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ORIGINAL PAPER

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Bounds to shakedown loads for a class of deviatoric plasticity models

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Abstract The problem of estimating bounds to shakedown loads for problems governed by a class of deviatoric plasticity models including those of Hill, von Mises, and Tresca is addressed. Assuming that an exact elastic solution is available, an upper bound to the elastic shakedown multiplier can be obtained relatively easily using the plastic shakedown theorem. A procedure for computing this upper bound for arbitrary load domains is presented. A number of problems are then examined and it is found that the elastic shakedown factor is given as the minimum of the plastic shakedown factor and the classical limit load factor. Finally, some exact solutions to a number of two dimensional problems are given.

Keywords Shakedown · Analytical solution · Upper bound · Lower bound · Finite elements

1 Introduction

Shakedown analysis is well-established as a tool for assessing the safety factor against instantaneous plastic collapse, fatigue and excessive accumulated strains in structures subjected to cyclic loading. Typically, the task is to find the factor by which a given set of cyclic loads of constant amplitude can be magnified without the structure suffering any of the above mentioned types of failure. If for a given load amplitude the structure is capable of accommodating the corresponding plastic strains it is said to shake down. The primary merit of shakedown analysis is that it enables the computation of the safety factor against failure without resorting to a full incremental load–displacement analysis. The trade-off is the simple linear elastic/perfectly plastic material model which must be assumed in order to enable a direct assessment of the safety factor. However, although the formulation of models capable of capturing the behaviour of elastoplastic materials when subjected to cyclic loading remains a largely unsolved problem, the linear elastic/perfectly plastic material model does in fact provide reasonable estimates to the shakedown load, at least for metals, see e.g. [17]. Thus, design by shakedown analysis is incorporated into a number of codes of practice and serves as a valuable engineering tool in the design of a wide variety of metal structures including pressure vessels [8] and rails [7]. Also, following the pioneering work of Sharp and Booker [26], shakedown concepts have in recent years been applied to the design of road pavements [18, 27, 2]. Although there is less experimental justification for application of shakedown theory to soils the results obtained seem to agree, at least qualitatively, with experiments [27].

The classical shakedown theorems lead to large, nonlinear optimization problems for which standard commercial finite element codes offer no solution procedures. Therefore, over the years a large number research codes have been developed. The development of such a code is a significant task, with the formulation of an efficient and robust optimization algorithm being particularly challenging. Moreover, the verification of a shakedown code is somewhat problematic in that only very few analytical solutions are available. Perhaps for this reason there is often significant deviation between the numerical results obtained by different research groups, for even the simplest problems.

In this paper we present a comprehensive review of numerical solutions to some common benchmark problems. Furthermore, we investigate the possibility of computing upper and lower bounds to the elastic shakedown factor without the use of optimization. This is achieved essentially by two ingredients. Firstly, we assume that the exact elastic stress fields corresponding to the vertices of the load domain are known. For general problems such elastic solutions are of course not available, but can, for problems without singularities, be computed to within an arbitrary accuracy using standard finite element techniques. Secondly, we make use of the *plastic shakedown theorem* and devise a systematic way of calculating bounds to the corresponding safety factor analytically. For load domains consisting of one or two

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points these bounds to the plastic shakedown factor coincide with the exact one whereas for arbitrary load domains rigorous upper and lower bounds are computed. Particularly the upper bound seems to furnish excellent results and for all the examples dealt with in this paper it coincides with the exact solution.

Although the plastic shakedown theorem is relatively well known, see e.g. [24, 36, 37], its potential significance is generally not appreciated. In this paper we show that for a number of typical problems of purely deviatoric plasticity the elastic shakedown multiplier coincides either with the plastic shakedown multiplier or with the classical plastic collapse multiplier. Although this can not be shown to hold in general, the trend is very consistent and no counterexamples have been produced despite an extensive search. The analytical procedures described should thus be of value in estimating bounds to the elastic shakedown factor and thus, aid in the verification of general shakedown codes.

2 Shakedown analysis

For linear elastic/perfectly plastic materials under cyclic loading three distinct modes of failure can be identified. These are commonly referred to as alternating plasticity (plastic non–shakedown), incremental collapse (ratcheting), and instantaneous collapse (plastic collapse). Roughly speaking, alternating plasticity is critical for problems where significant stress concentrations are present whereas for bending dominated problems failure tends to be by way of incremental or instantaneous collapse. The latter type of problem includes structures idealized by beam elements whereas the former usually involves solids containing imperfections, holes, or grooves. Classical (or elastic) shakedown analysis deals with the prevention of all three types of failure. In the following the elastic and plastic shakedown theorems are briefly discussed. For a more in-depth treatment we refer to the papers of Zouain and Silveira [36, 37].

2.1 Elastic shakedown

Consider a polyhedral load domain defined by V vertices. The elastic stresses corresponding to each of these vertices are denoted $\boldsymbol{\chi}_j(\mathbf{x})$, $j = 1, \dots, V$. All of the above mentioned failure modes are then prevented if there exists a residual stress field $\boldsymbol{\rho}(\mathbf{x})$ such that

$$\begin{aligned} &\nabla \cdot \boldsymbol{\rho}(\mathbf{x}) = \mathbf{0}, && \mathbf{x} \in \Omega \\ &\mathbf{n} \cdot \boldsymbol{\rho}(\mathbf{x}) = \mathbf{0}, && \mathbf{x} \in \partial\Omega_u \\ &F[\boldsymbol{\rho}(\mathbf{x}) + \alpha \boldsymbol{\chi}_j(\mathbf{x})] \leq 0, && (\mathbf{x}, j) \in \Omega \times \mathcal{V} \end{aligned} \tag{1}$$

where F is the yield function, $\partial\Omega_u$ is the unsupported part of the boundary, and $\mathcal{V} = (1, V)$ is the load domain. If these conditions are fulfilled exactly the multiplier α will be a lower bound to the true elastic shakedown multiplier. Thus, we seek to maximize α subject to the above constraints.

2.2 Plastic shakedown

The plastic shakedown theorem follows by simply neglecting the equilibrium constraints in (1) [24, 36, 37]. Thus, for any scalar α which satisfies

$$F[\boldsymbol{\rho}(\mathbf{x}) + \alpha \boldsymbol{\chi}_j(\mathbf{x})] \leq 0, \quad (\mathbf{x}, j) \in \Omega \times \mathcal{V} \tag{2}$$

collapse by alternating plasticity will not take place. Compared to (1) the plastic shakedown problem is much simpler. The residual stresses need no longer be self–equilibrated and the yield condition can be checked independently at each point in the combined space-load domain. In the following we consider a finite number of points in the spatial domain so that (2) is approximated as

$$F[\boldsymbol{\rho}_k + \alpha \boldsymbol{\chi}_{j,k}] \leq 0, \quad (j, k) \in \mathcal{V} \times \mathcal{P}, \tag{3}$$

where $\mathcal{P} = (1, P)$ with P being the total number of points under consideration. Obviously, since the plastic shakedown theorem appears as the special case of the elastic shakedown theorem where the equilibrium conditions are neglected, the resulting plastic shakedown multiplier will be an upper bound to the elastic shakedown multiplier, i.e.

$$\alpha_{\mathrm{AP}} \geq \alpha_{\mathrm{SD}}, \tag{4}$$

where α_{AP} is the plastic shakedown multiplier, or the safety factor against alternating plasticity, and α_{SD} is the elastic shakedown multiplier.

2.3 Plastic collapse

Consider the case where the load domain shrinks to a single point. By introducing a new variable

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\rho}(\mathbf{x}) + \alpha \boldsymbol{\chi}(\mathbf{x}) \tag{5}$$

the governing equations 1 can be written as

$$\begin{aligned} &\nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) = \alpha \nabla \cdot \boldsymbol{\chi}(\mathbf{x}), && \mathbf{x} \in \Omega \\ &\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{x}) = \alpha \mathbf{n} \cdot \boldsymbol{\chi}(\mathbf{x}), && \mathbf{x} \in \partial\Omega_u \\ &F[\boldsymbol{\sigma}(\mathbf{x})] \leq 0, && (\mathbf{x}, j) \in \Omega \times \mathcal{V}. \end{aligned} \tag{6}$$

The elastic stresses fulfill the equilibrium conditions

$$\begin{aligned} &\nabla \cdot \boldsymbol{\chi}(\mathbf{x}) = \mathbf{0}, && \mathbf{x} \in \Omega \\ &\mathbf{n} \cdot \boldsymbol{\chi}(\mathbf{x}) = \hat{\mathbf{t}}(\mathbf{x}), && \mathbf{x} \in \partial\Omega_u \end{aligned} \tag{7}$$

for some prescribed boundary tractions $\hat{\mathbf{t}}(\mathbf{x})$. The conditions 6 then reduce to

$$\begin{aligned} &\nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}, && \mathbf{x} \in \Omega \\ &\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{x}) = \alpha \hat{\mathbf{t}}(\mathbf{x}), && \mathbf{x} \in \partial\Omega_u \\ &F[\boldsymbol{\sigma}(\mathbf{x})] \leq 0, && (\mathbf{x}, j) \in \Omega \times \mathcal{V}. \end{aligned} \tag{8}$$

These are the conditions imposed by the lower bound theorem of limit analysis. It is thus clear that the exact plastic collapse multiplier is an upper bound to the exact elastic shakedown multiplier, i.e.

$$\alpha_{\mathrm{PC}} \geq \alpha_{\mathrm{SD}}, \tag{9}$$

where α_{PC} is the plastic collapse multiplier. Furthermore, since the safety factor against alternating plasticity is also an upper bound to the elastic shakedown multiplier, we have

$$\alpha_{UB} = \min(\alpha_{AP}, \alpha_{PC}) \geq \alpha_{\mathrm{SD}}. \tag{10}$$

That is, an upper bound to the elastic shakedown multiplier is found as the minimum of the alternating plasticity and plastic collapse multipliers.

2.4 Kinematic theorems

All of the above static theorems have a dual kinematic counterpart which may be derived using standard methods of convex analysis. In the present, however, the static theorems are of primary interest and the kinematic theorems will not be treated any further. For a comprehensive review on static–kinematic duality theory for shakedown analysis we refer [36, 37].

2.5 Yield condition

In the following we consider problems governed by yield conditions of the type

$$F(\boldsymbol{\sigma}) = \phi(\boldsymbol{\sigma}) - c = 0, \tag{11}$$

where c is a reference uniaxial yield strength and

$$\phi(\boldsymbol{\sigma}) = \sqrt{\frac{1}{2}\boldsymbol{\sigma}^{\mathsf{T}}\mathbf{A}\boldsymbol{\sigma}} \tag{12}$$

with $\mathbf{A}$ being a symmetric and positive definite matrix. The most general yield criterion which can be expressed by this quadratic form is probably that of Hill (of which the von Mises criterion is a special case). Also the plane strain Tresca criterion can be expressed by (12). We note that $\phi(\boldsymbol{\sigma})$ has the property that

$$\phi(\lambda\boldsymbol{\sigma}) = |\lambda|\phi(\boldsymbol{\sigma}) \tag{13}$$

for an arbitrary scalar λ.

3 Lower bounds

Provided the exact elastic stress distribution is known, a lower bound to the elastic shakedown multiplier can be obtained by setting all residual stresses equal to zero. The lower bound elastic shakedown theorem (1) then simplifies to

$$\begin{array}{ll} \text{maximize} & \alpha_{\mathrm{LB}} \\ \text{subject to} & \phi(\alpha_{\mathrm{LB}}\boldsymbol{\chi}_{j,k}) - c_k \leq 0, \quad (j,k) \in \mathcal{V} \times \mathcal{P}, \end{array} \tag{14}$$

where a spatial discretization analogous to that in (3) has been assumed. At each point k within the spatial domain and for each vertex j in the load domain, we can compute the factor

$$\alpha_{j,k} = \frac{c_k}{\phi(\boldsymbol{\chi}_{j,k})} \tag{15}$$

and a lower bound then follows as

$$\alpha_{LB} = \alpha_E = \min \alpha_{j,k}, \quad (j,k) \in \mathcal{V} \times \mathcal{P}, \tag{16}$$

where α_E is the elastic limit multiplier. Thus, the lower bound simply corresponds to the elastic limit.

4 Upper bounds

Whereas it is on the safe side to set all residual stresses equal to zero so that the equilibrium conditions are automatically fulfilled, it is clearly on the unsafe side to ignore the equilibrium conditions altogether. Thus, upper bounds to the shakedown multiplier may be calculated from the following simplified problem

$$\begin{array}{ll} \text{maximize} & \alpha_{\mathrm{AP}} \\ \text{subject to} & \phi(\boldsymbol{\rho}_k + \alpha_{\mathrm{AP}}\boldsymbol{\chi}_{j,k}) - c_k \leq 0, \quad (j,k) \in \mathcal{V} \times \mathcal{P}. \end{array} \tag{17}$$

The resulting multiplier is the safety factor against alternating plasticity and is bounded from below by the elastic shakedown multiplier [24]. Again, the simplified problem (17) can be decoupled into a number of smaller problems which deal with each individual point in the spatial domain. However, in contrast to what was the case for the lower bound estimates, these individual optimization problems are generally nontrivial to solve. For yield criteria of the type (12), however, it is possible to compute an estimate analytically as will be shown in the following.

4.1 Single-point load domain

For the case where the load domain consists of a single point (17) can be decoupled into P problems of the type

$$\begin{array}{ll} \text{maximize} & \alpha_k \\ \text{subject to} & \phi(\boldsymbol{\rho}_k + \alpha_k\boldsymbol{\chi}_k) - c_k \leq 0 \end{array} \tag{18}$$

The residual stress can here be chosen for example as $\boldsymbol{\rho}_k = -\alpha_k\boldsymbol{\chi}_k$ so that the yield criterion is always fulfilled. We therefore have

$$\alpha_{\mathrm{UB}} = \infty \tag{19}$$

for this particular case. Thus, for the problem of limit analysis (i.e. shakedown analysis with the load domain consisting of a single point) only the lower bound estimate (16) is of potential use.

4.2 Two-point load domain

The problem of determining the plastic shakedown multiplier for the case where the load domain consists of two points may be formulated as

$$\begin{array}{ll} \text{maximize} & \alpha_k \\ \text{subject to} & \phi(\boldsymbol{\rho}_k + \alpha_k\boldsymbol{\chi}_{1,k}) - c_k \leq 0, \\ & \phi(\boldsymbol{\rho}_k + \alpha_k\boldsymbol{\chi}_{2,k}) - c_k \leq 0 \end{array} \tag{20}$$

which is to be solved for each k. The geometric interpretation of (20) is shown in Fig. 1. At each point the multiplier that brings about complete separation between two translating yield surfaces is found. At the point of separation the unit normals of the two surfaces are aligned in opposite directions. This gives the condition

$$\mathbf{A}(\boldsymbol{\rho}_k + \alpha_k\boldsymbol{\chi}_{1,k}) = -\mathbf{A}(\boldsymbol{\rho}_k + \alpha_k\boldsymbol{\chi}_{2,k}) \tag{21}$$

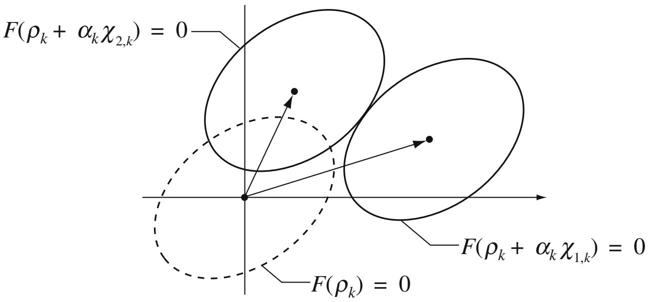

Fig. 1 Geometric interpretation of alternating plasticity

with the solution

$$\boldsymbol{\rho}_k = -\frac{1}{2}\alpha_k(\boldsymbol{\chi}_{1,k} + \boldsymbol{\chi}_{2,k}). \tag{22}$$

This is inserted into (20) to give

$$\begin{array}{ll} \text{maximize} & \alpha_k \\ \text{subject to} & \phi\left[\frac{1}{2}\alpha_k(\boldsymbol{\chi}_{1,k} - \boldsymbol{\chi}_{2,k})\right] - c_k \leq 0, \\ & \phi\left[\frac{1}{2}\alpha_k(\boldsymbol{\chi}_{2,k} - \boldsymbol{\chi}_{1,k})\right] - c_k \leq 0. \end{array} \tag{23}$$

Using the property (13) the optimal value of α_k is easily found as

$$\alpha_k = \frac{2c_k}{\phi(\boldsymbol{\chi}_{1,k} - \boldsymbol{\chi}_{2,k})} = \frac{2c_k}{\phi(\boldsymbol{\chi}_{2,k} - \boldsymbol{\chi}_{1,k})}. \tag{24}$$

Finally, the alternating plasticity multiplier is determined as

$$\alpha_{\text{AP}} = \min \alpha_k, \quad k \in \mathcal{P}. \tag{25}$$

In the important special case where one of the points in the load domain corresponds to a stress free state, i.e. $\boldsymbol{\chi}_1 = \boldsymbol{\chi}$ and $\boldsymbol{\chi}_2 = \mathbf{0}$, the alternating plasticity multiplier is given by

$$\begin{aligned} \alpha_{\text{AP}} &= \min \alpha_k = \min \frac{2c_k}{\phi(\boldsymbol{\chi}_k)}, \quad k \in \mathcal{P}, \\ &= 2\alpha_E, \end{aligned} \tag{26}$$

where α_E is given by (16). Thus, the frequently cited rule that the alternating plasticity limit is twice the elastic limit applies only to this special case.

4.3 General polyhedral load domains

The problem of plastic shakedown analysis for general polyhedral load domains can be formulated as

$$\begin{array}{ll} \text{maximize} & \alpha_k \\ \text{subject to} & \mathcal{F}_1 : \phi(\boldsymbol{\rho}_k + \alpha_k \boldsymbol{\chi}_{1,k}) - c_k \leq 0, \\ & \vdots \\ & \mathcal{F}_V : \phi(\boldsymbol{\rho}_k + \alpha_k \boldsymbol{\chi}_{V,k}) - c_k \leq 0. \end{array} \tag{27}$$

Again the geometrical interpretation is one of V translating yield surfaces and the maximum value of α_k is that which first leads to an empty intersection of the V yield sets. This does not necessarily imply separation between the yield surfaces as was the case for the two-point load domain. An example of this (for a three-point load domain) is shown in Fig. 2 where all three yield contraints are active. However, an upper bound to α_k, and thereby to the plastic shakedown multiplier, can be computed by finding the multiplier for which any two yield surfaces first separate. Thus, for each point k the following factors can be calculated

$$\begin{aligned} &\alpha_k^{1,2} = \frac{2c_k}{\phi(\boldsymbol{\chi}_{1,k} - \boldsymbol{\chi}_{2,k})}, \quad \alpha_k^{1,3} = \frac{2c_k}{\phi(\boldsymbol{\chi}_{1,k} - \boldsymbol{\chi}_{3,k})}, \ldots \\ &\alpha_k^{V-1,V} = \frac{2c_k}{\phi(\boldsymbol{\chi}_{V-1,k} - \boldsymbol{\chi}_{V,k})}. \end{aligned} \tag{28}$$

Finally, the best upper bound estimate is found as

$$\alpha_{\text{UB}} = \min(\alpha_k^{1,2}, \alpha_k^{1,3}, \ldots, \alpha_k^{V-1,V}), \quad k \in \mathcal{P} \tag{29}$$

Thus, for a load domain defined by V vertices a total of $\frac{1}{2}V(V-1)P$ factors must be computed and the best upper bound taken as the least of these.

4.3.1 *Verification of upper bounds as exact solutions*

Stein and Huang [31] have previously developed an analytical procedure for determining the exact plastic shakedown factor for the special case of a four-point load domain. In addition to the possibility of having only two active yield surfaces as implied above, their method considers the entire scope of possible situations. However, we generally find that the upper bound found by assuming only two yield constraints active coincides with the exact solution. In any given situation the validity of this claim can be verified as follows. Consider again the general plastic shakedown problem (27). The associated Lagrangian is given by

$$\begin{aligned} \mathcal{L} = \alpha_k &- \lambda_1\left[\phi(\boldsymbol{\rho}_k + \alpha_k\boldsymbol{\chi}_{1,k}) - c_k\right] \\ &+ \cdots\cdots - \lambda_V\left[\phi(\boldsymbol{\rho}_k + \alpha_k\boldsymbol{\chi}_{V,k}) - c_k\right], \end{aligned} \tag{30}$$

where λ_j are Lagrange multipliers. The first-order necessary and sufficient Karush–Kuhn–Tucker optimality conditions follow from the Lagrangian as

$$\begin{aligned} &\lambda_1\boldsymbol{\chi}_{1,k}^{\mathsf{T}}\nabla\phi(\boldsymbol{\rho}_k + \alpha_k\boldsymbol{\chi}_{1,k}) \\ &\quad + \cdots + \lambda_V\chi_{V,k}^{\mathsf{T}}\nabla\phi(\boldsymbol{\rho}_k + \alpha_k\boldsymbol{\chi}_{V,k}) = 1, \end{aligned} \tag{31}$$

$$\lambda_1\nabla\phi(\boldsymbol{\rho}_k+\alpha_k\boldsymbol{\chi}_{1,k}) + \cdots + \lambda_V\nabla\phi(\boldsymbol{\rho}_k+\alpha_k\boldsymbol{\chi}_{V,k}) = \mathbf{0}, \tag{32}$$

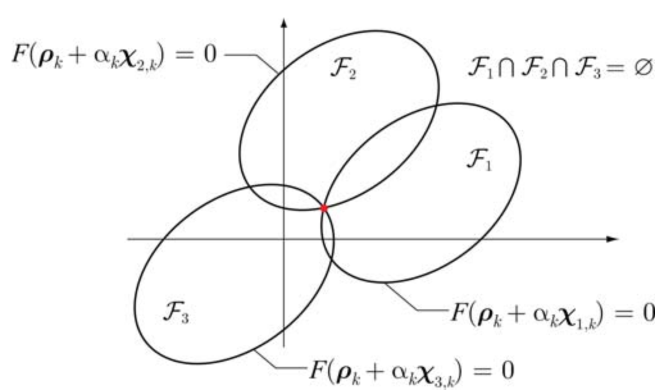

Fig. 2 Alternating plasticity without yield surface separation.

$$\begin{aligned}
&\lambda_1 \left[\phi(\boldsymbol{\rho}_k + \alpha_k \boldsymbol{\chi}_{1,k}) - c_k\right] = 0,\\
&\vdots \\
&\lambda_V \left[\phi(\boldsymbol{\rho}_k + \alpha_k \boldsymbol{\chi}_{V,k}) - c_k\right] = 0,
\end{aligned} \tag{33}$$

$$\lambda_j \geq 0, \quad j \in \mathcal{V}, \tag{34}$$

$$\begin{aligned}
&\phi(\boldsymbol{\rho}_k + \alpha_k \boldsymbol{\chi}_{1,k}) - c_k \leq 0,\\
&\vdots \\
&\phi(\boldsymbol{\rho}_k + \alpha_k \boldsymbol{\chi}_{V,k}) - c_k \leq 0.
\end{aligned} \tag{35}$$

By assuming only two yield surfaces active, we find a solution to (31)–(34). It remains then only to verify that all yield constraints (35) are fulfilled. If this is the case, the exact solution has been found and if not, the solution is an upper bound. In any case, the resulting multiplier constitutes an upper bound to the elastic shakedown multiplier.

5 Exact solutions

There are two types of problems for which exact solutions to the elastic shakedown problem can be derived immediately if the exact elastic solution is known. The first concerns the situation where one or more of the deviatoric elastic stress components tend to infinity. In two and three dimensional problems such stress singularities typically occur at sharp edges and at external supports. From (16) and (29) it follows immediately that the exact shakedown multiplier is given by

$$\alpha_{\mathrm{LB}} = \alpha_{\mathrm{UB}} = \alpha_{\mathrm{SD}} = 0 \tag{36}$$

Although this result is perhaps obvious, there are numerous examples in the literature where it is overlooked, particularly in connection with stress singularities at supports. Some recent examples can be found in [34, 19, 9] where the erroneous solutions presented must be attributed to an inadequate spatial discretization in the vicinity of the supports.

The other special case where it is possible to infer the exact solution from the elastic solution alone occurs when the load domain is such that the elastic stresses come in pairs of equal magnitude and opposite sign. That is, for a load domain defined by $2V$ vertices the corresponding elastic stresses are $\pm\boldsymbol{\chi}_1, \ldots, \pm\boldsymbol{\chi}_V$. An example of such a load domain is shown in Fig. 3.

The plastic shakedown factor can here be computed from V uncoupled problems of the type

$$\begin{aligned}
\text{maximize} \quad & \alpha_k \\
\text{subject to} \quad & \mathcal{F}_1^+ : \phi(\boldsymbol{\rho}_k + \alpha_k \boldsymbol{\chi}_{1,k}) - c_k \leq 0,\\
& \mathcal{F}_1^- : \phi(\boldsymbol{\rho}_k - \alpha_k \boldsymbol{\chi}_{1,k}) - c_k \leq 0,\\
& \vdots \\
& \mathcal{F}_V^+ : \phi(\boldsymbol{\rho}_k + \alpha_k \boldsymbol{\chi}_{V,k}) - c_k \leq 0,\\
& \mathcal{F}_V^- : \phi(\boldsymbol{\rho}_k - \alpha_k \boldsymbol{\chi}_{V,k}) - c_k \leq 0.
\end{aligned} \tag{37}$$

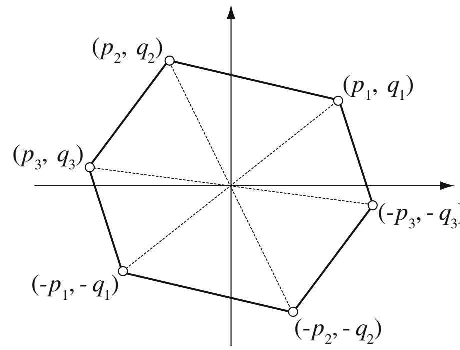

Fig. 3 Load domain enabling determination of the exact elastic shakedown multiplier

Following (16), and using the property (13), a lower bound is given by

$$\alpha_{\mathrm{LB}} = \alpha_E = \min\left(\frac{c_k}{\phi(\boldsymbol{\chi}_{j,k})}\right), \quad (j,k) \in \mathcal{V} \times \mathcal{P} \tag{38}$$

An upper bound follows by considering only the separation between yield sets $\mathcal{F}_j^+$ and $\mathcal{F}_j^-$. This gives

$$\begin{aligned}
\alpha_{UB} &= \min\left(\frac{2c_k}{\phi(\pm 2\boldsymbol{\chi}_{j,k})}\right), \quad (j,k) \in \mathcal{V} \times \mathcal{P}\\
&= \alpha_{\mathrm{LB}} = \alpha_{\mathrm{SD}}
\end{aligned} \tag{39}$$

and so, the exact solution has been established.

6 Applications

Firstly, a number of problems are analyzed and comparisons are made with existing numerical solutions. Secondly, a number of predictions are made for problems that to our knowledge have not been treated in detail elsewhere. Finally, some exact solutions are derived. In all cases we assume linear elastic/perfectly plastic material behaviour with yielding governed by the von Mises criterion. Unless indicated otherwise Poisson's ratio is $\nu = 0.3$ and the yield strengths refer to the uniaxial strengths.

6.1 Perforated plate in biaxial tension I

A common benchmark problem is that of a square, perforated plate in biaxial tension under plane stress assumptions, Fig 4. The load domain consists of four points: $(0, 0)$, $(p_x, 0)$, $(0, p_y)$, and (p_x, p_y) as shown in Fig. 4. This problem, with $D/L = 0.2$, was first studied by Belytschko [1] more than 30 years ago and has since then been used extensively to verify numerical shakedown procedures. However, judging from the past and recent literature on the problem, no exact (or pseudo-exact) solution is commonly agreed upon. This is highlighted by the selection of solutions originally compiled by Gross–Weege [11] and expanded with more recent results in Table 1.

Table 1 Elastic shakedown multipliers for perforated plate with $D/L = 0.2$ subjected to independently varying loads

	$p_x = 1, p_y = 0$	$p_x = 1, p_y = 1$	$p_x = 1, p_y = \frac{1}{2}$
Nguyen-Dang and Palgen [23]	0.557	0.431	0.514
Belytschko [1]	0.571	0.431	0.501
Zhang and Raad [33]	0.574	0.494	N/A
Zouain et al. [35]	0.594	0.429	0.500
Present (AP)	**0.595**	**0.430**	**0.499**
Zhang [32]	0.596	0.431	0.514
Khoi et al. [14]	0.599	N/A	N/A
Garcea et al. [9]	0.604	0.438	0.508
Gross-Weege [11]	0.614	0.446	0.524
Hamadouche [12]	0.623	0.490	N/A
Zhang [32]	0.624	0.453	0.539
Liu et al.[19]	0.647	0.477	0.549
Zhang et al.[34]	0.647	0.477	0.549
Genna [10]	0.653	0.478	0.566
Corradi and Zavelani [6]	0.654	0.504	0.579
Chen and Ponter [5]	0.666	N/A	N/A
Carvelli et al. [4]	0.696	0.518	N/A

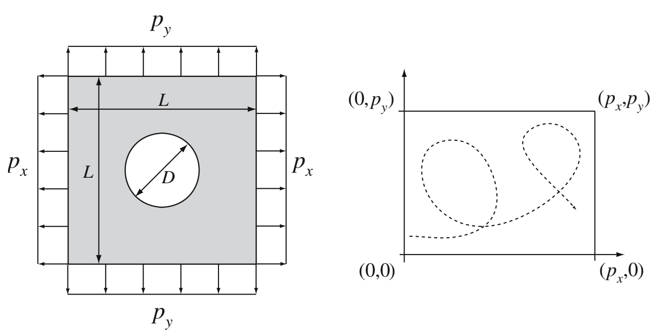

Fig. 4 Perforated plate subjected to independently varying cyclic loads

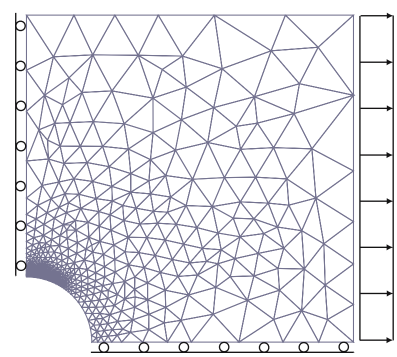

Fig. 5 Mesh used to compute the elastic solution ($D/L = 0.2$)

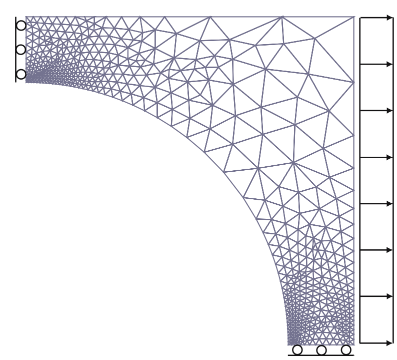

Fig. 6 Mesh used to compute the elastic solution ($D/L = 0.8$)

There is clearly a large scatter in these results. With the exception of the work of Zouain et al. [35], no particular effort to compute an accurate elastic solution is mentioned in any of the references of Table 1. In the present work the elastic stresses were computed using linear strain triangles with adaptive mesh refinement. In this way we obtain plastic shakedown factors which are in very close agreement with elastic shakedown factors presented by Zouain et al. [35]. It is therefore tempting to conclude that these solutions for all practical purposes are exact, possibly with uncertainty in the third digit. Thus, for all three load cases dealt with in Table 1, the mode of failure is alternating plasticity and as such the full elastic shakedown procedure usually employed for this problem is unnecessary (although this of course is not known a priori).

6.2 Perforated plate in biaxial tension II

As an example of a problem where alternating plasticity is not always critical we again consider the perforated plate, now with a load domain consisting of only two points: (0, 0) and (p_x, p_y). By keeping $p_x = 1$ constant and varying p_y between 0 and 1 the interaction diagrams shown in Fig. 7 for various values of D/L can be constructed. In the figures the alternating plasticity and plastic collapse domains are shown together with the elastic shakedown domain. The elastic shakedown problems have been solved using an algorithm based on previous work in limit analysis Lyamin and Sloan [20, 21] Krabbenhoft and Damkilde [15]. Again, quadratic displacement (velocity) triangles were used and the meshes refined to capture the elastic solution with optimal

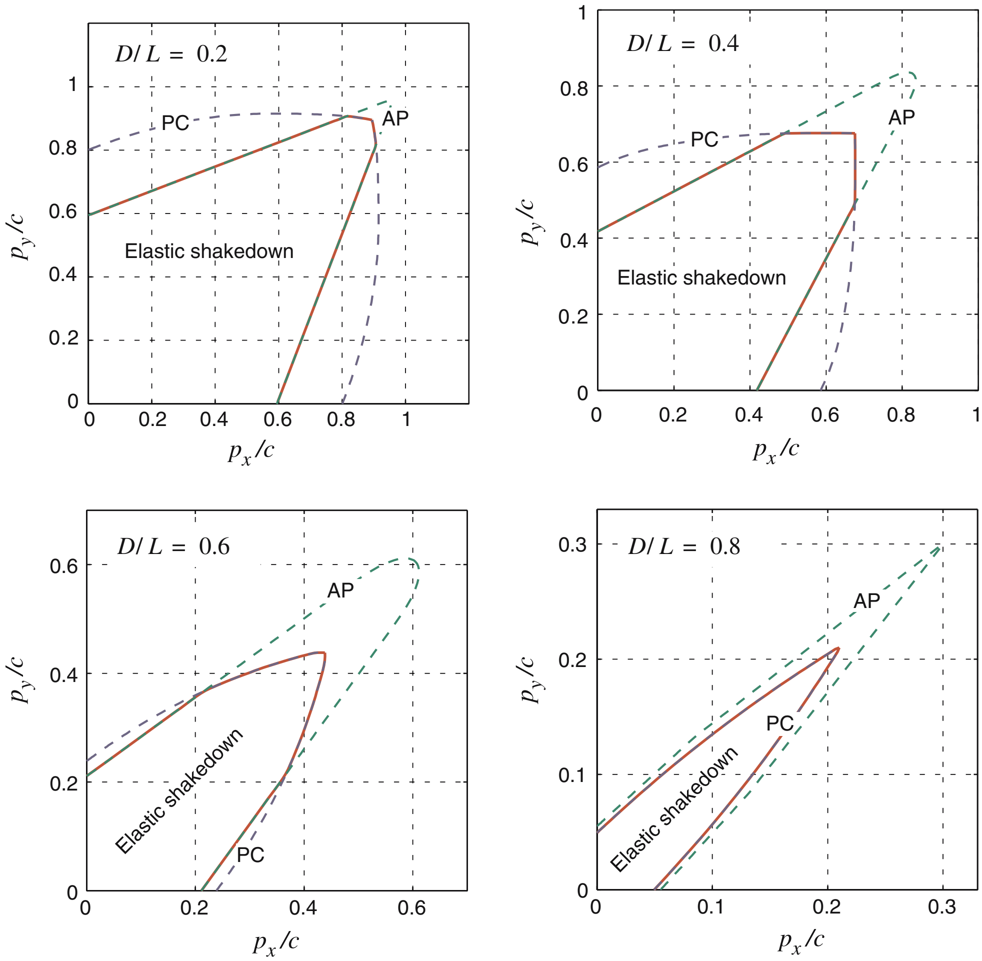

Fig. 7 Alernating plasticity (*AP*), plastic collapse (*PC*) and elastic shakedown domains for perforated plate subjected to proportional loading

accuracy. Two of the resulting meshes, for the case where $p_y = 0$, are shown in Figs. 5 and 6.

The results show that the elastic shakedown factor is given as the minimum between the alternating plasticity and plastic collapse factors. Thus, for this example we can again conclude that solving the elastic shakedown problem is unnecessary as all information can be inferred from a linear elastic and a classical limit analysis solution. This is rather remarkable since the problems are physically quite different which is also reflected by the change in the shape of the various domains as the hole diameter is increased. Thus, whereas the plastic collapse problem for small values of *D*/*L* is governed by shear failure, bending becomes increasingly dominant as *D*/*L* increases. Therefore, for values of *D*/*L* close to unity, failure by incremental collapse might be expected. However, the numerical results obtained here indicate that this is not the case, but rather that the elastic shakedown multiplier is identical to the plastic collapse multiplier. A similar trend has been observed for a large number of other problems, i.e. the elastic shakedown factor seems to be given as the smallest of the alternating plasticity and plastic collapse factors. In most works on shakedown analysis only the elastic shakedown multiplier, and perhaps the plastic collapse multiplier, is computed. An exception is found in the recent paper of Makrodimopoulos [22]. For the problem of a pressure vessel head, both elastic and plastic shakedown multipliers, as well as the plastic collapse multiplier are computed. The conclusions with respect to the relation between these three multipliers are identical to those found in the present paper.

6.3 Shakedown of strip footing

We now consider the problem of a cyclically loaded strip footing on a deep layer of soil shown in Fig. 8. The soil is undrained clay and the state of deformation is plane. Yielding is governed by Tresca's criterion:

$$F(\boldsymbol{\sigma}) = \phi(\boldsymbol{\sigma}) - c, \quad \phi(\boldsymbol{\sigma}) = \sqrt{\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2}\,, \qquad (40)$$

where c is the cohesion. For a uniform unit pressure the plastic collapse multiplier is given by the well-known Prandtl (or Hill) solution as

$$\alpha_{PC} = (2 + \pi)c. \qquad (41)$$

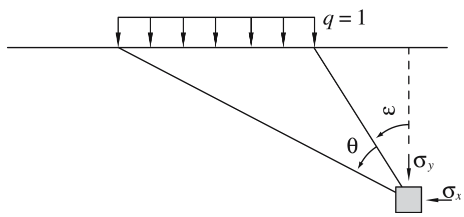

Fig. 8 Strip footing problem

It should be noted that the limit load for the more common case where the footing is rigid, and the stress distribution underneath it thus unknown, also is given by (41). The shakedown factor, however, is equal to zero due to the stress singularity at the edge of the rigid footing.

The problem where the stress distribution is prescribed to be uniform is one case where a complete elastic solution is available. With reference to Fig. 8, the exact elastic solution is given by Poulos and Davis [25]

$$\begin{aligned} \sigma_x &= \pi^{-1}[\theta + \sin\theta\cos(\theta + 2\omega)], \\ \sigma_y &= \pi^{-1}[\theta - \sin\theta\cos(\theta + 2\omega)], \\ \tau_{xy} &= \pi^{-1}\sin\theta\sin(\theta + 2\omega). \end{aligned} \tag{42}$$

Following the soil mechanics sign convention compressive stresses are taken as being positive. To our knowledge this exact elastic stress distribution has not previously been exploited. Rather, a number of numerical solutions have been presented where both the linear elastic solution and the shakedown multiplier are found using the same mesh (or so it seems—details on this crucial issue are rarely provided).

For the case of a two–point load domain with the one point being zero Zhang and Raad [33] find a shakedown multiplier which is slightly lower than the plastic collapse multiplier. This multiplier is claimed to be an upper bound. Recently, however, Boulbibane and Ponter [3] have presented numerical results suggesting that the shakedown multiplier is equal to the plastic collapse multiplier. In the present work we have obtained further numerical evidence to this effect. Boulbibane and Ponter [3] also perform another test where the load is varied cyclically between two values of equal magnitude and opposite sign. For this problem they find a shakedown factor which is "approximately half" of that given by (41). With the elastic solution available this problem is easily solved analytically.

The location of the critical stress state is first found by maximizing the yield function, i.e.

$$\text{maximize } \phi(\boldsymbol{\sigma}) = \sqrt{\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2} \tag{43}$$

with the result being

$$\phi_{\max} = \pi^{-1} \tag{44}$$

for $\theta = 90°$ and arbitrary ω. This corresponds to the points at edges of the loaded segment. Using (39) the exact solution is found to be

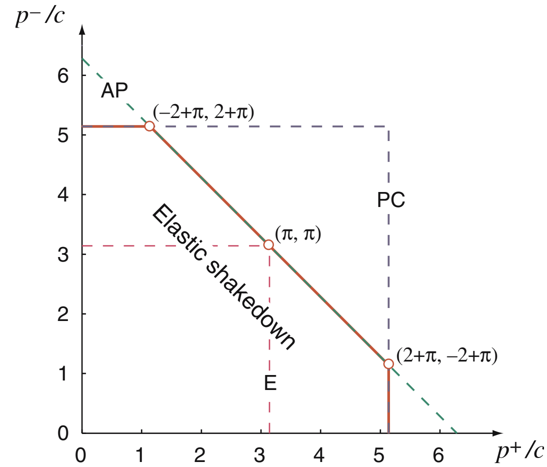

Fig. 9 Elastic (*E*), alternating plasticity (*AP*), plastic collapse (*PC*) and elastic shakedown domains for strip footing subjected to cyclic loading

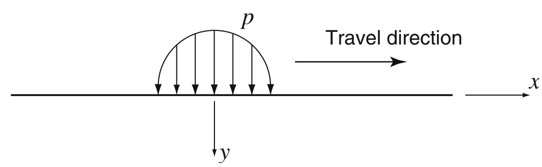

Fig. 10 Rolling contact

$$\alpha_{\mathrm{SD}} = \frac{c}{\phi_{\max}} = \pi c. \tag{45}$$

It should be noted that since the critical points are located at ground level, an increase in strength with depth will not lead to an increase in shakedown capacity.

Finally, we consider the more general case where the load varies between two extremes $-|p^-|$ and $|p^+|$. For this case the diagram shown in Fig. 9 can be constructed. As the exact elastic solution is available, the rigorous upper and lower bound methods previously developed for limit analysis by Sloan and his coworkers [28–30, 20, 21, 16] can be used to compute pseudo-upper and lower bounds. Since the lower bound theorem requires that the yield condition is enforced at every point of the domain, which is obviously not possible using a discrete method, the bound can not be seen as being completely rigorous. Similar features prohibit the computation of completely rigorous upper bound solutions. However, the error involved in this approximation is generally small and can be all but eliminated by appropriate refinement of the mesh.

The results, Fig. 9, follow the same trend as that observed for the perforated plate problems: the elastic shakedown domain is the intersection of the alternating plasticity and plastic collapse domains. In other words, the shakedown multiplier is found as the minimum of the alternating plasticity and plastic collapse multipliers. The first of these is determined by a simple post-processing of the elastic solution and the second is found using methods of classical limit analysis.

6.4 Rolling contact

Rolling contact is one of the most successful applications of shakedown analysis and the shakedown limit is well established as the rational design criterion for rails, roller bearings, and traction drives [13]. In the following, the problem sketched in Fig. 10 is considered. Assuming a state of plane strain this problem corresponds to the rolling of an infinitely long cylinder on the surface of a half–space. At a given section $x = \text{const}$ the surface will be continuously loaded and unloaded so that the load domain consists of the two points 0 and p. Also, since all such sections experience the same loading the residual stresses must be independent of x, i.e.

$$\frac{\partial \rho_x}{\partial x} = \frac{\partial \rho_y}{\partial x} = \frac{\partial \rho_{xy}}{\partial x} = 0. \tag{46}$$

The equilibrium conditions then reduce to

$$\frac{\partial \rho_{xy}}{\partial y} = 0, \quad \frac{\partial \rho_y}{\partial y} = 0 \tag{47}$$

and taking the boundary and symmetry conditions into account it is clear that

$$\rho_x = \rho_x(y), \quad \rho_y = \rho_{xy} = 0. \tag{48}$$

Thus, for this problem the equilibrium conditions are particularly simple. Using these conditions the problem to be solved is

$$\begin{array}{ll} \text{maximize} & \alpha \\ \text{subject to} & \partial \rho_x / \partial x = 0, \\ & |\rho_x| \le 2c, \\ & \sqrt{\frac{1}{4}(\rho_x + \alpha\chi_x - \alpha\chi_y)^2 + \alpha^2\chi_{xy}^2} \le c. \end{array} \tag{49}$$

As for all the problems treated so far, lower bounds are found by setting $\rho_x = 0$ whereas upper bounds are found by ignoring the equilibrium condition.

Lower bounds are then given by (16) as

$$\alpha_{\text{LB}} = \frac{c}{\phi_{\max}}. \tag{50}$$

Upper bounds are found by

$$\begin{array}{ll} \text{maximize} & \alpha \\ \text{subject to} & \phi' = \dfrac{1}{4}(\rho_x + \alpha\chi_x - \alpha\chi_y)^2 + \alpha^2\chi_{xy}^2 \le c^2, \end{array} \tag{51}$$

where the first yield constraint of (49) has been left out since it is always less restrictive than the second one. From this problem the optimal value of ρ_x is found as

$$\frac{\partial \phi'}{\partial \rho_x} = \frac{1}{2}(\rho_x + \alpha\chi_x - \alpha\chi_y) = 0 \Rightarrow \rho_x = \alpha(\chi_y - \chi_x) \tag{52}$$

so that an upper bound is given by

$$\alpha_{\text{UB}} = \frac{c}{\chi_{xy}^{\max}}. \tag{53}$$

Assuming a Hertzian contact stress distribution given by

$$p = \sqrt{1 - x^2}, \quad |x| \le 1 \tag{54}$$

we have [13]

$$\phi_{\max} \simeq 0.300, \quad \chi_{xy}^{\max} = \frac{1}{4} \tag{55}$$

which gives

$$\alpha_{\text{LB}} \simeq 3.33c, \quad \alpha_{\text{UB}} = 4c. \tag{56}$$

Johnson [13] showed that the exact solution is equal to the upper bound given by (56). If the contact stress is assumed to be uniform it is easily verified from (42) that $\phi_{\max} = \chi_{xy}^{\max} = \pi^{-1}$ so that the exact solution in this case is given by

$$\alpha_{\text{SD}} = \pi c. \tag{57}$$

The critical points are here located on the surface at the ends of the loaded segment whereas the Hertzian contact stress distribution implies sub–surface failure.

Obviously, the bounds given by (50) and (53) are valid for any contact stress distribution and, as was the case for all the previous examples, we should expect the upper bound to be closer (and probably in most cases identical) to the exact solution. Thus, we can expect all information about the shakedown load to be contained in the elastic solution.

7 Conclusions

Traditionally, computational shakedown analysis takes its point of origin in computational limit analysis, the idea being that the former is a generalization of the latter. In this way optimization algorithms initially developed for limit analysis can usually be extended to shakedown analysis in a relatively straightforward manner. However, despite their mathematical similarities, the two problems of limit and shakedown analysis are physically very different. Indeed, these differences call for techniques which are quite foreign to limit analysis. Thus, in shakedown analysis the elastic solution is of paramount importance and requires much more attention to the finite element discretization than is usually the case in limit analysis. Unawareness of this fact will inevitably lead to unsatisfactory, and in most cases, it seems, unsafe solutions.

For many problems it is not practically feasible to compute sufficiently accurate elastic solutions using mesh refinement strategies based on simple systematic subdivision of structured meshes. The uniformly loaded half–space treated in Sects. 6.3 and 6.4 are examples of such problems. It seems therefore that adaptive mesh refinement has much to offer in computational shakedown analysis. This is the case both for the problems treated here, as well as for more complicated ones including, for example, temperature dependent yield strengths. Besides being crucial in obtaining accurate solutions, an exact (or a good approximate) elastic solution also enables the computation of pseudo upper and lower bounds. This is achieved by application of the rigorous upper and lower finite element methods previously developed for limit analysis.

Finally, concerning the specific numerical and analytical results obtained, we can summarize the following facts:

- If the elastic stress distribution contains any singularities, the exact shakedown load is equal to zero. This situation arises, for example, where the boundary conditions change abruptly, such as at supports.
- If the load domain consists of two points with one of them being zero, the plastic shakedown limit is twice the elastic limit. This rule applies only to this special case.
- If the exact elastic stress field is known the exact shakedown multiplier can be computed in the case where a single load set varies cyclically between two extremes of equal magnitude and opposite sign. The shakedown limit is here equal to the elastic limit. This result generalizes to load domains of the type shown in Fig. 3.
- Since both the plastic shakedown and the plastic collapse multipliers constitute upper bounds on the elastic shakedown multiplier, the minimum of these will also be an upper bound. For all the problems treated here this bound coincides with the elastic shakedown multiplier. Although this result can not be shown to hold in general, we do not know of any examples (homogeneous two or three dimensional structures modeled by solid elements and subjected to mechanical loading) which contradicts it.

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