linear elastic modulus produces an elasticity theory that appears to capture the observed physical behaviour of a number of soils subjected to various static loadings (see also Reference 5).

Although the proposed hypoelastic model does not generate or dissipate energy in a closed stress loop, it does produce residual strains. This may be seen from Figure 3 of the paper, in which contours of constant Young's modulus are shown. A closed stress loop will cause the elastic modulus to vary in such a way that residual shear strains and volumetric strains remain upon return to the initial stress point. On the other hand, if the model is exercised in a closed strain loop then the final state of stress will not be the same as the initial state. Since no energy is spent in a closed strain loop, it is possible to change the state of stress without expending any energy. This is an effect of the hypoelastic formulation of the elastic behaviour. If the stress loop collapses to a line (i.e. the area within the stress loop reduces to zero), then no residual strains are obtained at the end of the loop. In view of the hypoelastic formulation, it is clear that the material does not have memory and that closed stress loops will produce residual strains at the conclusion of each cycle. This model is therefore not suitable for cyclic loading conditions in which the loading cycle encloses an area in stress space. However, all known models of the non-linear elastic behaviour of soils exhibit residual strains or residual stresses when exercised in closed stress or strain loops, respectively.

It can be shown that the assumption of constant Poisson's ratio combined with a requirement of no residual strain leads to a constant Young's modulus (i.e. linear elasticity) when derived from a potential function. If, on the other hand, the elastic parameters are allowed to vary and cross-terms (coupling effects) between normal stresses and shear strains and between shear stresses and normal strains are admitted, then complicated differential equations are obtained. For the general case, the elastic potential function can be derived only in terms of infinite series. Because material parameters become very difficult to determine, and because the use of such models would be very complicated, such expressions are not useful for practical engineering purposes.

In view of the fact that the material is fundamentally an inelastic medium and that an elastic component is required for purposes of analysis, it seems prudent to select an elasticity model which captures as much of the observed behaviour as possible. None of the available models are ideal in all respects, but the proposed model appears to come closest to the desirable characteristics of a non-linear elastic model.

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REFERENCES

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2. J. J. Allen and M. R. Thompson, 'Resilient response of granular materials subjected to time-dependent lateral stresses', *Transport. Res. Record*, **510,** 1–13 (1974).
3. S. E. Brown and A. F. L. Hyde, 'Significance of cyclic confining stress in repeated-load triaxial testing of granular material', *Transport. Res. Record*, **537,** 49–58 (1975).
4. H. B. Seed and I. M. Idriss, 'Soil moduli and damping factors for dynamic analyses', Report No. EERC 70-10, University of California, Berkeley, 1970.
5. P. V. Lade, 'Model and parameters for the elastic behavior of soils', in Proc. 6th Int. Conf. on Numerical Methods in Geomechanics, Innsbruck, Vol. 1, 1988, pp. 359–364.