

# A volume–stress model for sands under isotropic and critical stress states

Daichao Sheng, Yangping Yao, and John P. Carter

**Abstract:** A simple volume–stress model for granular soils under isotropic and critical stress states is presented. The model is formulated in the double logarithmic space of void ratio versus mean stress. It has the same number of parameters as used in the Cam Clay models to describe isotropic compression, with one additional parameter to define the critical state curve. The model can qualitatively describe a number of unique features of sand behaviour. Comparison with experimental data indicates that the model is able to predict well the volume change of a range of different sands subjected to isotropic and triaxial compression.

*Key words:* sand, constitutive modelling, volumetric behaviour, particle crushing.

**Résumé :** On présente un simple modèle volume-contrainte pour les sols pulvérulents dans des états isotropes et de contrainte critique. Le modèle est formulé dans le double espace logarithmique de l'indice des vides versus la contrainte moyenne. Il a le même nombre de paramètres qu'utilisés dans les modèles de « Cam Clay » pour décrire la compression isotrope, mais un paramètre additionnel pour définir la courbe de l'état critique. Il peut décrire qualitativement un certain nombre de caractéristiques uniques du comportement du sable. La comparaison avec des données expérimentales indiquent que le modèle est capable de bien prédire le changement de volume d'une plage de différents sables assujettis à une compression triaxiale isotrope.

*Mots-clés :* sable, modèle de comportement, comportement volumétrique, broyage de particules.

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## Introduction

The mechanical behaviour of sands is distinctively different from that of clays, particularly at low stress levels. For example, the typical triaxial test results reported by Lee and Seed (1967), as summarized by Muir Wood (1990) and shown in Fig. 1, suggest the following:

- (1) Sand under isotropic loading is relatively incompressible at low stresses, and large volume changes only occur at very high stress levels where, presumably, particle crushing becomes the dominant mechanism of volume change.
- (2) The isotropic compression curves (ICCs) of sand cannot be represented by straight lines in the space of specific volume  $v$  versus the logarithm of mean stress  $\ln p'$ . How-

ever, these curves seem to approach an asymptotic line at high stresses.

- (3) Sands under triaxial shearing also tend to reach a steady state (continuous shear deformation under constant volume and stress) or a critical state, but the slope of this critical state line in the  $v - \ln p'$  space changes significantly at low stresses.
- (4) The critical state line cannot generally be represented by a straight line in the  $v - \ln p'$  space.
- (5) The slope of the ICC in the  $v - \ln p'$  space is usually smaller than that of the critical state line at low stress levels, but the two may become parallel at very high stresses.

Such behaviour has also been observed for a variety of sands, for example, by Lee and Farhoomand (1969); Carter and Airey (1994); Pestana and Whittle (1995); Verdugo and Ishihara (1996); Yamamuro and Lade (1996); Chuhan et al. (2003); and Lade and Bopp (2005), and has been summarized by Muir Wood (1990); Potts and Zdravkovic (2001); and Mitchell and Soga (2005).

In the literature, the volume change of granular soils is typically modelled by three methods: (i) the single logarithmic approach, (ii) the power or double logarithmic approach, and (iii) the nonlinear function approach in hypoplasticity. The single logarithmic approach assumes a linear or bilinear relationship in the space of void ratio versus the logarithmic mean stress. Such an approach is considered to be more suitable for clays, but has also been used for

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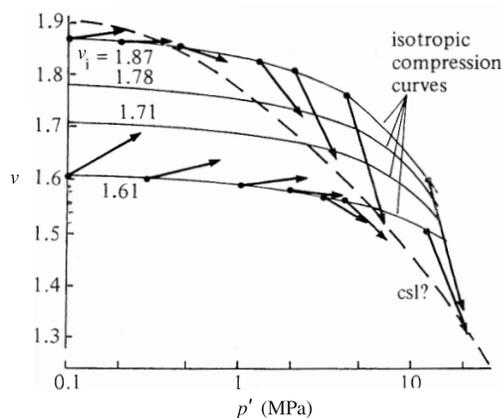
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**Fig. 1.** Triaxial test data on Sacramento River sand (from Muir Wood 1990 based on the data of Lee and Seed 1967).



granular soils (e.g., Been and Jefferies 1985; Yu 1998). The power approach assumes that the elastic bulk modulus is proportional to the mean or vertical stress raised to some power other than unity (Schultze and Moussa 1961). Li and Wang (1998) generalized this approach to express the critical (or steady) state void ratio as a function of the power of the normalized mean stress (normalized against the atmospheric pressure). Because a power function can be represented by a straight line in double logarithmic space, linear or bilinear relationships between the logarithm of void ratio and the logarithm of mean stress have also been used to represent the limit states of sands (Pestana and Whittle 1995). The double logarithmic approach also ensures that the void ratio will never become negative at high stresses. However, one shortcoming with this approach is that it cannot describe the volume change of sands at low stresses, at least not in a simple way. The third approach is based on nonlinear functions in hypoplastic models for granular materials (Bauer and Wu 1993; Bauer 1996; Gudehus 1996). These functions seem to be quite capable of predicting volumetric behaviour of granular soils and have recently been widely adopted (e.g., Wan and Guo 2004). Hypoplastic models have proved to be a very useful alternative to the classic critical state plasticity, particularly for granular soils.

This technical note suggests a volumetric stress–strain model for sands based on the framework of critical state plasticity. The model adopts the double logarithmic approach, but instead of normalizing the mean stress against the atmospheric pressure (as in Li and Wang 1998) or the reference stress defined at a void ratio of one (as in Pestana and Whittle 1995), the model uses a shifting stress to capture the curvature of the ICCs. It can also recover the classic Cam Clay model by adjusting its parameters. The model predictions are then compared with experimental data in the literature.

### Volumetric stress–strain model

As shown by Pestana and Whittle (1995), the asymptotic line that all ICCs approach at high stresses is more or less a straight line in the space defined by the logarithm of void ratio versus the logarithm of mean effective stress. This line is called the limit compression curve (LCC)

$$[1] \quad \text{LCC} : \quad \ln e = \ln N - \lambda \ln p'$$

where  $e$  is the void ratio of the soil,  $\lambda$  is the slope of the LCC in  $\ln e - \ln p'$  space, and  $N$  is the void ratio on the LCC when  $p' = 1$  (unit stress). The existence of the LCC for sands is also supported by the argument that successive fragmentation of particles will eventually lead to an ultimate grain size distribution at which the soil particles are no longer crushable (McDowell et al. 1996; Einav 2007). The LCC then corresponds to the ICC for the soil at this ultimate particle size distribution, and can therefore be approximated by a straight line in  $\ln e - \ln p'$  space (Pestana and Whittle 1995).

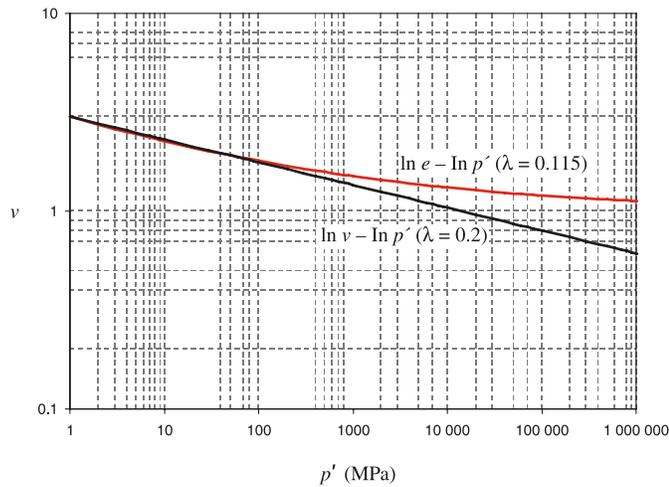
We note that a linear or bilinear relationship in the  $v - \ln p'$  space (Schofield and Wroth 1968) or in the  $\ln v - \ln p'$  space (Butterfield 1979; Hashiguchi 1995), where  $v$  denotes the specific volume ( $v = 1 + e$ ), is often used to approximate the volume change of a clay. Following the same arguments as proposed by Pestana and Whittle (1995), the void ratio  $e$  is preferred instead of the specific volume in the present model. One of these arguments is that the void ratio cannot become negative at high stresses in a linear  $\ln e - \ln p'$  relationship. McDowell (2005) also proposed a justification for the linear  $\ln e - \ln p'$  relationship based on the theory of fractal crushing. Because the change of the specific volume is usually small in sands, generally less than one order of magnitude, a linear  $\ln v - \ln p'$  prediction can be approximated reasonably well by a linear  $\ln e - \ln p'$  prediction, particularly for relatively low stress levels. Figure 2 demonstrates a  $\ln v - \ln p'$  line with  $\lambda = 0.2$  and a  $\ln e - \ln p'$  line with  $\lambda = 0.115$ . The difference between the two becomes apparent only at higher stresses. However, it is noted that the two alternatives lead to different expressions for the elastic bulk modulus, and different values for the parameter  $\lambda$  would be deduced from the same set of experimental data. The definition of the elastic bulk modulus assuming a linear  $\ln e - \ln p'$  relationship will be discussed later.

Unlike the unique isotropic compression line of a normally consolidated clay, the ICCs of a sand are not unique and depend on the initial density of the soil. The term “isotropic compression curve” is used here instead of the “normal compression line” or the “virgin consolidation line” because the concept of overconsolidation is less meaningful for sands. The ICCs for a sand cannot usually be approximated by straight lines in the  $e - \ln p'$  or  $\ln e - \ln p'$  spaces. They are generally very flat at relatively low stresses, but large volume change can occur when particle crushing becomes dominant at high stresses. The particular shape of an ICC also depends on the initial void ratio of the soil. For example, for dense quartz sand, the void ratio does not change much until very high stresses. At these very high stresses, the ICCs all approach the LCC. A simple function that has these properties takes the form

$$[2] \quad \text{ICC} : \quad \ln e = \ln N - \lambda \ln(p' + p'_r)$$

where  $p'_r$  is a shifting stress controlling the curvature of the ICC. It depends on the initial void ratio of the soil as well as on  $\lambda$  and  $N$

**Fig. 2.** Comparison between linear  $\ln e - \ln p'$  and  $\ln v - \ln p'$  predictions ( $N = 3$ ).



$$[3] \quad p'_r = \left(\frac{N}{e_0}\right)^{1/\lambda} - p'_0$$

where  $e_0$  is the void ratio at a known mean stress  $p'_0$  on the ICC. The parameters  $e_0$  and  $p'_0$  can also be regarded as the initial state of the soil. If the initial stress  $p'_0$  is taken as the unit stress where  $N$  is calculated, the ratio between  $N$  and  $e_0$  then reflects the curvature of the ICC. For the same sand under the same initial mean stress, a smaller initial void ratio leads to a larger  $p'_r$  value. Setting  $p'_r = 0$  in eq. [2] recovers the normal compression line of the Cam Clay theory of elastoplasticity in the double logarithmic,  $\ln e - \ln p'$ , space.

The LCC and ICCs are plotted in Fig. 3 for  $N = 1.1$  at  $p' = 1$  MPa and  $\lambda = 0.2$ . It is clear that the slope of the LCC is not a constant in the  $e - \ln p'$  space, but instead it decreases with increasing stress. The LCC is a straight line in the  $\ln e - \ln p'$  space. The ICCs are not straight lines in either  $e - \ln p'$  or  $\ln e - \ln p'$  space. All the ICCs approach the LCC at very high stress levels.

In addition, we also define a critical state curve (CSC)

$$[4] \quad \text{CSC} : \quad \ln e = \ln \Gamma - \lambda \ln(p' + p'_{cr})$$

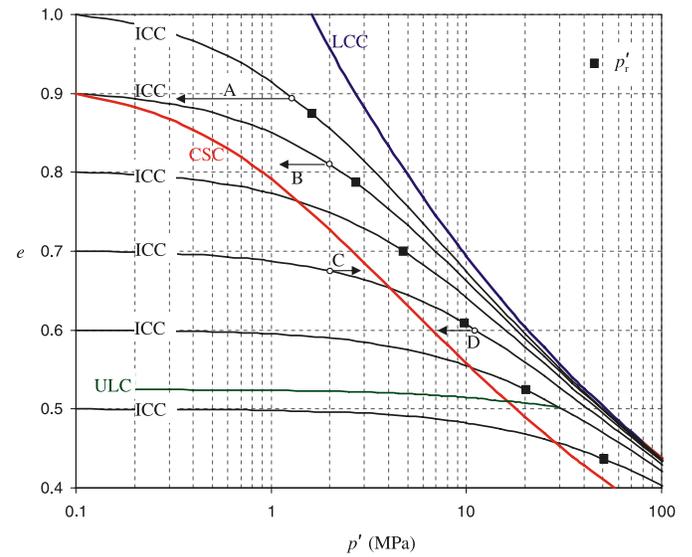
where  $\Gamma$  is the void ratio at critical state when  $p' + p'_{cr} = 1$  (unit stress), and

$$[5] \quad p'_{cr} = \left(\frac{\Gamma}{e_{c0}}\right)^{1/\lambda} - p'_{c0}$$

where  $e_{c0}$  is the void ratio at a known mean stress  $p'_{c0}$  on the CSC. Setting  $p'_{cr} = 0$  recovers the critical state line as defined in the Cam Clay models in the double logarithmic,  $\ln e - \ln p'$ , space.

The CSC is shown in Fig. 3 for a sand with  $\Gamma = 0.9$  at  $p' = 1$  MPa,  $e_{c0} = 0.9$ , and  $p'_{c0} = 0.1$  MPa. As for the ICCs, the CSC is not a straight line in either  $e - \ln p'$  or  $\ln e - \ln p'$  space. However, it approaches a straight line parallel to the limit compression line at very high stresses in the  $\ln e - \ln p'$  space. The vertical distance between the LCC and the asymptotic line of the CSC at very high stresses is

**Fig. 3.** Definition of the reference compression line (LCC), critical state curve (CSC), and normal compression lines (ICC) ( $N = 1.1$ ,  $\lambda = 0.2$ ,  $\Gamma = 0.9$ ,  $e_{c0} = 0.9$  at  $p'_{c0} = 0.1$  MPa).



$$[6] \quad \chi = \ln N - \ln \Gamma$$

In the context of the theory of work hardening plasticity, the value of  $\chi$  depends on the yield function of the specific model used to describe the soil. For example, in the Modified Cam Clay model  $\chi = \lambda \ln 2$ . However, in this note we are only concerned with the volumetric behaviour of the soil.

If the soil is unloaded from  $p'_1$  to  $p'_2$ , the volume change is assumed to be given by

$$[7] \quad \text{Unloading curve} : \quad \ln\left(\frac{e_2}{e_1}\right) = \kappa \ln\left(\frac{p'_1 + p'_r}{p'_2 + p'_r}\right)$$

where  $\kappa$  is a material constant, and  $p'_r$  is defined by eq. [3]. The unloading curve (ULC) defined by eq. [7] is not a straight line in either  $e - \ln p'$  or  $\ln e - \ln p'$  space. Its slope approaches zero at low stresses. One ULC with  $\kappa = 0.05$  is shown in Fig. 3 where the soil with an initial void ratio of 0.6 at  $p' = 1$  MPa was isotropically compressed to 30 MPa and then unloaded to 0.2 MPa. The elastic bulk modulus is then defined as

$$[8] \quad K = \frac{\partial p'}{\partial \varepsilon_v^e} = -(1 + e) \frac{\partial p'}{\partial e} = \left(\frac{p' + p'_r}{\kappa}\right) \left(\frac{1 + e}{e}\right)$$

in which  $e^e$  and  $\varepsilon_v^e$  are the elastic (recoverable) components of void ratio change and volumetric strain. In the previous definition, it is assumed that the total volume occupied by solid particles of the soil does not change, even though the particles crush and the void volume changes.

Reloading can be treated as purely elastic and will follow the same ULC (such as in the Cam Clay models, Vermeer 1978; Yu 1998) or be treated as elastoplastic (such as in Dafalias 1986; Pestana and Whittle 1995; Asaoka 2003; Yao et al. 2004). A good review of constitutive models for granular materials can be found in Potts and Zdravkovic (2001).

The volumetric model presented here contains four parameters:  $N$ ,  $\Gamma$ ,  $\lambda$ , and  $\kappa$ , as well as one critical state void

ratio  $e_{c0}$  at a known mean stress  $p'_{c0}$ . The first four parameters are common to most critical state models (such as Modified Cam Clay), but have slightly different definitions. The parameters  $\lambda$  and  $\kappa$  are measures of soil compressibility, and  $N$  and  $T$  are measures of void ratio at a given stress level. As the proposed volumetric model follows the same framework as that of the critical state models, it can easily be incorporated into a complete constitutive model for sand.

Some qualitative observations can be made from Fig. 3:

- (1) The slopes of the ICCs in the  $e - \ln p'$  space are typically very small at low stresses, and large volume change starts to occur when the mean stress is around  $p'_r$ . Therefore,  $p'_r$  can be interpreted as the stress level where significant particle crushing occurs. A denser sand has a higher  $p'_r$  than a looser sand. Similarly, stronger sand particles (e.g., quartz sands) will tend to crush at higher stresses than weaker particles (e.g., calcareous sands) by having a larger value of the parameter  $N$ . These features will be illustrated by the data to be presented later.
- (2) All ICCs approach the same LCC at very high stress levels. The ICC for a denser sand approaches this LCC at a higher stress than that for a looser sand. The slope of the LCC in the  $e - \ln p'$  space decreases with increasing stress.
- (3) The slope of the CSC changes significantly at low stresses (between 0.1 and 10 MPa for the sand in Fig. 3) and approaches the slope of the LCC in the  $\ln e - \ln p'$  space at very high stresses.
- (4) When an extremely loose sand (with an ICC above the CSC) is sheared under undrained (constant volume) conditions, the stress path can only approach the CSC from the right hand side (path A in Fig. 3), and hence the mean stress decreases and may approach zero, resulting in static liquefaction.
- (5) When a sand is sheared under undrained triaxial conditions at a fixed confining pressure, the stress path can approach the CSC either from the right hand side (path B in Fig. 3), or from the left hand side (path C), depending on the initial void ratio of the soil. The mean stress will decrease for an initially loose sand, but increase for an initially dense sand.
- (6) When a sand is sheared under undrained triaxial conditions at a fixed initial void ratio, the stress path can approach the critical state line either from its right side (path D in Fig. 3), or from its left side (path C), depending on the confining pressure. The mean stress will increase for a low confining pressure (path C), but decreases for an initially high confining pressure (path D).
- (7) When sheared under drained conditions, most sands (with an ICC below the CSC) will show some dilatancy at low stress levels (on the left side of CSC). The sand may become contractive at higher stresses. The stress level at which dilation changes to contraction, or the so-called characteristic state point, depends on the initial void ratio of the soil. The characteristic state point corresponds to the intersection between the ICC and the CSC.
- (8) When sheared under drained conditions, an extremely loose sand (with an ICC above the CSL) can only contract.

## Comparisons between test data and model predictions

The volumetric model presented in this paper is compared to, and validated against, experimental data in this section. It should be noted that the degree of parameterisation of a model is usually equivalent to its flexibility in capturing experimental behaviour. In this context, the presented model uses the same number of parameters as the Cam Clay models for isotropic compression, but one more parameter is added for the critical state prediction.

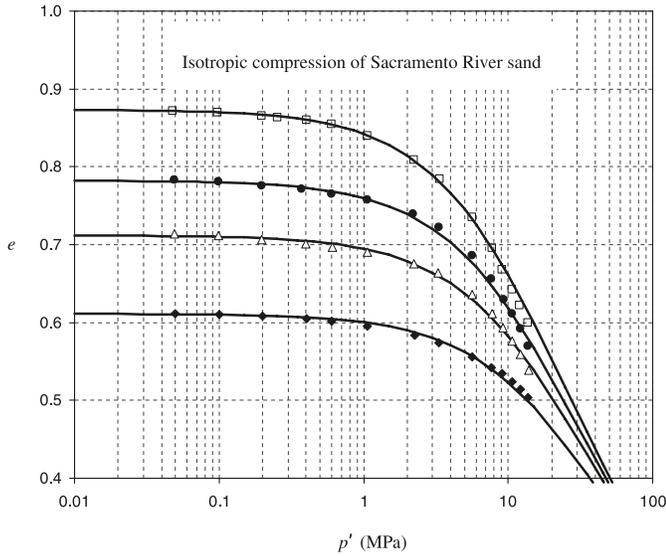
The predicted ICCs are first compared with the test data presented by Lee and Seed (1967) in Fig. 4. The Sacramento River sand was thoroughly washed between the No. 50 (0.297 mm) and the No. 100 (0.149 mm) sieves. The sand grains were mostly feldspar and quartz minerals with subangular and subrounded shapes. The soil was tested under four different initial void ratios. Two parameters ( $\lambda$  and  $N$ ) as well as the initial void ratios were used to generate the predictions in Fig. 4. The parameters  $\lambda$  and  $N$  were determined by best fitting the experimental data. The measured void ratios at mean stress of 0.1 MPa were used to find the shifting stress  $p'_r$ . It is shown that the proposed function for the ICCs, that is, eq. [2], fits the data of Lee and Seed very well. Lee and Seed (1967) also presented triaxial compression data, but their data on critical states are rather scattered, which makes a comparison less meaningful. It is noted that accurate determination of the critical state condition is often difficult in dense sands in which the development of shear bands can often mask or preclude the development of a critical state condition throughout the entire sand sample.

In Fig. 5, eq. [2] is used to model the ICCs of Cambria sand published by Lade and Bopp (2005). The soil was described as a coarse, uniform sand consisting of subangular to well-rounded (mostly quartz) grains with diameters between 0.83 and 2 mm. Again, two parameters used in the prediction (i.e.,  $\lambda = 0.7$  and  $N = 6.7$ ) are determined by best fitting the experimental data. The void ratios at mean stress 0.1 MPa, either interpolated or extrapolated from the experimental data, are used to determine the shifting stress  $p'_r$ . The predictions are not as good as those for the data by Lee and Seed (1967), particularly for the loose sample with an initial void ratio of 0.7. For the medium dense and dense sands, the predicted ICCs are in relatively good agreement with the test data.

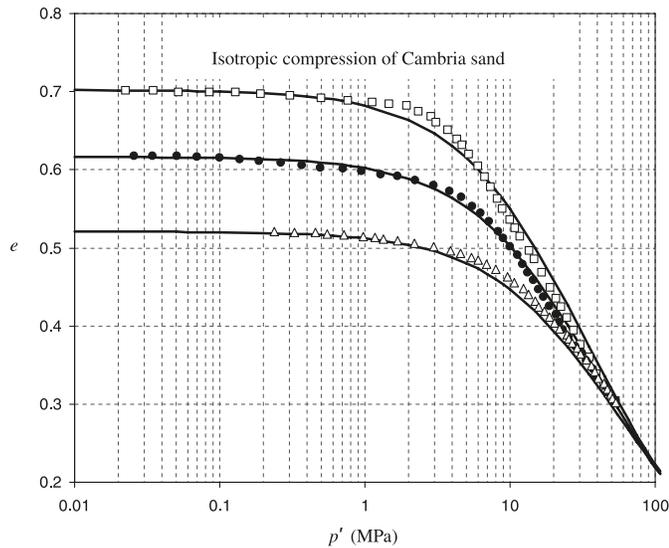
The sands tested by Lee and Seed (1967) and Lade and Bopp (2005) consist mostly of quartz grains. In Fig. 6, the isotropic compression results for a more compressible carbonate sand are compared with the prediction of the proposed model. The material parameters used in the prediction (i.e.,  $\lambda = 0.6$  and  $N = 3.4$ ) are obtained by best fitting the test curve for the loosest sample. The measured void ratios at mean stress of 1 MPa are used to find the shifting stress  $p'_r$ . It is shown that the predictions are again in relatively good agreement with the measured data. It should be noted that eq. [2] applies to freshly prepared samples and does not consider any unloading-reloading or overconsolidation effects on the volume change. If unloading and reloading are assumed to cause only elastic volume change, the initial portions of the predicted curves in Fig. 6 would be even flatter.

In Fig. 7, eq. [4] is used to predict the critical state data

**Fig. 4.** Predicted isotropic compression curves (solid lines) compared with measured data (points) from Lee and Seed (1967) (parameters used in the prediction:  $\lambda = 0.5, N = 3.2$  at  $p' = 1$  MPa).



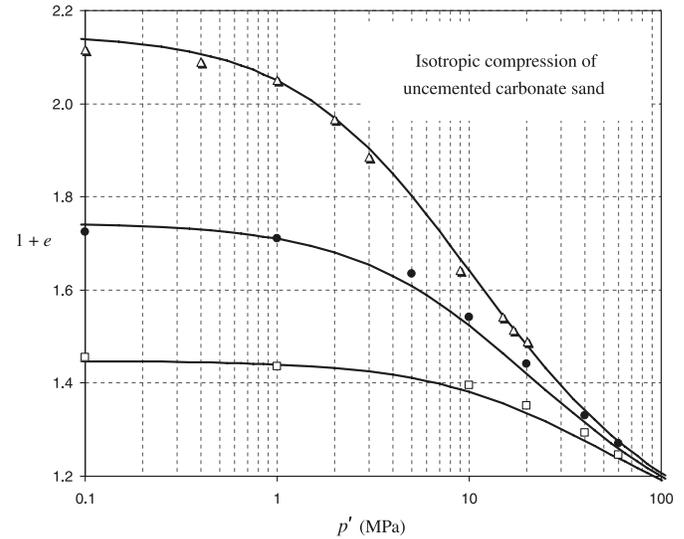
**Fig. 5.** Predicted isotropic compression curves (solid lines) compared with measured data (points) by Lade and Bopp (2005) (parameters:  $\lambda = 0.7, N = 6.7$  at  $p' = 1$  MPa).



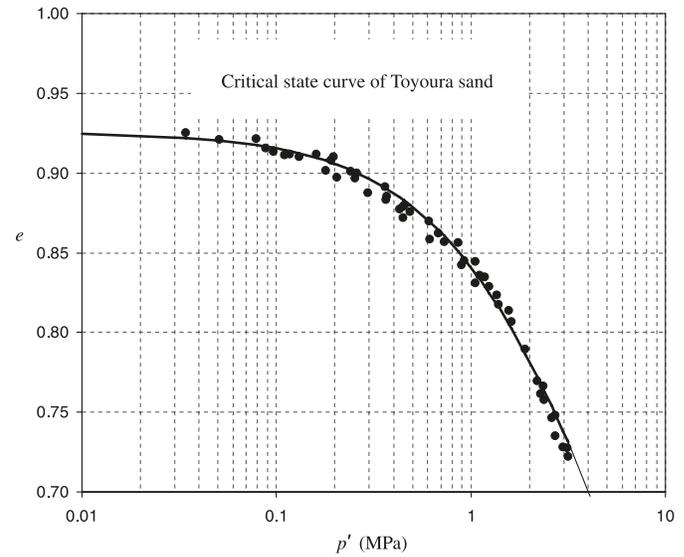
obtained by Verdugo and Ishihara (1996). The soil is the standard Japanese Toyoura sand with a mean grain size  $D_{50} = 0.17$  mm and a coefficient of uniformity  $C_u = 1.7$ . The critical (steady) state data in Fig. 7 were obtained from undrained triaxial compression tests. The two parameters used in the prediction are  $\lambda = 0.3$  and  $\Gamma = 1.24$ , as well as the critical state void ratio  $e_{c0} = 0.925$  at  $p'_{c0} = 0.01$  MPa. It is shown that the proposed equation can fit well the critical state data of Verdugo and Ishihara (1996), at least for the testing stress range between 0.3 and 3 MPa.

In Fig. 8, the test data on Tung-Chung sand obtained by Li and Wang (1998) were used to validate the proposed model. The soil is a fine-to-coarse sand mixed with a small fraction of fragmented shells, with a mean grain size  $D_{50} = 0.33$  mm and a coefficient of uniformity  $C_u = 4.36$ . The pa-

**Fig. 6.** Predicted isotropic compression curves (solid lines) compared with measured data (points) by Carter and Airey (1994) (parameters:  $\lambda = 0.6, N = 3.4$  at  $p' = 1$  MPa).



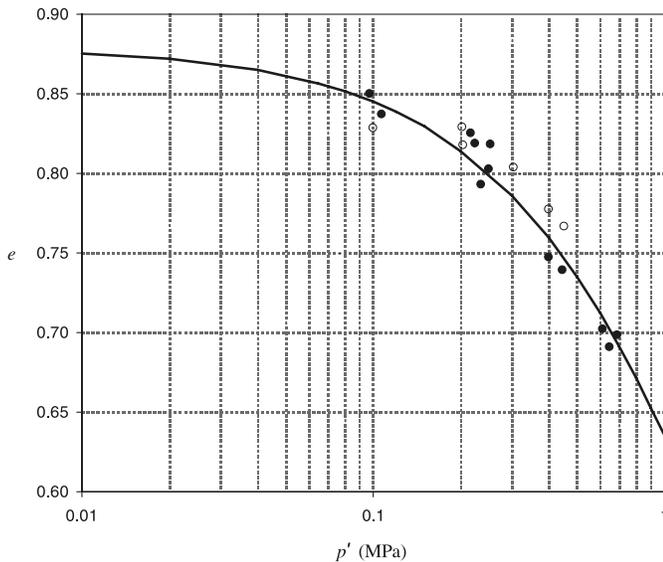
**Fig. 7.** Predicted critical state curve compared with measured data by Verdugo and Ishihara (1996) (parameters used in the prediction:  $\lambda = 0.3, \Gamma = 1.24, e_{c0} = 0.925$  at  $p'_{c0} = 0.01$  MPa).



rameters used in the prediction are  $\lambda = 0.8, \Gamma = 1.53, e_{c0} = 0.875$  at  $p'_{c0} = 0.01$  MPa. The agreement between the prediction and the test data is relatively good. However, it is noted that the stress range used in the tests is relatively small, between 0.1 and 0.7 MPa, which makes the prediction less convincing.

Overall, the proposed volumetric stress–strain model seems to be able to fit experimental data for a variety of sands tested under isotropic and triaxial compression. The model contains the same number of parameters as the Cam Clay models, but unlike the latter it is defined in the double logarithmic,  $\ln e - \ln p'$ , space. The comparisons with experimental data focused on the stress range from low to high (less than 100 MPa). At extremely high stresses, the soil is likely to become effectively incompressible. The current

**Fig. 8.** Predicted critical state curve compared with measured data by Li and Wang (1998) (parameters used in the prediction:  $\lambda = 0.8$ ,  $\Gamma = 1.53$ ,  $e_{c0} = 0.875$  at  $p'_{c0} = 0.01$  MPa). open circle, undrained; solid circle, drained.



model implies a limiting void ratio of zero when the mean stress becomes infinitely large. However, a finite limiting void ratio can easily be incorporated into the model so that the limiting compression curve takes the alternative form

$$[9] \quad \text{LCC} : \ln(e - e_{\text{lim}}) = \ln N - \lambda \ln p'$$

where  $e_{\text{lim}}$  is the limiting void ratio when  $p' \rightarrow \infty$ . In this case, eqs. [2] and [4] have to be modified accordingly

$$[10] \quad \text{ICC} : \ln(e - e_{\text{lim}}) = \ln N - \lambda \ln(p' + p'_t)$$

$$[11] \quad \text{CSC} : \ln(e - e_{\text{lim}}) = \ln \Gamma - \lambda \ln(p' + p'_{\text{cr}})$$

This version of the volumetric model is likely to be more flexible in capturing the behaviour at extremely high stresses, at the cost of one additional model parameter. More experimental validation of the model is certainly desirable to resolve such issues.

## Summary

A new volumetric stress–strain model for sands has been proposed in this technical note. The model is defined in the space of logarithmic void ratio versus logarithmic mean stress. A limiting compression curve is assumed to be a straight line in the double logarithmic space, and this curve corresponds to the ultimate grain size distribution of the soil when its particles are fully crushed. A shifting stress that depends on the initial void ratio and the critical state properties of the soil is used to capture the initial curvature of the ICC. The proposed model uses the same number of soil parameters as the Cam Clay models for isotropic compression and one additional parameter for critical state void ratios. It follows the same framework as most critical state models and can be incorporated easily into a complete constitutive model for sands. It has been shown that this simple model can capture the experimental behaviour of

several types of sands tested under isotropic and triaxial compression.

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