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# Equivalent stress approach in modelling unsaturated soils

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SUMMARY

This article presents an equivalent stress approach that can be used in many elastoplastic constitutive models for unsaturated soils. The use of the equivalent stress leads to a modified yield locus that is independent of the suction. In addition, the equivalent stress becomes the major stress variable, with suction required only as an additional variable in calculations. The model on the basis of equivalent stress predicts exactly the same soil behaviour, with the sole difference being the use of equivalent stress instead of original stress variables. This article also presents the equivalent stress formulations of several constitutive models for unsaturated soils, including the Barcelona Basic Model. The predictions from these models remain unchanged, with the only difference being in their implementation. Finally, the equivalent stress approach and the net stress approach are compared for the Barcelona Basic Model. Copyright © 2011 John Wiley & Sons, Ltd.

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## 1. CHALLENGES IN THE CONSTITUTIVE MODELLING OF UNSATURATED SOILS

In saturated soils, it is assumed that all the pores between the soil grains are filled with porous fluid and no gas phase is present. Such an assumption is often valid, especially for soils below the water table. However, soils above the water table are frequently only partially saturated, which means that the pores between soil grains are filled with not only porous fluid but also some air bubbles or even a continuous gas phase. In such soil, commonly known as unsaturated soil, additional effects due to unsaturation are present, mostly because of capillary effects in the pore fluid. They exhibit a suction, $s$, defined as the difference between pore air and pore water pressure, which replaces the pore water pressure of saturated soils. Generally, for aggregated soils, the presence and the increase of suction lead to stiffer and more shear-resistant soil; however, when suction is reduced, this additional stiffness is also lowered correspondingly.

One of the most important features of unsaturated soils behaviour is the occurrence of a collapse. For a collapse to occur, the soil with high suction must be loaded with a mean net stress beyond the plastic limit of the saturated soil. Such soil will exhibit much lower deformation during loading as compared with the saturated soil because of the effects of unsaturation. Once the unsaturated soil is wetted (without load reduction), it may collapse; therefore, the volume of the soil must be reduced although the external load is constant. Finally, in a fully saturated state, the specific volume of the soil would correspond to that of the saturated soil. The ability to model collapse reliably is required for a constitutive model for unsaturated soils.

To model collapse, constitutive models most often assume that the yield locus enlarges when the suction is increased and contracts when the suction is reduced. In contrast to saturated soils, this change in the size of the yield locus is not associated with any plastic deformations; the plastic deformations occur only when the stress state fulfils the suction-dependent yield locus equation.

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The models for unsaturated soils use either a traditional net stress as the stress variable or a combination of net stress and suction as the new stress variable. This choice has been the subject of intense research, including contributions by Bishop [1], Jennings and Burland [2], Houlsby [3], Hutter *et al*. [4], Li [5] and Khalili *et al*. [6]. Additional information on the subject may be found in a recent review article by Laloui and Nuth [7].

Several constitutive models for unsaturated soils exist. Elastoplastic constitutive models on the basis of the net stress approach have been proposed by Alonso *et al*. [8], Cui and Delage [9], Sun *et al*. [10], Sheng *et al*. [11] and Kohler and Hofstetter [12], among others, whereas models with complex stress variables have been given by Bolzon *et al*. [13], Jommi [14], Wheeler *et al*. [15], Gallipoli *et al*. [16], Russel and Khalili [17] and Romero and Jommi [18].

The article shows that many unsaturated soil models can be reformulated into equivalent stress models. The reformulated models predict exactly the same soil behaviour as the original models; the only difference is that their yield loci are independent of suction and depend on equivalent stresses only.

## 2. INTRODUCING EQUIVALENT STRESSES INTO A CONSTITUTIVE MODEL

The yield loci of constitutive models can often be normalised, that is, divided by a factor to reduce them to a standard size. These normalised yield loci have better numerical properties for the implementation of finite element codes (e.g. see [19, 20]). If such a normalisation is possible for a constitutive model for unsaturated soils, one can normalise the yield locus to the corresponding yield locus for the saturated soil instead of normalising it to a constant size. This normalisation would effectively mean that the new normalised yield locus will not depend on suction, that is, suction changes will not lead to changes in the yield locus size. Assuming that the hardening parameter for saturated conditions is the maximum mean stress, the normalisation of the yield locus can be generally achieved by multiplying the original yield locus by a factor composed of the saturated hardening parameter divided by the total length of the yield locus.

The second step is to choose the equivalent stress variables such that the yield locus will depend solely on them. This, again, can usually be achieved by multiplying the original stress variables by a factor that is similar to that applied to the yield locus. To complete the formulation, the original stress variables must be replaced by the equivalent stresses in the model equations. As an example, the complete formulation of the Barcelona Basic Model [8] in equivalent stresses is given below, as well as the partial formulations of the constitutive models proposed by Romero and Jommi [18] and Kohler and Hofstetter [12]. The equivalent stress formulation is relatively simple for most of the cases presented, but for some constitutive models, it may be computationally difficult or even not possible (e.g. when the yield surface is a combination of yield surfaces, a single equivalent stress cannot describe yielding in a form that is independent of suction, unless it is defined separately for each yield surface).

### *2.1. Normalisation of the yield locus for the Barcelona Basic Model*

In the case of the Barcelona Basic Model, the original yield locus reads

$$F_{\mathrm{BBM}} = q^2 - M^2(p + p_{\mathrm{s}})(p_0 - p) = 0 \tag{1}$$

where

$$p_{\mathrm{s}} = ks, \qquad p_0 = p^c\left(\frac{p_0^*}{p^c}\right)^{[\lambda(0)-\kappa]/[\lambda(s)-\kappa]}, \qquad \lambda(s) = \lambda(0)\left[(1-r)e^{-\beta s} + r\right] \tag{2}$$



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and $\lambda(0)$, $\kappa$, M, $p^c$, $k$, $r$ and $\beta$ are the model parameters and $p_0^*$ is the hardening variable (see [8]). The yield locus can be scaled by dividing it by a factor that is dependent on the hardening parameter $p_0^*$ and the size of the yield locus $p_0$+$p_s$

$$\left(\frac{p_0 + p_s}{p_0^*}\right)^2 \tag{3}$$

to obtain the normalised yield locus

$$F_{\mathrm{BBMn}} = \left(\frac{qp_0^*}{p_0 + p_s}\right)^2 - M^2 p_0^{*2} \frac{(p + p_s)}{p_0 + p_s} \frac{(p_0 - p)}{p_0 + p_s} = 0 \tag{4}$$

Expressing Equation (4) in terms of the equivalent stresses $p'$ and $q'$

$$q'_{\mathrm{BBM}} = \frac{p_0^* q}{p_0 + p_s} \qquad p'_{\mathrm{BBM}} = \frac{p_0^*(p + p_s)}{p_0 + p_s} \tag{5}$$

the final yield locus equation is obtained as

$$F_{\mathrm{BBMn}} = {q'_{\mathrm{BBM}}}^2 - M^2 p'_{\mathrm{BBM}} \left(p_0^* - p'_{\mathrm{BBM}}\right) = 0 \tag{6}$$

Note that when the full stress tensor is required for calculations, the equivalent stress components become

$$\sigma'_{ij\mathrm{BBM}} = \frac{p_0^*}{p_0 + p_s}\ \sigma'_{ij} + \frac{1}{3}\delta_{ij} \frac{p_s p_0^*}{p_0 + p_s} \tag{7}$$

The normalised yield locus is independent of the suction, and its size depends only on the hardening parameter $p_0^*$.

## 2.2.. *Barcelona Basic Model formulation with equivalent stresses*

Net stresses can be recovered from equivalent stresses as

$$q = q'_{\mathrm{BBM}} \frac{p_0 + p_s}{p_0^*} \qquad p = p'_{\mathrm{BBM}} \frac{p_0 + p_s}{p_0^*} - p_s \tag{8}$$

Then, all the equations of the original model need to have the net stresses replaced in line with Equation (8). The specific volume is calculated as

$$v = N(s) - \lambda(s) \ln\left(\frac{p_0}{p^c}\right) + \kappa \ln\left(\frac{p_0}{p}\right) = N(s) - \lambda(s) \ln\left(\frac{p_0}{p^c}\right) + \kappa \ln\left(\frac{p_0 p_0^*}{p'_{\mathrm{BBM}}(p_0 + p_s) - p_s p_0^*}\right) \tag{9}$$

with the volumetric strain increment following as

$$\Delta\varepsilon_v = \ln\frac{v}{v_0} \tag{10}$$



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The increment of the volumetric plastic strain is the same as in the original formulation of the model because it depends on the change of the hardening parameter only. The change of volume due to suction and the suction yield locus also remain unaffected. The flow rule is

$$\frac{\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{p}}}{\mathrm{d}\varepsilon_{\mathrm{q}}^{\mathrm{p}}}=\frac{\frac{\partial F_n}{\partial p}}{\alpha\frac{\partial F_n}{\partial q}}=\frac{1}{\alpha}\;\frac{\frac{\partial F_n}{\partial p'_{\mathrm{BBM}}}\frac{\partial p'_{\mathrm{BBM}}}{\partial p}+\frac{\partial F_n}{\partial q'_{\mathrm{BBM}}}\frac{\partial q'_{\mathrm{BBM}}}{\partial p}}{\frac{\partial F_n}{\partial p'_{\mathrm{BBM}}}\frac{\partial p'_{\mathrm{BBM}}}{\partial q}+\frac{\partial F_n}{\partial q'_{\mathrm{BBM}}}\frac{\partial q'_{\mathrm{BBM}}}{\partial q}} \tag{11}$$

where $\alpha$ is

$$\alpha=\frac{M(M-9)(M-3)}{9(6-M)}\frac{1}{1-\kappa/\lambda(0)} \tag{12}$$

The relevant differentials are now

$$\begin{aligned}
&\frac{\partial F_n}{\partial p'_{\mathrm{BBM}}}=M^2\left(2p'_{\mathrm{BBM}}-p_0^*\right) &&\frac{\partial F_n}{\partial q'_{\mathrm{BBM}}}=2q'_{\mathrm{BBM}}\\
&\frac{\partial p'_{\mathrm{BBM}}}{\partial p}=\frac{p_0^*}{p_0+p_{\mathrm{s}}} &&\frac{\partial p'_{\mathrm{BBM}}}{\partial q}=0\\
&\frac{\partial q'_{\mathrm{BBM}}}{\partial p}=0 &&\frac{\partial q'_{\mathrm{BBM}}}{\partial q}=\frac{p_0^*}{p_0+p_{\mathrm{s}}}
\end{aligned} \tag{13}$$

Thus, the ratio of the plastic strains is

$$\frac{\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{p}}}{\mathrm{d}\varepsilon_{\mathrm{q}}^{\mathrm{p}}}=\frac{1}{\alpha}\frac{M^2\left(2p'_{\mathrm{BBM}}-p_0^*\right)\frac{p_0^*}{p_0+p_{\mathrm{s}}}}{2q'_{\mathrm{BBM}}\frac{p_0^*}{p_0+p_{\mathrm{s}}}}=\frac{1}{\alpha}\frac{M^2\left(2p'_{\mathrm{BBM}}-p_0^*\right)}{2q'_{\mathrm{BBM}}} \tag{14}$$

The infinitesimal change of volumetric elastic strain is computed from two parts, which stem from the mean net stress change and the suction change:

$$\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{el}}=\frac{\kappa}{v}\frac{\mathrm{d}p}{p},\qquad \mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{es}}=\frac{\kappa_{\mathrm{s}}}{v}\frac{\mathrm{d}s}{s+p_{\mathrm{at}}} \tag{15}$$

Under the usual assumption of constant specific volume in the increment, the elastic strain increment can be integrated as

$$\Delta\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{el}}\cong\frac{\kappa}{v}\ln\frac{p+\Delta p}{p},\qquad \Delta\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{es}}\cong\frac{\kappa_{\mathrm{s}}}{v}\ln\frac{s+\Delta s+p_{\mathrm{at}}}{s+p_{\mathrm{at}}} \tag{16}$$

The previous equation can be written in terms of equivalent stress as

$$\Delta\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{el}}=\frac{\kappa}{v}\ln\frac{p'_{\mathrm{BBM,final}}\left(p_{0,\mathrm{final}}+p_{\mathrm{s,final}}\right)-p_0^*p_{\mathrm{s,final}}}{p'_{\mathrm{BBM}}\left(p_0+p_{\mathrm{s}}\right)-p_0^*p_{\mathrm{s}}} \tag{17}$$

where the subscript final refers to values at the end of the increment. The elastic shear strain increment can be calculated as

$$\mathrm{d}\varepsilon_{\mathrm{q}}^{\mathrm{el}} = \frac{1}{3}Gdq, \qquad \Delta\varepsilon_{\mathrm{q}}^{\mathrm{el}} = \frac{1}{3}G\Delta q \tag{18}$$

which, in terms of equivalent stresses, can be written as

$$\Delta\varepsilon_{\mathrm{q}}^{\mathrm{el}} = \frac{1}{3}\frac{G}{p_0^*}\left(q'_{\mathrm{BBM,final}}\left(p_{0,\mathrm{final}} + p_{\mathrm{s,final}}\right) - q'_{\mathrm{BBM}}\left(p_0 + p_{\mathrm{s}}\right)\right) \tag{19}$$

Equations (17) and (19) for the elastic strain increment due to changes in the net stress are probably less convenient to use in the form of equivalent stress. This is, to some degree, because they were created to be simple and intuitive to use in the net stress space.

To implement the model in a finite element code, an elastoplastic tangent matrix is also required. Here, the matrix will be similar to that of the Barcelona Basic Model (e.g. see [20]), except that the differentials of the yield locus and potential surface need to be computed in terms of the equivalent stresses.

### *2.3. Normalisation of the yield locus for the Romero and Jommi model*

The model proposed by Romero and Jommi [18] for unsaturated soils extends the work of Gallipoli *et al*. [16] and allows for an anisotropic response by incorporating the ideas of Dafalias [21]. The yield locus equation reads

$$F_{\mathrm{RJ}} = \left(q - M_\alpha \hat{p}\right)^2 - \left(M^2 - M_\alpha^2\right)\hat{p}\left(\hat{p}_0 - \hat{p}\right) = 0 \tag{20}$$

where

$$\hat{p}_0 = p_0^*\left[1 + b_1\left(e^{b_2(1-S_r)} - 1\right)\right] \tag{21}$$

$$\mathrm{d}M_\alpha = c\left|\mathrm{d}\varepsilon_v^p\right|\left(q/\hat{p} - \xi M_\alpha\right) \tag{22}$$

$$\hat{p} = (p - u_a) + S_r s \overset{u_a=0}{=} p + S_r s \tag{23}$$

and $b_1$, $b_2$, c, $\xi$ and $M$ are the model parameters, $p_0^*$ is a hardening parameter for saturated conditions, $\hat{p}$ is the mean skeleton stress, $S_r$ denotes the degree of saturation, $s$ denotes suction and $\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{pl}}$ is the increment of plastic volumetric strain. After the division of the yield locus equation by the factor

$$\left(\frac{\hat{p}_0}{p_0^*}\right)^2 \tag{24}$$

the normalised yield locus becomes

$$F_{\mathrm{RJn}} = \left(\frac{p_0^* q - M_\alpha p_0^* \hat{p}}{\hat{p}_0}\right)^2 - \left(M^2 - M_\alpha^2\right)\frac{\hat{p}p_0^*}{\hat{p}_0}\left(p_0^* - \frac{\hat{p}p_0^*}{\hat{p}_0}\right) = 0 \tag{25}$$

Introducing the equivalent stresses

$$q'_{\mathrm{RJ}} = \frac{p_0^* q}{\hat{p}_0} \qquad p'_{\mathrm{RJ}} = \frac{p_0^*\hat{p}}{\hat{p}_0} \tag{26}$$



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the normalized yield locus can be written as

$$F_{\mathrm{RJn}} = (q'_{\mathrm{RJ}} - M_\alpha p'_{\mathrm{RJ}})^2 - \left(M^2 - M_\alpha^2\right)p'_{\mathrm{RJ}}\left(p_0^* - p'_{\mathrm{RJ}}\right) = 0 \tag{27}$$

with

$$\mathrm{d}M_\alpha = c\left|\mathrm{d}\varepsilon_v^p\right|(q'_{\mathrm{RJ}}/p'_{\mathrm{RJ}} - \xi M_\alpha) \tag{28}$$

because the ratio of the equivalent stresses is the same as that of the original stresses $\hat{p}$ and $q$. The resulting yield locus does not depend on suction, and its equation is the same as for saturated soils, although the stresses $\hat{p}$ and $q$ are replaced by the equivalent stresses. Should the full stress tensor be needed, the equivalent stress components can be computed as

$$\sigma'_{ij} = \frac{p_0^*}{\hat{p}_0}\hat{\sigma}_{ij} \tag{29}$$

Finally, for use in the constitutive model equations, net stresses can be recovered from equivalent stresses using the following relations:

$$q = \frac{\hat{p}_0 q'_{\mathrm{RJ}}}{p_0^*} \qquad \hat{p} = \frac{\hat{p}_0 p'_{\mathrm{RJ}}}{p_0^*} \tag{30}$$

The remaining equations may be obtained in a similar way to that shown in the Barcelona Basic Model by substituting $\hat{p}$ and $q$ with the equivalent stresses $p'_{\mathrm{RJ}}$ and $q'_{\mathrm{RJ}}$.

The specific volume computed as

$$v = N(0) - \lambda \ln\frac{p_0^*}{\hat{p}_c} - \kappa \ln\frac{\hat{p}}{p_0^*} \tag{31}$$

when described in the equivalent stresses reads

$$v = N(0) - \lambda \ln\frac{p_0^*}{\hat{p}_c} - \kappa \ln\frac{\hat{p}_0 p'_{\mathrm{RJ}}}{\left[p_0^*\right]^2} \tag{32}$$

The elastic equations

$$\Delta q' = 3G\Delta\varepsilon_{\mathrm{s}}^{\mathrm{e}} \qquad \hat{p}_{\mathrm{new}} = \hat{p}e^{\frac{v}{\kappa}\Delta\varepsilon_{\mathrm{v}}^{\mathrm{e}}} \tag{33}$$

after introducing the constitutive stresses become

$$\Delta q'_{\mathrm{RJ}} = 3G\frac{p_0^*}{p_0}\Delta\varepsilon_{\mathrm{s}}^{\mathrm{e}} \qquad p'_{\mathrm{RJ,new}} = p'_{\mathrm{RJ}}e^{\frac{v}{\kappa}\Delta\varepsilon_{\mathrm{v}}^{\mathrm{e}}} \tag{34}$$

As the hardening rule is dependent on the hardening parameter $p_0^*$ only, it remains unaffected

$$\mathrm{d}p_0^* = \frac{v p_0^*}{\lambda - \kappa}\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{p}} \tag{35}$$

The plastic strain ratio can be obtained as

$$\frac{\mathrm{d}\varepsilon_{\mathrm{v}}^{\mathrm{p}}}{\mathrm{d}\varepsilon_{\mathrm{q}}^{\mathrm{p}}}=\frac{\frac{\partial F_{\mathrm{RJn}}}{\partial p}}{\frac{\partial F_{\mathrm{RJn}}}{\partial q}}=\frac{\frac{\partial F_{\mathrm{RJn}}}{\partial p'_{\mathrm{RJ}}}\frac{\partial p'_{\mathrm{RJ}}}{\partial \hat{p}}+\frac{\partial F_{\mathrm{RJn}}}{\partial q'_{\mathrm{RJ}}}\frac{\partial q'_{\mathrm{RJ}}}{\partial \hat{p}}}{\frac{\partial F_{\mathrm{RJn}}}{\partial p'_{\mathrm{RJ}}}\frac{\partial p'_{\mathrm{RJ}}}{\partial q}+\frac{\partial F_{\mathrm{RJn}}}{\partial q'_{\mathrm{RJ}}}\frac{\partial q'_{\mathrm{RJ}}}{\partial q}} \tag{36}$$

where

$$\begin{aligned}&\frac{\partial p'_{\mathrm{BBM}}}{\partial \hat{p}}=\frac{p_0^*}{\hat{p}_0} && \frac{\partial p'_{\mathrm{BBM}}}{\partial q}=0\\ &\frac{\partial q'_{\mathrm{BBM}}}{\partial p}=0 && \frac{\partial q'_{\mathrm{BBM}}}{\partial q}=\frac{p_0^*}{\hat{p}_0}\end{aligned} \tag{37}$$

Further details on the stress integration algorithms suitable for the model may be found in Cattaneo *et al*. [22].

### 2.4. Normalisation of the yield locus for the model proposed by Kohler and Hofstetter

This example shows that the normalised yield locus is also possible for the unsaturated cap model of Kohler and Hofstetter [12], although it is more complex than the other cases. This is because the evolution of the cap model surface with suction increase is not homothetic, that is, the resulting surface is not a simple scaling of the initial surface but rather involves the individual scaling of each of the surfaces. The yield loci (the shear failure surface for the hardening parameter $I''_1 \leqslant \kappa(s)$ and the cap for $I''_1 > \kappa(s)$) are

$$F_{\mathrm{KHs}}=L(\vartheta)||\mathbf{s}||-F_{\mathrm{e}}\left(I''_1\right)-F_{\mathrm{s}}(s)=\left(\frac{1-\omega\cos3\vartheta}{1-\omega}\right)^{-\eta}||\mathbf{s}||-\alpha-\theta I''_1-ks=0 \tag{38}$$

$$F_{\mathrm{KHc}}=F_{\mathrm{c}}\left(||\mathbf{s}||,I''_1,\vartheta,\kappa(s)\right)-F_{\mathrm{e}}(\kappa(s))-F_{\mathrm{s}}(s)=\sqrt{L^2(\vartheta)||\mathbf{s}||^2+\left(\frac{I''_1-\kappa(s)}{R}\right)^2}-\alpha-\theta\kappa(s)-ks=0 \tag{39}$$

The latter equation may be written in the more usual form of an ellipse equation:

$$\frac{L(\vartheta)||\mathbf{s}||}{\alpha+\theta\kappa(s)+ks}=\sqrt{1-\left(\frac{I''_1-\kappa(s)}{R(\alpha+\theta\kappa(s)+ks)}\right)^2} \tag{40}$$

The same equation for the saturated case is

$$\frac{L(\vartheta)||\mathbf{s}||}{\alpha+\theta\kappa(0)}=\sqrt{1-\left(\frac{I''_1-\kappa(0)}{R(\alpha+\theta\kappa(0))}\right)^2} \tag{41}$$

Comparing these two forms, it is clear that the right choice of equivalent stresses is

$$\sigma'_{ij}=\frac{\sigma_{ij}(\alpha+\theta\kappa(s)+ks)}{\alpha+\theta\kappa(0)}-\frac{1}{3}\delta_{ij}\left[\kappa(0)\frac{(\alpha+\theta\kappa(s)+ks)}{\alpha+\theta\kappa(0)}+\kappa(s)\right] \tag{42}$$

which leads to the equivalent stresses

$$I'_{1,\mathrm{KH}}=\frac{\left(I''_1-\kappa(0)\right)(\alpha+\theta\kappa(s)+ks)}{\alpha+\theta\kappa(0)}+\kappa(s) \tag{43}$$



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$$||\boldsymbol{s}'_{\mathrm{KH}}|| = \frac{||\boldsymbol{s}||(\alpha + \theta\kappa(s) + ks)}{\alpha + \theta\kappa(0)} \tag{44}$$

and the yield locus equation

$$\frac{L(\vartheta)||\boldsymbol{s}'_{\mathrm{KH}}||}{\alpha + \theta\kappa(0)} = \sqrt{1 - \left(\frac{I'_{1,KH} - \kappa(0)}{R(\alpha + \theta\kappa(0))}\right)^2} \tag{45}$$

Note that the calculation of Lode's angle in $L(\vartheta)$ may be different should the equivalent stresses be used.

Unfortunately, the other part of the yield locus defined by Equation (38) requires another set of equivalent stresses, which are

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\left(\frac{ks}{\theta}\right) \tag{46}$$

This leads to

$$I'_1 = I''_1 - \frac{ks}{\theta} \tag{47}$$

$$||\mathbf{s}'|| = ||\mathbf{s}|| \tag{48}$$

Accordingly, the equations describing the elastic behaviour will be different depending on which yield locus (and set of equivalent stresses) is active. The transition between the two surfaces happens when the original stress $I''_1$ is equal to $\kappa(s)$. This means that the equations change when $I''_1$ is reducing within the cap region of the yield locus and

$$I'_{1,\mathrm{KH}} = \frac{[\kappa(s) - \kappa(0)](\alpha + \theta\kappa(s) + ks)}{\alpha + \theta\kappa(0)} + \kappa(s) \tag{49}$$

or when

$$I'_1 = \kappa(s) - \frac{ks}{\theta} \tag{50}$$

and $I''_1$ is increasing in the conical part of yield locus. As a result, two models (with different sets of equivalent stress–strain relationships) must be considered. This will result in a cumbersome algorithm; therefore, the equivalent stress approach is not recommended for the model proposed by Kohler and Hofstetter [12].

## 3. VALIDATION AND ASSESSMENT OF THE EQUIVALENT STRESS APPROACH

It has been shown that the equivalent stress approach is feasible for several constitutive models for unsaturated soils. The clear advantage of the equivalent stress approach is that the yield locus is independent of the suction. However, the equivalent stress does not have an obvious physical interpretation, and its form is dependent on the constitutive model adopted. To investigate more closely the behaviour of a constitutive formulation using the equivalent stress, the Barcelona Basic Model is now studied. Three tests are considered: the volumetric collapse test with no deviator stress



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and the two shearing and wetting tests. In these tests, the original net stress formulation and the equivalent stress formulation are used. The Barcelona Basic Model parameters are shown in Table I, and the stresses at the given points of all the tests are given in Table II.

The three tests considered are as follows: (i) the isotropic compression test; (ii) the standard shear test; (iii) and the pure shear test. The isotropic compression test (see Figure 1) consists of saturated loading (A–B), drying (B–C), loading (C–D), unloading (D–E), wetting (E–F) and loading (F–G). The results of the test in terms of the net stress and the equivalent stress formulations are given in Figures 2 and 3, respectively.

The shearing test (Figures 4 and 5) consists of loading (A–B), drying (B–C), shearing on a standard triaxial path (C–D), removing the deviator stress (D–E), wetting (E–F) and loading (F–G). The results of the test in terms of the net stress and the equivalent stress formulations are given in Figures 6 and 7, respectively.

The pure shear test (Figures 8 and 9) consists of loading (A–B), drying (B–C), loading (C–D), shearing with deviator stress increment only (D–E) and finally wetting (E–F). The results of the test are given in Figures 10–13.

From the tests, it is clear that the occurrence of yield depends only on the equivalent stress. This is especially evident in the case of isotropic compression because no shear stress is present. However, the other tests do confirm that this is indeed the case. As such, the model plastic behaviour is explicitly dependent not on the suction but rather on the equivalent stresses, which are, in turn, dependent on model parameters, original stress variables and suction.

The normal compression line is maintained and is never exceeded when the equivalent stress approach is used. Such behaviour is similar to that of saturated soils. However, with the net stress approach and yield locus expansion (without any plastic work being done), the normal compression line can be crossed (e.g. compare Figure 2 with Figure 3).

Table I. Barcelona Basic Model Parameters.

| $N(0)$ | $M$ | $G$ (MPa) | $p_c$ (kPa) | $\kappa$ | $\lambda(0)$ | $p_{at}$ (kPa) | $k$ | $\kappa_s$ | $\beta$ (kPa$^{-1}$) | $r$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 2.8 | 0.5 | 20 | 1 | 0.02 | 0.2 | 100 | 0.6 | 0.012 | 0.01 | 0.75 |

Table II. Stress state and hardening parameter values at specified points.

| | Point | $p$ (kPa) | $q$ (kPa) | $s$ (kPa) | $p'$ (kPa) | $q'$ (kPa) | $p_0^*$(kPa) |
|---|---|---|---|---|---|---|---|
| Initial state | A | 10 | 0 | 0 | 10 | 0 | 15 |
| Test 1: volumetric collapse | B | 20 | 0 | 0 | 20 | 0 | 20 |
| | C | 20 | 0 | 200 | 16.3 | 0 | 20 |
| | D | 80 | 0 | 200 | 27.9 | 0 | 27.9 |
| | E | 60 | 0 | 200 | 25.1 | 0 | 27.9 |
| | F | 60 | 0 | 0 | 60 | 0 | 60 |
| | G | 95 | 0 | 0 | 95 | 0 | 95 |
| Test 2: standard triaxial shear | B | 20 | 0 | 0 | 20 | 0 | 20 |
| | C | 20 | 0 | 200 | 16.3 | 0 | 20 |
| | D | 60 | 120 | 200 | 32.8 | 21.9 | 91.2 |
| | E | 60 | 0 | 200 | 32.8 | 0 | 91.2 |
| | F | 60 | 0 | 0 | 60 | 0 | 91.2 |
| | G | 100 | 0 | 0 | 100 | 0 | 100 |
| Test 3: pure shear test | B | 40 | 0 | 0 | 40 | 0 | 40 |
| | C | 40 | 0 | 200 | 25.8 | 0 | 40 |
| | D | 40 | 100 | 200 | 29.0 | 18.1 | 74.3 |
| | E | 40 | 0 | 200 | 29.0 | 0 | 74.3 |
| | F | 40 | 0 | 0 | 40 | 0 | 74.3 |
| | G | 90 | 0 | 0 | 90 | 0 | 90 |

 
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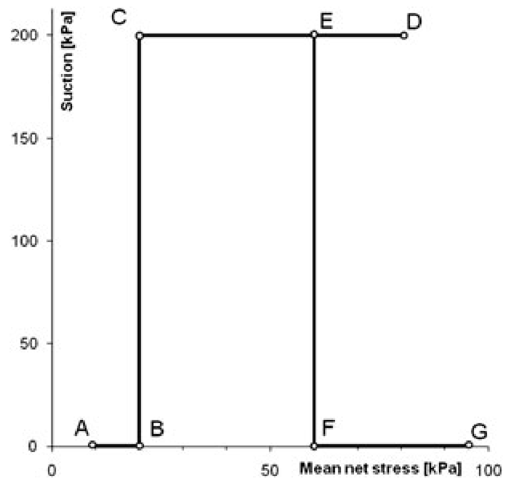


Figure 1. Mean net stress and suction values during isotropic compression test.

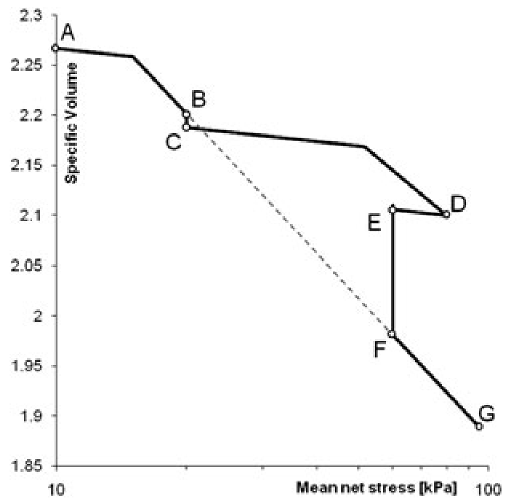


Figure 2. Specific volume versus mean net stress during isotropic compression test.

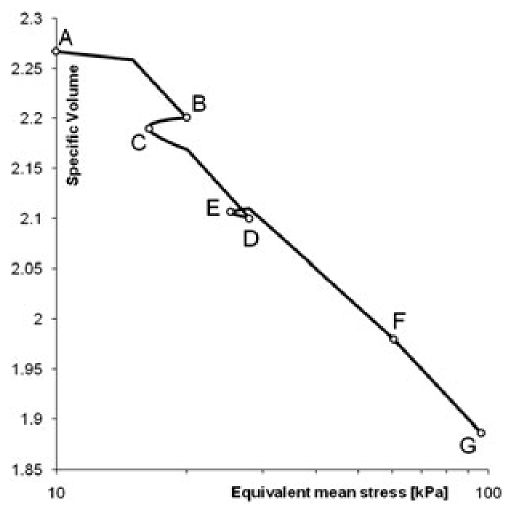


Figure 3. Specific volume versus equivalent mean stress during isotropic compression test.



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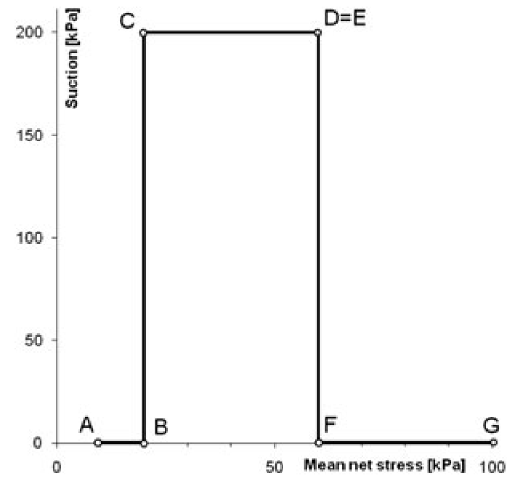


Figure 4. Mean net stress and suction values during standard shear test.

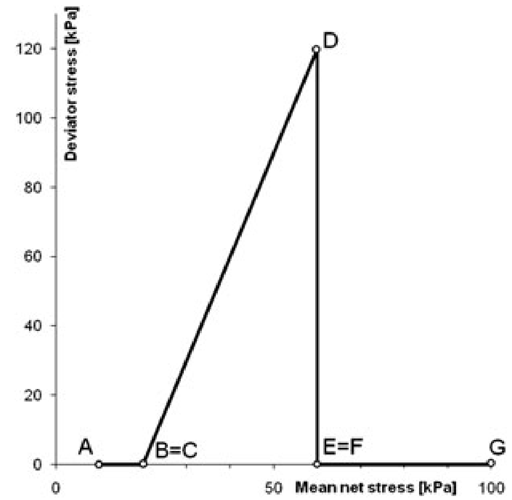


Figure 5. Mean net stress and deviator stress values during standard shear test.

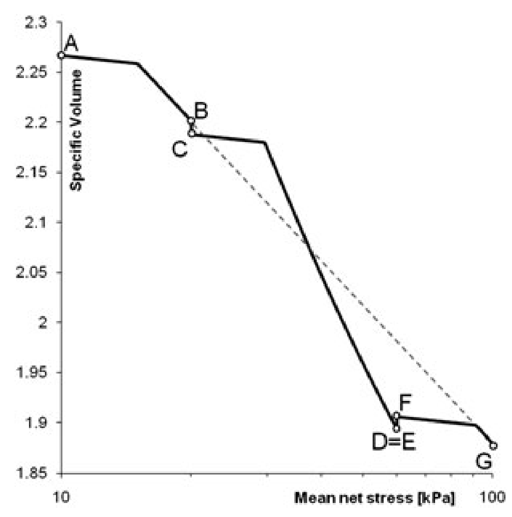


Figure 6. Variation of specific volume and mean net stress during standard shear test.



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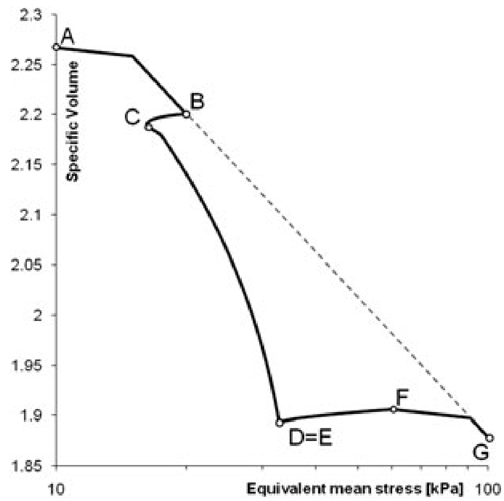


Figure 7. Variation of specific volume and equivalent mean stress during standard shear test.

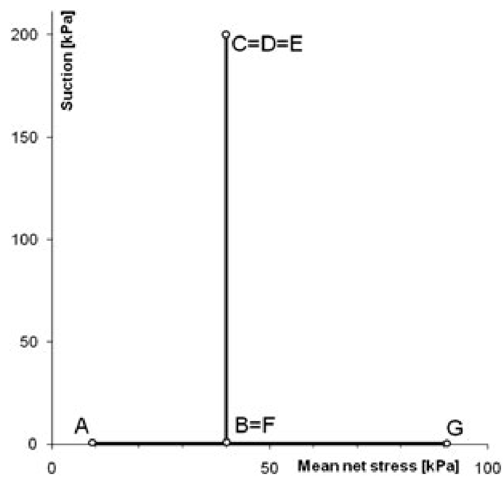


Figure 8. Mean net stress and suction values in pure shear test.

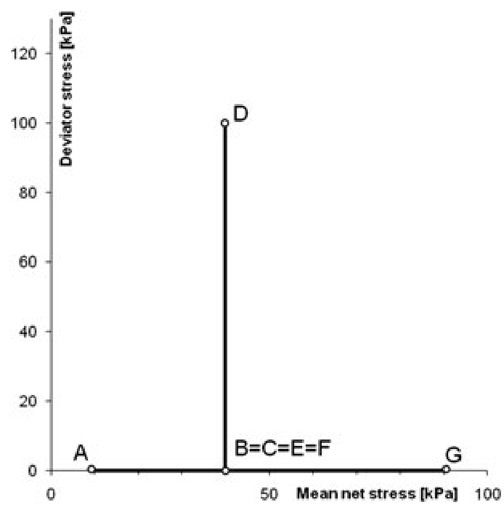


Figure 9. Mean net stress and deviator stress values in pure shear test.



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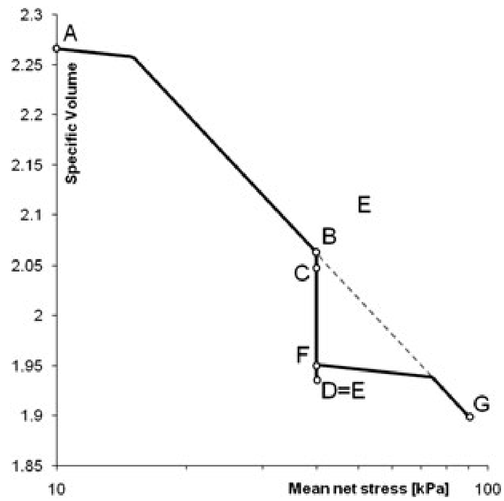


Figure 10. Variation of specific volume and mean net stress in pure shear test.

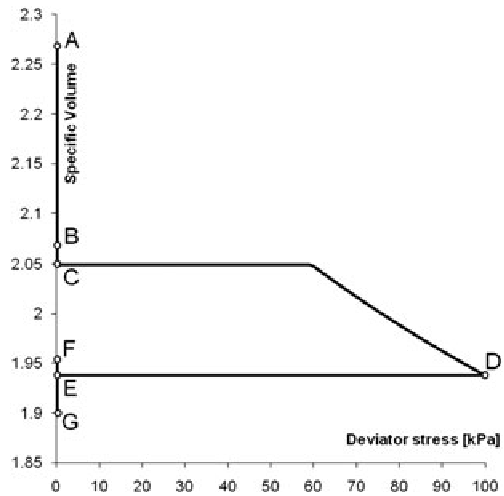


Figure 11. Variation of specific volume and deviator stress in pure shear test.

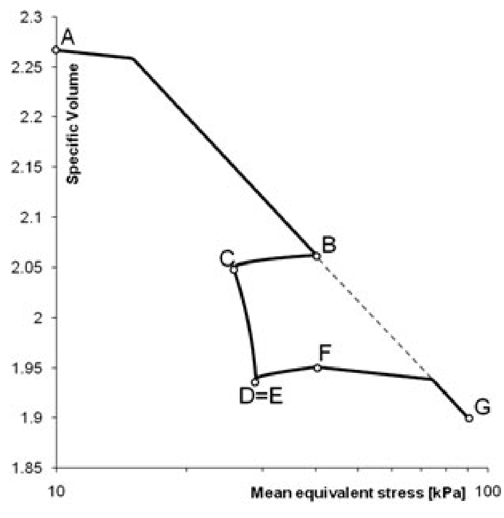


Figure 12. Variation of specific volume and mean equivalent stress in pure shear test.



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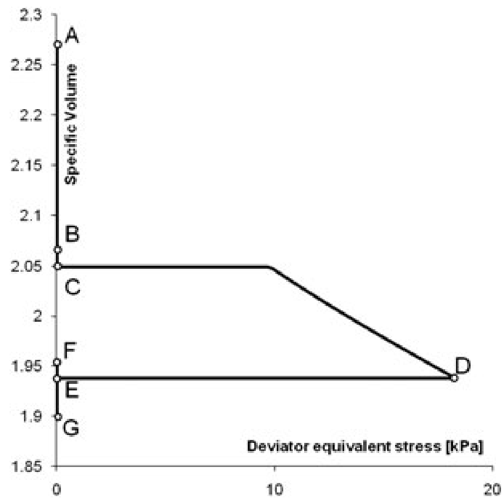


Figure 13. Variation of specific volume and deviator equivalent stress in pure shear test.

It is interesting to note that the equivalent stress generally decreases with increasing suction. This leads to sophisticated equations for predicting the elastic behaviour of the material. It also shows that the equivalent stress is very different to the microstructural stress, which generally increases with increasing suction.

## 4. CONCLUDING REMARKS

The article presents an equivalent stress framework that may be used for unifying constitutive models for unsaturated soils. It is shown that several important models may be formulated in terms of equivalent stress. However, the equivalent stress formulations for models with yield surfaces that are nonhomothetic as the suction increases (such as the Kohler and Hofstetter model) are more complex compared with those models that are homothetic. The equivalent stress concept can be introduced into models that are based on either net stress or more complex stress variables (such as the Bishop stress).

The introduction of the equivalent stress leads to a yield locus that is independent of suction. As such, the yield locus evolution is always connected to plastic work and resembles that of saturated soils. Nonetheless, suction may be still present in the remaining model equations and influence the soil behaviour.

The general feature of the equivalent stress is that it reduces with the application of a suction increment. This is very different to the microstructural stress in soils that—at least at relatively high degrees of saturation—increases with suction increase. It also implies that the equivalent stress does not have any physical interpretation but is rather a mathematical construct dependent on the given model, the stress state and the suction. It also may be interpreted as a coordinate transformation approach where, at the cost of increased stress variable complexity, typical yield behaviour is obtained.

The models considered are not simpler when the equivalent stress is introduced. However, there are some possible gains due to the use of a standard plasticity framework (i.e. the yield locus now depends on a single variable rather than both the hardening parameter and the suction). Therefore, in some cases, an equivalent stress formulation may be numerically beneficial. More importantly, it allows for a new interpretation and unification of elastoplastic models for unsaturated soils that—when described in terms of equivalent stresses—become very similar to each other and to the models for saturated soils.

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