LIMIT ANALYSIS OF ANISOTROPIC SOILS USING FINITE ELEMENTS AND LINEAR PROGRAMMING

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ABSTRACT: The main purpose of this paper is to present a finite element formulation of the bound theorems which allows for the variation of soil strength with direction. To achieve this objective, the conventional isotropic Mohr–Coulomb yield criterion is generalised to include the effect of strength anisotropy. The finite element limit analysis formulation using the modified anisotropic yield criterion is then developed. Several examples are given in the paper to illustrate the capability and effectiveness of the proposed numerical procedure for computing rigorous bounds for anisotropic soils.

INTRODUCTION

It is generally accepted that the strength parameters of natural soils are anisotropic. Measurements of the fabric of clays have shown that the particles tend to become oriented in the horizontal direction during one dimensional deposition and subsequent loading. This preferred particle orientation causes an inherent anisotropy which can lead to changes in soil properties as the direction of the major principal stress varies during shear. It is well known that measurement of inherent anisotropy is difficult but its effects have been inferred from results of laboratory tests performed on specimens cut at different orientations. Some studies of inherent anisotropy have also been made on sand using prepared samples in the laboratory. In particular, anisotropic behaviour has been measured in consolidated–drained shear tests and it is noted that a slightly higher friction angle occurs when the major principal stress coincides with the direction of sand deposition than when the major principal stress acts perpendicular to the direction of formation.

Existing data seems to suggest that the variation of cohesion (undrained shear strength for undrained loadings) with direction due to inherent anisotropy is much more significant than the effects of anisotropy on friction angles (with typical order of 2–3 degrees, see for example, Arthur and Menzies, 1972; Arthur and Phillips, 1975). Partly because of this and more significantly due to the fact that more extensive studies have been carried out in the past on the influence of anisotropy in clays than in sands, this paper will mainly concentrate on the effects of anisotropic cohesion in stability analysis for cohesive–frictional materials.

Although the variation of undrained shear strength with direction has been taken into account in stability analysis by a number of researchers (see, for example, Lo, 1965; Davis and Christian, 1971; Reddy and Srinivasan, 1970; Chen, 1975), very little work has been done on the effects of variation of cohesion with direction in cohesive–frictional materials.

It is recognized that the lower bound theorem has been applied less frequently than the upper bound theorem as it is relatively easier to construct a good kinematically admissible failure mechanism than it is to construct a good statically admissible stress field. An upper bound solution is often a good estimate of the collapse load, but a lower bound solution is more valuable in engineering practice as it results in a safe design. Furthermore, a lower bound solution combined with an upper bound solution serves to bracket the actual collapse load. Due to the obvious difficulties in constructing a good statically admissible stress field manually, a numerical method for lower bound calculations is therefore particularly desirable.
This paper aims to develop a general numerical method which can be used to calculate rigorous bound solutions for soils whose cohesion varies with direction. To achieve this objective, the conventional isotropic Mohr–Coulomb yield criterion has been generalised to include the effect of anisotropy which is caused by the variation of cohesion with direction. The numerical formulation of the lower and upper bound theorems using the modified anisotropic yield criterion is then developed. It is found that by using a suitable linear approximation of the yield surface, the application of the bound theorems leads to a linear programming problem. As will be shown in this paper, a major advantage of using a numerical formulation of the bound theorems is that both complex loading geometry and soil behaviour can be easily dealt with.

**PROPOSED ANISOTROPIC STRENGTH THEORY**

Based on the earlier studies of Lo (1965), the cohesion is assumed to vary with direction according to:

\[
c_{\theta} = c_h + (c_v - c_h) \sin^2 \theta
\]

where \(c_h\), \(c_v\) denote the values of cohesion on the horizontal and vertical planes respectively and \(\theta\) represents the angle between the direction of a plane where the cohesion \(c_\theta\) is measured and the horizontal direction.

![Figure 1: Resolution of stresses into normal and shear components acting on a plane](image)

Because of the anisotropic nature of cohesion strength, the conventional isotropic Mohr–Coulomb failure criterion is no longer valid. The shear strength that can be developed on a plane such as \(ab\) shown in Figure 1 is:

\[
s = c_{\theta} - \sigma_n \tan \phi = c_h + (c_v - c_h) \sin^2 \theta - \sigma_n \tan \phi
\]

where the normal and shear stress components on the plane \(ab\) are:

\[
\sigma_n = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - \tau_{xy} \sin 2\theta
\]

\[
\tau = -\sigma_x \sin \theta \cos \theta + \sigma_y \sin \theta \cos \theta + \tau_{xy} \cos 2\theta
\]

There will be a critical plane on which the available shear strength \(s\) will be first reached as the shear stress \(\tau\) increases. The orientation of this critical plane can be obtained by solving for \(\frac{\partial (\tau - s)}{\partial \theta} = 0\). Solving for the critical orientation angle and substituting (2) and (4) in \(\tau - s = 0\) gives the following anisotropic strength criterion:

\[
\text{Critical orientation angle} = \text{Solving for the critical orientation angle...}
\]
\[ F = X^2 + Y^2 - R^2 = 0 \]  

where

\[
X = \sigma_y - \sigma_x - 2\tau_{xy}\tan\phi
\]

\[
Y = c_v - c_h + 2\tau_{xy} - \sigma_x\tan\phi + \sigma_y\tan\phi
\]

\[
R = c_v + c_h - \sigma_x\tan\phi - \sigma_y\tan\phi
\]

**LOWER BOUND LIMIT ANALYSIS**

In this section, a very brief outline of the finite element formulation of the lower bound theorem is presented. The basic idea of the technique is to use a linear variation for the stress field and an internal linear approximation of the generalized Mohr–Coulomb yield criterion so that the application of the lower bound theorem will lead to a linear programming problem.

![Elements for lower bound analysis](image)

The lower bound formulation under conditions of plane strain uses the three types of elements shown in Figure 2. The stress field for each of these is assumed to vary linearly. The variation of the stress throughout each element is linear and each node is associated with 3 unknown stresses, \(\sigma_x, \sigma_y, \tau_{xy}\). The extension elements are used to extend the solution over a semi–infinite domain and therefore provide a complete statically admissible stress field. In contrast to the usual displacement type of finite element analysis, several nodes may have identical co–ordinates and each node is unique to a particular element. Statically admissible stress discontinuities are permitted at edges shared by adjacent triangles and also along borders between adjacent extension elements.

The equality constraints imposed on the stresses by equilibrium and stress boundary conditions for anisotropic soils will not be discussed here as they are identical to those presented by Sloan (1988) for an isotropic soil. As discussed in the previous section, the inclusion of variation of cohesion strength with direction in a soil will, however, modify the nature of the yield criterion of the soil. To ensure that the yield condition (5) is satisfied, it is necessary to impose \(F \leq 0\). It is readily known that \(F \leq 0\) results in a number of non–linear constraints on stresses. Since we
wish to formulate the lower bound theorem as a linear programming problem, it is necessary to approximate (5) by a yield criterion which is a linear function of the unknown stress variables. The linearized yield surface must lie inside the generalized Mohr–Coulomb yield surface in stress space so that the solution obtained is a rigorous lower bound.

If \( p \) is the number of sides used to approximate the yield function (5), the linearized yield function can be shown to be:

\[
A_k \sigma_x + B_k \sigma_y + C_k \tau_{xy} \leq D_k, \quad k = 1,2,...,p
\]

where

\[
A_k = -\cos\left(\frac{2\pi k}{p}\right) - \tan\phi \sin\left(\frac{2\pi k}{p}\right) + \tan\phi \cos\left(\frac{\pi}{p}\right)
\]

\[
B_k = \cos\left(\frac{2\pi k}{p}\right) + \tan\phi \sin\left(\frac{2\pi k}{p}\right) + \tan\phi \cos\left(\frac{\pi}{p}\right)
\]

\[
C_k = -2\tan\phi \cos\left(\frac{2\pi k}{p}\right) + 2\sin\left(\frac{2\pi k}{p}\right)
\]

\[
D_k = (c_v + c_h) \cos\left(\frac{\pi}{p}\right) + (c_h - c_v) \sin\left(\frac{2\pi k}{p}\right)
\]

With the linearized yield function (6), it can be easily proved that it is sufficient to enforce these linear constraints at each nodal point in order that they are satisfied throughout the mesh.

Once the various constraints and objective terms have been assembled, the problem of finding a statically admissible stress field which maximizes the collapse load \( C^T X \) may be expressed as:

\[
\text{Minimise} \quad - C^T X
\]

\[
\text{Subject to} \quad A_1 X = B_1
\]

\[
A_2 X \leq B_2
\]

where \( X \) denotes the global vector of unknown nodal stress variables; \( A_1, B_1 \) represent the coefficients for linear constraints due to equilibrium and stress boundary conditions; and \( A_2, B_2 \) are due to yield conditions. The details of the numerical procedure for solving the above linear programming problem may be found in Sloan (1988).

**UPPER BOUND LIMIT ANALYSIS**

The 3-noded triangular element used in the finite element formulation of the upper bound theorem is shown in Figure 3. Each element has 6 nodal velocities and \( p \) plastic multiplier rates (where \( p \) is the number of sides in the linearized yield polygon of (5)). In addition, each discontinuity is typically defined by four nodes and requires four unknowns to describe the tangential velocity jumps along its length. To ensure that the computed velocity field is kinematically admissible, the unknowns are subject to constraints which are generated by the plastic flow rule and boundary conditions. Unlike an upper calculation which is based on a rigid block mechanism, plastic deformation is permitted to occur through the soil mass. To remove the stress terms from the flow rule equations, and thus provide a linear relationship between the unknown velocities and plastic multiplier rates, it is again necessary to linearize the generalized Mohr–Coulomb yield criterion (5). To preserve the bounding property of the solution, the upper bound formulation uses an external linear approximation of the Mohr–Coulomb yield surface.
The externally linearized yield function takes the following form:

\[ F_k = A_k \sigma_x + B_k \sigma_y + C_k \tau_{xy} - D_k = 0; \quad k = 1, 2, \ldots, p \]  

where

\[ A_k = - \cos(\frac{2\pi k}{p}) - \tan \phi \sin(\frac{2\pi k}{p}) + \tan \phi \]

\[ B_k = \cos(\frac{2\pi k}{p}) + \tan \phi \sin(\frac{2\pi k}{p}) + \tan \phi \]

\[ C_k = - 2 \tan \phi \cos(\frac{2\pi k}{p}) + 2 \sin(\frac{2\pi k}{p}) \]

\[ D_k = [1 - \sin(\frac{2\pi k}{p})] c_v + [1 + \sin(\frac{2\pi k}{p})] c_h \]

To define the objective function, the dissipated power is expressed in terms of the unknown velocities and plastic multiplier rates. As the soil deforms, power dissipation may occur in the velocity discontinuities as well as in the triangles. Once the solution to the upper bound linear programming problem has been found, a rigorous upper bound on the exact collapse load is found in the usual way by equating the rate of work of the external forces to the rate of dissipation of internal power.

After assembling the necessary constraints and objective function coefficients, the task of finding a kinematically admissible velocity field, which minimizes the internal power dissipation for the prescribed boundary conditions, may be written as

\[ \text{Minimise} \quad C_1^T X_1 + C_2^T X_2 \]

\[ \text{Subject to} \quad A_{11} X_1 + A_{12} X_2 = B_1 \]

\[ A_2 X_1 \leq B_2 \]

\[ A_3 X_1 = B_3 \]

\[ A_4 X_1 = B_4 \]

\[ X_2 \geq 0 \]

where \( X_1 \) is a global vector of nodal velocities and \( X_2 \) is a global vector of element plastic multiplier rates; \( A_{11}, A_{12}, B_1 \) denote the coefficients for linear constraints due to plastic flow rule within each triangular element; \( A_2, A_3, B_2, B_3 \) represent the coefficients for linear constraints imposed
by velocity flow rule; and $A_4$, $B_4$ are the constraint coefficients caused by velocity boundary conditions. The details of the numerical procedure for solving the above linear programming problem may be found in Sloan (1989).

**NUMERICAL EXAMPLES**

**Stability of a smooth trapdoor in anisotropic undrained clays**

The first problem that is to be analyzed is the stability of a plane strain trapdoor in undrained clay. The geometry of the problem is shown in Figure 4. Both the upper and lower bound solutions to this problem have been obtained by Sloan et. al. (1990) for the case when the undrained shear strength does not vary with direction. The stability factor for the trapdoor can be shown to be $N = (\gamma H + \sigma_s - \sigma_t)/c_h$, where $\gamma$ is the unit weight of the soil.

The upper bound mesh used to model a trapdoor with $H/B = 5$ and a perfectly smooth contact is shown in Figure 5. The grid has one vertical discontinuity. Due to symmetry, only one half of the problem needs to be considered and the velocity boundary conditions are as shown. Overall, there are 494 nodes, including the dual nodes along the discontinuity and 896 triangles. To compute the stability number for a fixed $H/B$ value, the surcharge and soil unit weight are set to zero and a unit downward velocity is imposed on the trapdoor. The resulting linear programming problem is then solved to furnish the kinematically admissible velocity field which dissipates the least amount of power internally.

![Figure 4: The trapdoor problem](image-url)
Figure 5: Upper bound mesh used for analysis of the trapdoor problem

Figure 6: Lower bound mesh used for analysis of the trapdoor problem
Numerical upper bounds
Numerical lower bounds

Figure 7: Stability factor against the value of $c_v/c_h$

$\sigma_n = \tau = 0$

Figure 8: Lower bound mesh for footing analysis

The lower bound mesh, also for the case of $H/B = 5$ and a perfectly smooth interface, is shown in Figure 6. This grid has 616 nodes, 196 triangular elements, 7 rectangular extension elements and 293 stress discontinuities, and is subject to the stress boundary conditions shown. The rectangular extension elements enable the stress field to be extended indefinitely in the horizontal direction without violating equilibrium, the stress boundary conditions or yield function, and the solution is thus said to be rigorous. In the actual calculations, the surcharge and unit weight are set to zero and the tensile stress over the surface of the trapdoor is maximized to give the stability number directly.
A summary of the stability bounds obtained for $H/B=5$ and various ratios of $c_v/c_h$, using an 18-sided approximation of the yield criterion, is shown in Figure 7. As expected, the stability number increases with increasing $c_v/c_h$. In general, the upper bounds agree well with the lower bound solutions and the difference is typically less than 15%.
Bearing capacity of a smooth strip footing on anisotropic cohesive-frictional soils

The second problem to be analyzed is the bearing capacity of a strip footing on cohesive-frictional soils. The finite element meshes used for the lower and upper bound calculations are plotted in Figures 8 and 9 respectively. For the case when self-weight of the soil is ignored, the bearing capacity is \( q = N_c c_h \), where \( N_c \) depends on the values of the soil friction angle and \( c_v/c_h \).

Figure 10 presents the upper and lower bounds obtained with an 18-sided yield surface linearization for the case where the soil friction angle is 20 degrees. The results are plotted as the bearing capacity factor against the anisotropic cohesion ratio \( c_v/c_h \). The upper bound solutions for the bearing capacity agree well with the lower bounds, and typically differ from their mean by about ten to fifteen percent. It should also be noted that the difference between the numerical upper and lower bounds will be reduced if the number of sides used to approximate the yield function is increased.

CONCLUSIONS

This paper presents a finite element formulation of the lower and upper bound theorems for a soil whose cohesion varies with direction. To develop the numerical formulations for anisotropic soil, the conventional isotropic Mohr–Coulomb yield criterion is generalized to include the effect of anisotropy which is caused by a variation of cohesion with direction. Numerical examples presented in the paper illustrate that the proposed numerical procedure can be used to compute rigorous bound solutions of anisotropic soils.

REFERENCES