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ANALYSIS AND CALCULATION OF THE NONLINEAR STABILITY OF THE ROTATIONAL COMPOSITE SHELL *

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Abstract: *By adopting the energy method, a new method to calculate the stability of the composite shell of revolution is presented. This method takes the influence of nonlinear prebuckling deformations and stresses on the buckling of the shell into account. The relationships between the prebuckling deformations and strains are calculated by nonlinear Kármán equations. The numerical method is used to calculate the energy of the total system. The nonlinear equations are solved by combining gradient method and amendatory Newton iterative method. The computer program is also developed. An example is given to demonstrate the accuracy of the method presented.*

Key words: composite material; rotational shell; stability; nonlinear
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Introduction

By adopting the energy method, a new method to calculate the stability of the composite shell of revolution is presented. This method takes the influence of nonlinear prebuckling deformations and stresses on the buckling of the shell into account. The relationships between the prebuckling deformations and strains are calculated by nonlinear Kármán equations. The numerical method is used to calculate the energy of the total system. Displacement and internal force before buckling are calculated by using the minimum potential energy theory. First, linear buckling load P^* (External pressure), F^* (Axial compression), T^* (Combination of axial compression and external pressure) are calculated. Secondly, displacement and internal force under loads between $[0, 2P^*]$, $[0, 2F^*]$, $[0, 2T^*]$ are calculated by an increment of N, and then put them into the buckling equations, a series of buckling loads with combination of half wave mumber (m, n) are obtained. Then decrease N to get the buckling loads with higher precision. The minimum of these loads is the buckling load.

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Biography: Huang Jinsong (1967 ~), Associate Professor, Doctor

1 Stiffnesses of Composite Laminated[1]

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ & & A_{66} & B_{12} & B_{16} & B_{66} \\ & & & D_{11} & D_{12} & D_{16} \\ & & & & D_{22} & D_{26} \\ & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \tag{1}$$

It also is

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{Bmatrix},$$

where $[A]$ is extensional stiffnesses, $[D]$ is bending stiffnesses, $[B]$ is coincidence stiffnesses between extension and bending.

$[A]$, $[B]$, $[D]$ are given as

$$[A_{ij}] = \sum_{k=1}^{n} (\bar{Q}_{ij})_k [h_k - h_{k-1}],$$

$$[B_{ij}] = \frac{1}{2} \sum_{k=1}^{n} (\bar{Q}_{ij})_k [h_k^2 - h_{k-1}^2],$$

$$[D_{ij}] = \frac{1}{3} \sum_{k=1}^{n} (\bar{Q}_{ij})_k [h_k^3 - h_{k-1}^3],$$

where k is number of the lamina, h_k and h_{k-1} are radial coordinates of lamina k, $(\bar{Q}_{ij})_k$ is corresponding transformed stiffnesses of lamina k.

2 Strain Displacement Relation[2]

$$\left.\begin{aligned}
\varepsilon_{10} &= \frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} v - \frac{w}{R_1} + \frac{1}{2}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1}\right)^2, \\
\varepsilon_{20} &= \frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} u - \frac{w}{R_2} + \frac{1}{2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2}\right)^2, \\
\gamma_{120} &= \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{v}{A_2}\right) + \frac{1}{A_1 A_2}\frac{\partial w}{\partial \alpha_1}\frac{\partial w}{\partial \alpha_2}, \\
\chi_1 &= \frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1}\right) + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2}\right), \\
\chi_2 &= \frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2}\right) + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1}\right), \\
\chi_{12} &= \frac{2}{A_1 A_2}\left(\frac{\partial^2 w}{\partial \alpha_1 \partial \alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial \alpha_2}\frac{\partial w}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial \alpha_1}\frac{\partial w}{\partial \alpha_2}\right).
\end{aligned}\right\} \tag{2}$$

3 Energy Variation Method

Potential energy of the system[3] :

$$\Pi = \frac{1}{2}\iint (N_1\varepsilon_{10} + N_2\varepsilon_{20} + N_{12}\gamma_{120} + M_1\chi_1 + M_2\chi_2 + M_{12}\chi_{12})A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 -$$

$$\left\{\iint qwA_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 + \int\left[N_1^0 u + N_{12}^0 v - M_1^0\frac{\partial w}{\partial x} + V_1^0 w\right]_0^{2\pi} A_2\mathrm{d}\alpha_2 + \right.$$

$$\left.\int\left[N_2^0 v + N_{12}^0 u - M_2^0\frac{\partial w}{\partial y} + V_2^0 w\right]_{\varphi_1}^{\varphi_2} A_1\mathrm{d}\alpha_1 + S^0 w\,|_s\right\}, \tag{3}$$

where M_1^0, M_2^0, V_1^0, V_2^0, S^0, $w\,|_s$ respectively are the moment, shear force, wrestle and angle of torsion given at the edge.

First-order derivative of the strain takes the form

$$\left.\begin{aligned}
\delta\varepsilon_{10} &= \frac{1}{A_1}\frac{\partial(\delta u)}{\partial\alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\delta v - \frac{\delta w}{R_1} + \frac{1}{A_1^2}\frac{\partial w}{\partial\alpha_1}\cdot\frac{\partial(\delta w)}{\partial\alpha_1},\\
\delta\varepsilon_{20} &= \frac{1}{A_2}\frac{\partial(\delta v)}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\delta u - \frac{\delta w}{R_2} + \frac{1}{A_2^2}\frac{\partial w}{\partial\alpha_2}\frac{\partial(\delta w)}{\partial\alpha_2},\\
\delta\gamma_{120} &= \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\delta u}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\delta v}{A_2}\right) + \frac{1}{A_1A_2}\frac{\partial w}{\partial\alpha_1}\frac{\partial(\delta w)}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial w}{\partial\alpha_2}\frac{\partial(\delta w)}{\partial\alpha_1},\\
\delta\chi_1 &= \frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial(\delta w)}{\partial\alpha_1}\right) + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial(\delta w)}{\partial\alpha_2}\right),\\
\delta\chi_2 &= \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial(\delta w)}{\partial\alpha_2}\right) + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial(\delta w)}{\partial\alpha_1}\right),\\
\delta\chi_{12} &= \frac{2}{A_1A_2}\left(\frac{\partial^2(\delta w)}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial(\delta w)}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial(\delta w)}{\partial\alpha_2}\right).
\end{aligned}\right\} \tag{4}$$

Second-order derivative of the strain takes the form

$$\left.\begin{aligned}
\delta^2\varepsilon_{10} &= \frac{1}{A_1^2}\left(\frac{\partial(\delta w)}{\partial\alpha_1}\right)^2,\\
\delta^2\varepsilon_{20} &= \frac{1}{A_2^2}\left(\frac{\partial(\delta w)}{\partial\alpha_2}\right)^2,\\
\delta^2\varepsilon_{120} &= \frac{2}{A_1A_2}\frac{\partial(\delta w)}{\partial\alpha_1}\frac{\partial(\delta w)}{\partial\alpha_2}.
\end{aligned}\right\} \tag{5}$$

Apply the linear stress-strain relation yields the following equations

$$\left.\begin{aligned} \delta N_1 \varepsilon_{10} + \delta N_2 \varepsilon_{20} + \delta N_{12} \gamma_{120} &= N_1 \delta \varepsilon_{10} + N_2 \delta \varepsilon_{20} + N_{12} \delta \gamma_{120}, \\ \delta M_1 \chi_1 + \delta M_2 \chi_2 + \delta M_{12} \chi_{12} &= M_1 \delta \chi_1 + M_2 \delta \chi_2 + M_{12} \delta \chi_{12}. \end{aligned}\right\} \tag{6}$$

Substitution of Eqs. (6) into the first-order derivative of Eq. (3) yields the following equation

$$\delta \Pi = \iint (N_1 \delta \varepsilon_{10} + N_2 \delta \varepsilon_{20} + N_{12} \delta \gamma_{120} + M_1 \delta \chi_1 + M_2 \delta \chi_2 + M_{12} \delta \chi_{12}) A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 -$$

$$\left\{ \iint q \delta w A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 + \int \left[N_1^0 \delta u + N_{12}^0 \delta v - M_1^0 \frac{\partial(\delta w)}{\partial x} + V_1^0 \delta w \right]_0^{2\pi} A_2 \mathrm{d}\alpha_2 + \right.$$

$$\left. \int \left[N_2^0 \delta v + N_{12}^0 \delta u - M_2^0 \frac{\partial(\delta w)}{\partial y} + V_2^0 \delta w \right]_{\varphi_1}^{\varphi_2} A_1 \mathrm{d}\alpha_1 + S^0 \delta w \mid_s \right\}. \tag{7}$$

Substitution of Eqs. (4) into Eqs. (7) and season $\delta \Pi = 0$ yields the equilibrium equations and boundary conditions before buckling.

Second-order derivative of the Potential energy of the system is given by

$$\delta^2 \Pi = \iint \{ [N_1 \delta^2 \varepsilon_{10} + N_2 \delta^2 \varepsilon_{20} + N_{12} \delta^2 \gamma_{120}] + [\delta N_1 \delta \varepsilon_{10} + \delta N_2 \delta \varepsilon_{20} + \delta N_{12} \delta \gamma_{120} +$$

$$\delta M_1 \delta \chi_1 + \delta M_2 \delta \chi_2 + \delta M_{12} \delta \chi_{12}] \} A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 . \tag{8}$$

At critical point, Eq. (8) takes the form

$$\delta^* (\delta^2 \Pi) = 0.$$

Because at critical point $\delta^2 \Pi$ takes the minimum value, to all possible virtual displacement $\delta^* (\delta u)$, $\delta^* (\delta v)$, $\delta^* (\delta w)$ the buckling equations are got as

4 Prebuckling analysis

The displacement function before buckling is given as

$$\begin{cases} u = A_0 x, \\ w = B_0 x^2 + C_0 \sin^2 \pi x e^{\beta x}, \end{cases} \tag{9}$$

where $x = \dfrac{\varphi - \varphi_1}{\varphi_2 - \varphi_1}$, $\beta = -\sqrt[4]{\dfrac{3(1-\mu^2) R_2^4}{R_1^2 h^2}}$, A_0, B_0, C_0 are constants.

Boundary conditions given by

when $\varphi = \varphi_1$, $x = 1$, $u \neq 0$, $w \neq 0$; $\varphi = \varphi_2$, $x = 0$, $u = 0$, $w = 0$, $\partial w / \partial \varphi = 0$.

Substitution of Eqs. (9) into Eqs. (4) yields the following equations:

$$\left.\begin{aligned}
\varepsilon_{10} &= \frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}v - \frac{w}{R_1} + \frac{1}{2}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1}\right)^2 = \\
&- \frac{1}{R_1(\varphi_2-\varphi_1)}\cdot A_0 - \frac{x^2}{R_1}\cdot B_0 - \frac{1}{R_1}\sin^2\pi x e^{\beta x}\cdot C_0 + \\
&\frac{2x^2}{R_1^2(\varphi_2-\varphi_1)^2}\cdot B_0^2 + \frac{2x}{R_1^2(\varphi_2-\varphi_1)^2}(2\pi\sin\pi x\cos\pi x e^{\beta x} + \beta e^{\beta x}\sin^2\pi x)\cdot B_0C_0 + \\
&\frac{1}{2R_1^2(\varphi_2-\varphi_1)^2}(2\pi\sin\pi x\cos\pi x e^{\beta x} + \beta e^{\beta x}\sin^2\pi x)^2\cdot C_0^2, \\
\varepsilon_{20} &= \frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}u - \frac{w}{R_2} + \frac{1}{2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2}\right)^2 = \\
&\frac{\cot\varphi}{R_1}x\cdot A_0 - \frac{x^2}{R_2}\cdot B_0 - \frac{\sin^2\pi x e^{\beta x}}{R_2}\cdot C_0, \\
\gamma_{120} &= \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{v}{A_2}\right) + \frac{1}{A_1A_2}\frac{\partial w}{\partial \alpha_1}\frac{\partial w}{\partial \alpha_2} = 0, \\
\chi_1 &= \frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1}\right) + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2}\right) = \\
&\frac{2}{R_1^2(\varphi_2-\varphi_1)^2}\cdot B_0 + \frac{e^{\beta x}}{R_1^2(\varphi_2-\varphi_1)^2}[2\pi^2(\cos^2\pi x - \sin^2\pi x) + \\
&4\pi\beta\sin\pi x\cos\pi x\beta + \beta^2\sin^2\pi x]\cdot C_0, \\
\chi_2 &= \frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2}\right) + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1}\right) = \\
&- \frac{2x\cot\varphi}{R_1^2(\varphi_2-\varphi_1)}\cdot B_0 - \frac{\cot\varphi}{R_1^2(\varphi_2-\varphi_1)}(2\pi\sin\pi x\cos\pi x e^{\beta x} + \beta\sin^2\pi x e^{\beta x})\cdot C_0, \\
\chi_{12} &= \frac{2}{A_1A_2}\left(\frac{\partial^2 w}{\partial \alpha_1\partial \alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial \alpha_2}\frac{\partial w}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial \alpha_1}\frac{\partial w}{\partial \alpha_2}\right) = 0.
\end{aligned}\right\} \tag{10}$$

Substitution of Eqs. (10) into Eqs. (7) and yields the following equations:

$$\frac{\partial \Pi}{\partial A_0} = 0, \quad \frac{\partial \Pi}{\partial B_0} = 0, \quad \frac{\partial \Pi}{\partial C_0} = 0. \tag{11}$$

Solve Eqs. (11), displacement and internal force before buckling can be got.

5 Buckling Analysis

The buckling displacement function is given as

$$\begin{cases}
\delta u = A\cos m\pi x\sin n\theta, \\
\delta v = B\sin m\pi x\cos n\theta, \\
\delta w = C\sin m\pi x\sin n\theta,
\end{cases} \tag{12}$$

where A, B, C are constants.

Boundary conditions given by

when $\varphi = \varphi_1$, $x = 1$, $v = 0$; when $\varphi = \varphi_2$, $x = 0$, $v = 0$.

Substitution of Eqs.(12) into Eqs.(4) yields the following equations:

$$\delta\varepsilon_{10} = \frac{1}{A_1}\frac{\partial(\delta u)}{\partial\alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\delta v - \frac{\delta w}{R_1} + \frac{1}{A_1^2}\frac{\partial w}{\partial\alpha_1}\cdot\frac{\partial(\delta w)}{\partial\alpha_1} =$$

$$\frac{m\pi}{R_1(\varphi_2-\varphi_1)}\sin m\pi x\sin n\theta\cdot A - \frac{1}{R_1}\sin m\pi x\sin n\theta\cdot C +$$

$$\frac{2m\pi x}{R_1^2(\varphi_2-\varphi_1)^2}\cos m\pi x\sin n\theta\cdot B_0 C +$$

$$\frac{m\pi}{R_1^2(\varphi_2-\varphi_1)^2}(2\pi\sin\pi x\cos\pi x e^{\beta x} + \beta\sin^2\pi x e^{\beta x})\cos m\pi x\sin n\theta\cdot C_0 C, \tag{13a}$$

$$\delta\varepsilon_{20} = \frac{1}{A_2}\frac{\partial(\delta v)}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\delta u - \frac{\delta w}{R_2} + \frac{1}{A_2^2}\frac{\partial w}{\partial\alpha_2}\frac{\partial(\delta w)}{\partial\alpha_2} =$$

$$\frac{\cot\varphi}{R_1}\cos m\pi x\sin n\theta\cdot A - \frac{n}{R_2\sin\varphi}\sin m\pi x\sin n\theta\cdot B -$$

$$\frac{1}{R_2}\sin m\pi x\sin n\theta\cdot C, \tag{13b}$$

$$\delta\gamma_{120} = \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\delta u}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\delta v}{A_2}\right) + \frac{1}{A_1A_2}\frac{\partial w}{\partial\alpha_1}\frac{\partial(\delta w)}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial w}{\partial\alpha_2}\frac{\partial(\delta w)}{\partial\alpha_1} =$$

$$\frac{n}{R_2\sin\varphi}\cos m\pi x\cos n\theta\cdot A + \left(\frac{\cot\varphi}{R_1}\sin m\pi x\cos n\theta + \frac{m\pi}{R_1(\varphi_2-\varphi_1)}\cos m\pi x\cos n\theta\right)\cdot B -$$

$$\frac{2nx}{R_1R_2(\varphi_2-\varphi_1)\sin\varphi}\sin m\pi x\cos n\theta\cdot B_0 C -$$

$$\frac{2\pi\sin\pi x\cos\pi x e^{\beta x} + \beta\sin^2\pi x e^{\beta x}}{R_1R_2(\varphi_2-\varphi_1)\sin\varphi}\sin m\pi x\cos n\theta\cdot C_0 C, \tag{13c}$$

$$\delta\chi_1 = \frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial(\delta w)}{\partial\alpha_1}\right) + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial(\delta w)}{\partial\alpha_2}\right) =$$

$$-\frac{m^2\pi^2}{R_1^2(\varphi_2-\varphi_1)^2}\sin m\pi x\sin n\theta\cdot C, \tag{13d}$$

$$\delta\chi_2 = \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial(\delta w)}{\partial\alpha_2}\right) + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial(\delta w)}{\partial\alpha_1}\right) =$$

$$-\left[\frac{m\pi\cot\varphi}{R_1^2(\varphi_2-\varphi_1)}\cos m\pi x\sin n\theta + \frac{n^2}{R_2^2\sin^2\varphi}\sin m\pi x\sin n\theta\right]\cdot C, \tag{13e}$$

$$\delta\chi_{12} = \frac{2}{A_1A_2}\left(\frac{\partial^2(\delta w)}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial(\delta w)}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial(\delta w)}{\partial\alpha_2}\right) =$$

$$-\frac{2}{R_1R_2\sin\varphi}\left(\frac{\pi mn}{\varphi_2-\varphi_1}\cos m\pi x\cos n\theta + n\cot\varphi\sin m\pi x\cos n\theta\right)\cdot C. \tag{13f}$$

Substitution of Eqs.(12) into Eqs.(5) yields the following equations:

$$\left.\begin{aligned}
\delta^2\varepsilon_{10} &= \frac{1}{A_1^2}\left(\frac{\partial(\delta w)}{\partial\alpha_1}\right)^2 = \left(\frac{m\pi}{R_1(\varphi_2-\varphi_1)}\cos m\pi x\sin n\theta\right)\cdot C^2,\\
\delta^2\varepsilon_{20} &= \frac{1}{A_2^2}\left(\frac{\partial(\delta w)}{\partial\alpha_2}\right)^2 = \left(\frac{n\sin m\pi x\cos n\theta}{R_2\sin\varphi}\right)^2\cdot C^2,\\
\delta^2\varepsilon_{120} &= \frac{2}{A_1A_2}\frac{\partial(\delta w)}{\partial\alpha_1}\frac{\partial(\delta w)}{\partial\alpha_2} =\\
&\frac{2mn\pi}{R_1R_2(\varphi_2-\varphi_1)\sin\varphi}\sin m\pi x\cos m\pi x\sin n\theta\cos n\theta\cdot C^2.
\end{aligned}\right\}\qquad(14)$$

Substitution of Eqs.(13) and Eqs.(14) into Eqs.(8) and do δ^* variation yields the buckling equations.

6 Solution Technique

First, linear buckling loads P^* (External pressure), F^* (Axial compression), T^* (Combination of axial compression and external pressure) are calculated. Secondly, displacement and internal force under loads between $[0, 2P^*]$, $[0, 2F^*]$, $[0, 2T^*]$ are calculated by an increment of N. And then put them into the buckling equations, a series of buckling loads with combination of half wave number (m, n) are obtained. Then decrease N to get the buckling loads with higher precision. The minimum of these loads is the buckling load.

Using FORTRAN language the computer program is developed. The integral is evaluated numerically by Monte formula. Combine gradient method and amendatory Newton iterative method to solve the nonlinear equations. Gradient method is used to give the initial value. Newton iterative method is used to increase the precision.

In order to demonstrate the accuracy of the method presented in this paper, the module of no moment theory is put into the program. This module uses no moment theory to calculate the displacement and internal force before buckling and the buckling displacement function with the solution technique are the same with the main module.

The method presented in this paper is propitious to shell with positive gauss curvature. The load can be as axial compression, external pressure, combination of axial compression and external pressure, external pressure with given axial compression and axial compression with given external pressure.

7 Example

The example is taken from reference [1], The shell is cylindrical with length 500 mm, radius 500 mm, thickness of single layer 0.15 mm, Young's modulus 117.6 GPa, 5.88 GPa, shear modulus 2.94 GPa, Poisson's ratio 0.3. The type of laminated take as: ⅰ) perpendicular symmetrical 40 layers; ⅱ) ± 45° symmetrical 40 layers; ⅲ) ± 45° symmetrical 20 layers; ⅳ) ± 22.5° symmetrical 40 layers; ⅴ) ± 22.5° symmetrical 20 layers. Results are given in table 1. Because the displacement function before buckling does not comport with the displacement of cylindrical shell under axial load, the results in this case are not given.

Table 1 Unit axial compression: KN, external pressure: MPa

number	load case	Ref.[1] shear theory	Ref.[1] classic theory	no moment theory	method in this paper
1	axial compression	2 758.0(2,7)	2 804.8(2,6)	2 577.953(2,6)	
1	external pressure	0.590(1,7)	0.605(1,7)	0.586 377(1,7)	0.589 323(1,7)
2	axial compression	2 954.8(4,2)	2 988.9(4,2)	2 551.853(4,1)	
2	external pressure	0.627(1,8)	0.645(1,8)	0.573 606(1,7)	0.560 094(1,7)
3	axial compression	711.2(6,2)	717.4(6,2)	638.097(6,1)	
3	external pressure	0.097 6(1,9)	0.099 0(1,9)	0.088 278(1,8)	0.086 167(1,8)
4	axial compression	3 155.7(3,2)	3 235.8(3,2)	3 109.077(3,1)	
4	external pressure	0.336(1,11)	0.342(1,11)	0.312 382(1,10)	0.322 148(1,10)
5	axial compression	784.8(4,2)	792.2(4,2)	756.287(4,1)	
5	external pressure	0.052 7(1,13)	0.053 1(1,13)	0.047 807(1,12)	0.047 492(1,12)

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