



Incorporating a predefined limit condition in a hypoplastic model by means of stress transformation

Wenxiong Huang^{a,b,*}, Scott Sloan^b, Stephen Fityus^b

^a Department of Engineering Mechanics, Hohai University, Nanjing 210098, China

^b Centre for Geotechnical and Material Modelling, The University of Newcastle, NSW 2308, Australia

ARTICLE INFO

Article history:

Received 3 July 2007

Received in revised form 21 April 2008

Keywords:

Stress transformation

Hypoplasticity

Granular materials

ABSTRACT

Hypoplasticity, as an alternative approach of elastoplasticity for describing inelastic, irreversible behaviour of materials, appreciates the advantages of elegance of mathematical formulation and ease in numerical implementation. A successful hypoplastic model for sand proposed by Gudehus and Bauer, which consists of stress and void ratio as state variables, can capture qualitatively the pressure- and density-dependence with 8 state-independent parameters. In this paper, we demonstrate that by using a stress transformation technique, either the Matsuoka–Nakai failure criterion or the Lade failure criterion can be easily incorporated in the model for the description of critical states. Numerical results are presented to display the improvement in the model performance under three-dimensional stress condition.

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction

In last two decades, a new framework for constitutive modelling of soil behaviour known as hypoplasticity has been developed (e.g. Kolymbas, 2000). In this framework, the incremental stress–strain relation is described directly with a non-linear tensor-valued function without employing the conventional plasticity concept of decomposition of strain rate into elastic and plastic part. This type of constitutive equation was first proposed as an extension of hypoelastic equations (e.g. Kolymbas, 1985). A general formulation for describing the irreversible and rate-independent response of materials was then developed with the stress tensor as a single state variable (Wu and Kolymbas, 1990; Wu, 1992). Later on, the void ratio was added as an additional state variable and it was found (Wu, 1992; Wu and Bauer, 1993) that the pressure-dependency of the material response can be easily modelled by introducing a relevant density factor to the non-linear term. On the

basis of these works, Gudehus (1996) and Bauer (1996) proposed a hypoplastic equation for sands, which can capture, with a single set of material parameters, the main features of the pressure- and density-dependent responses of both loose and dense sand for a wide range of stress. The model can reproduce the contractive volumetric deformation and strain hardening response to shear of loose sand as well as shear dilatancy behaviour with strain softening for dense sand. A further improvement was due to Wolfersdorff (1996) who incorporated a predefined limit stress condition (Matsuoka and Nakai, 1974) into the model. This model has been used in many applications (e.g. Huang et al., 2002a,b; Huang et al., 2007).

Working with the current version of this Gudehus–Bauer model, we found that while the model can produce critical states obeying Matsuoka–Nakai (M–N) criterion, the predicted peak strengths in three-dimensional stress condition are not all satisfactorily. For a triaxial extension test, the model predicts a shear strength lower than that for triaxial compression, which is, compared with experimental results, not realistic. This trend is more significant when an initial denser state is assumed. As a consequence, the peak stress surface in the principal stress space becomes slightly concaved, as shown in Huang et al. (2007). We

* Corresponding author. Address: Centre for Geotechnical and Material Modelling, The University of Newcastle, NSW 2308, Australia. Tel.: +61 2 49216082; fax: +61 2 49216991.

E-mail address: wh670@hhu.edu.cn (W. Huang).

are also not satisfied with a non-constant strength parameter, which was purposely formulated as a function of the Lode angle to cope with the variation of the shear strength with respect to the Lode angle. In the present work, we propose to incorporate a predefined limit stress condition by using a stress transformation technique. The technique, developed in Matsuoka et al. (1999) and Yao and Sun (2000), was initially applied to improve the performance of the Cam clay type models in three-dimensional stress conditions. The technique allows the models to be calibrated in conventional triaxial tests and to reproduce realistic three-dimensional responses. The application of the technique in the hypoplastic model discussed here is straight forward and even simpler, as in hypoplasticity there is no need for computing the derivative of a potential function.

2. A brief description of the Gudehus–Bauer model

The hypoplastic model proposed by Gudehus (1996) and Bauer (1996) (termed as G-B model) is based on the assumption that the mechanical responses of a granular soil rely fully upon the current state, which is sufficiently described by the Cauchy stress σ and the void ratio e . In response to a strain rate input $\dot{\epsilon}$, the change of the state variables are governed by the following equations:

$$\dot{\sigma} = f_s[\hat{a}^2 \dot{\epsilon} + \hat{\sigma}(\hat{\sigma} : \dot{\epsilon}) + f_d \hat{a}(\hat{\sigma} + \hat{\sigma}_d)\|\dot{\epsilon}\|], \quad (1)$$

$$\dot{e} = (1 + e)\dot{e}_v. \quad (2)$$

Here, $\hat{\sigma} = \sigma/\text{tr}\sigma$ is the normalised stress tensor and $\hat{\sigma}_d = \hat{\sigma} - \mathbf{1}/3$ is the deviator of $\hat{\sigma}$, with $\mathbf{1}$ being the second order unit tensor; $\|\dot{\epsilon}\| = \sqrt{\dot{\epsilon} : \dot{\epsilon}}$ represents the Euclidean norm of the strain rate; $\dot{e}_v = \text{tr}\dot{\epsilon}$ is the volumetric strain rate; \hat{a} is the strength parameter whose role will be discussed in detail later. Two scalar factors in Eq. (1), f_s and f_d , are functions of the pressure (or mean stress) $p = -\text{tr}\sigma/3$ and the void ratio e , given as

$$f_d = \left(\frac{e - e_d}{e_c - e_d}\right)^\alpha, \quad f_s = \frac{1}{\hat{\sigma} : \hat{\sigma}} \left(\frac{e_i}{e}\right)^\beta \left(\frac{3p}{h_s}\right)^{1-n} f_b^*, \quad (3)$$

where e_i represents the maximum void ratio of a sand under an isotropic stress condition, e_c and e_d are void ratios of the sand at the critical and the densest states; α and β are two constants for scaling the density-dependent peak stress and the tangential stiffness, and f_b^* is a constant factor which can be determined from the consistent condition for an isotropic compression (Gudehus, 1996). Two material constants h_s and n are introduced to fit the experimental data from isotropic compression tests using the following exponential relation (Bauer, 1996):

$$e_i = e_{i0} \exp[-(3p/h_s)^n]. \quad (4)$$

The same relationship is postulated for e_c and e_d as $e_c/e_{c0} = e_d/e_{d0} = e_i/e_{i0}$. Here e_{c0} , e_{d0} and e_{i0} represent the critical, the densest and the loosest void ratios at a nearly stress-free state.

Eq. (1) defines the stress rate as a non-linear tensor-valued function of the strain rate. The constitutive equation is characterized by factorizing the pressure and density effects into a stiffness factor f_s and a density factor f_d with the tensor

terms being represented with the normalized stress tensor. With factor f_d , the relative density of a sand packing can be quantified by a density index $I_d = (e - e_d)/(e_c - e_d)$, which equals to 0 for a densest state and takes a value of 1 at a critical state. The implementation of the critical state is another salient property of Eq. (1). The critical state concept (e.g. Roscoe et al., 1963; Schofield and Wroth, 1968), which has been widely accepted as a basis of soil mechanics, asserts that a soil element under monotonic shearing will eventually reach a stationary state with a constant stress and a constant volume. The void ratio of a soil at a critical state is independent of the initial density of the soil packing, but a function of the mean stress only. The critical state concept has provided an insight into the soil behaviour, and it also allows for a shear strength that is independent of soil density to be defined for a soil. With respect to the current hypoplastic model Eqs. (1) and (2), a critical state corresponds to $\dot{\sigma} = 0$, $\dot{e} = 0$ and $f_d = 1$, which yields a critical void ratio $e = e_c$, a flow condition of $\hat{\epsilon} := \dot{\epsilon}/\|\dot{\epsilon}\| = -\hat{\sigma}_d/\hat{a}$ and a limit stress condition (see e.g. Bauer, 2000):

$$\|\hat{\sigma}_d\| = \hat{a}. \quad (5)$$

3. Description of shear strength of sand

Coulomb's friction law provided a fundamental understanding for the failure mechanism of frictional materials. The shear strength of soils is usually defined with a friction angle, often measured from conventional triaxial compression tests, in the sense of Mohr–Coulomb's theory. Although the Mohr–Coulomb's (M-C) criterion is widely used in engineering practice, it does not take into account the effect of the intermediate principal stress and it underestimates the shear strength of granular soils under stress conditions other than triaxial compression and extension. Another shortcoming of the M-C criterion is that the limit stress surface in the principal stress space is not smooth, which creates some difficulties for numerical implementation. Drucker and Prager (1952) proposed a simple variation for description of soil failure (D-P criterion), which corresponds to a circular cone in the principal stress space. Compared with the M-C criterion, when calibrated in a conventional triaxial compression test, the D-P criterion predicts a much too high shear strength for soils in stress conditions other than triaxial compression. Many experimental results, such as those presented in Fig. 2 in Lade (2006), show that the peak shear strength of soils under three-dimensional conditions may be described with the Matsuoka–Nakai failure (M-N) criterion or the Lade criterion (Lade, 1977).

The M-N criterion, which circumscribe tightly the M-C hexagon on the deviatoric stress plane (refer to Fig. 1) with the same friction angle for triaxial compression and extension, has a simple mathematical representation:

$$\hat{I}_2/\hat{I}_3 = C_{MN} = \text{const}. \quad (6)$$

Here $\hat{I}_2 = I_2/I_1^2 = (\hat{\sigma}_1\hat{\sigma}_2 + \hat{\sigma}_2\hat{\sigma}_3 + \hat{\sigma}_3\hat{\sigma}_1)$, $\hat{I}_3 = I_3/I_1^3 = \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3$ are, respectively, the second and the third invariant of the normalized stress tensor with $I_1 = \text{tr}\sigma$ being the first invariant of the stress tensor. The Lade criterion, which can be expressed in a simple expression of

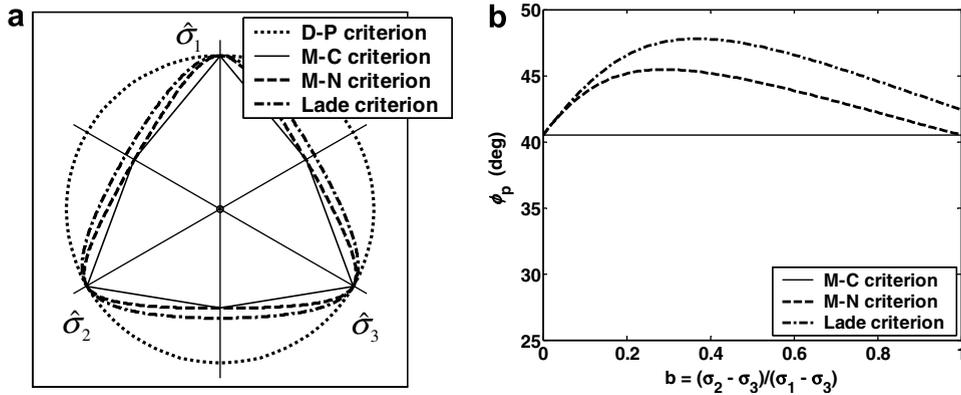


Fig. 1. Comparison of different limit stress criteria and represented (a) in the deviatoric stress plane and (b) as peak friction angle against the intermediate stress parameter b .

$$1/\hat{I}_3 = C_L = \text{const.} \quad (7)$$

predicts a slightly higher strength than the M-N criterion for tests other than triaxial compression. A comparison of the M-N criterion and Lade criterion on the deviatoric stress plane is given in Fig. 1a. The corresponding peak friction angles versus the intermediate principal stress parameter $b = (\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_3)$ are presented in Fig. 1b. We can see that the M-N criterion assumes the same friction angle for triaxial compression and triaxial extension while a greater friction angle is predicted by the Lade criterion. It may be noted that a triaxial extension test is usually more difficult to conduct than a triaxial compression test and the test results are more likely to be affected by other factors such as cross-anisotropy of the sand sample and strain localization. But nevertheless, from scrutinizing many experimental results (see e.g. Lade, 2006), we may conclude that for sands, a higher peak friction angle may be obtained in triaxial extension tests than that in triaxial compression tests. Therefore Lade's criterion can be a better description for the peak shear strength. However, there is no experimental evidence on the superiority of the Lade criterion over the M-N criterion for the description of the critical state. Due to the influence of shear localization, critical state is hard to achieve experimentally in a sand sample. Conceptually, the critical state is independent of the density. In contrast, the peak strength is strongly influenced by the density of the sample at failure. Referring to the true triaxial tests by controlling the mean stress at a constant value, the axial and volumetric strain at failure varies with the stress paths. In triaxial extension, the sand sample fails at a less volume dilation than that in a triaxial compression. In the present constitutive model, the critical state is predefined by incorporating a given limit stress condition and the peak strength will come out as model prediction. We assume that either of the M-N and the Lade criterion can be used as a reasonably good description for soil critical state. Later on, we will show that by incorporating the M-N criterion in the present G-B model using the proposed stress transformation approach, a higher peak strength for triaxial extension is also predicted for dense sand (refer to Fig. 4).

Whereas the peak strength is influenced by the initial density state of sand packing, the conceptual critical state is not, and a unique shear strength parameter such as the critical friction angle ϕ_c can therefore be defined based on triaxial compression testing results. Generally critical states for sand are not easy to obtain experimentally as strain localization often occurs in the specimen around or after the peak. But in well-conducted triaxial compression tests (e.g. Wang and Lade, 2001; Sun et al., 2008) uniform deformation can be achieved long after the peak (see also Huang et al., 2007), which provides a good approximate data for ϕ_c . Here we accept the assumption already used in the Wolffersdorff version of the G-B model, that the M-N criterion or the Lade criterion can also be used to describe the critical state. With ϕ_c being the unique strength parameter, the constant in the M-N criterion (6) and the Lade criterion (7) can be determined as $C_{MN} = (9 - \sin^2 \phi_c)/(1 - \sin^2 \phi_c)$ and $C_L = (1 - \sin^2 \phi_c)(1 - \sin \phi_c)/(3 - \sin \phi_c)^2$, respectively.

4. Incorporating a predefined limit condition by parameter interpolation

The Gudehus–Bauer model has combined the critical state concept with limit stress satisfying Eq. (5). If a constant value is assigned to parameter \hat{a} , Eq. (5) corresponds to a D-P type limit condition, which, as has been pointed out, will significantly overestimate the shear strength of sand in 3-dimensional stress conditions. Bauer (1996) suggested tackling this problem by formulating \hat{a} as a function of the Lode angle θ . By assuming the same friction angle for triaxial compression and extension, he proposed the following interpolation formulation for $\hat{a}(\theta) = \hat{a}_\theta$ (see also Bauer, 2000):

$$\hat{a}_\theta = \hat{a}_0 \left[1 - \sqrt{\frac{3}{8}} \|\hat{\sigma}_d\| (1 - \cos(3\theta)) \right]. \quad (8)$$

Here

$$\hat{a}_0 = \sqrt{\frac{8}{3}} \frac{\sin \phi_c}{3 - \sin \phi_c} \quad (9)$$

is the value of \hat{a} at $\theta = 0^\circ$ or at an isotropic compressive stress state, i.e. $\|\hat{\sigma}_d\| = 0$. The Lode angle θ is defined by (refer to Fig. 2)

$$\cos(3\theta) = \sqrt{6} \frac{\text{tr} \hat{\sigma}_d^3}{(\text{tr} \hat{\sigma}_d^2)^{3/2}}. \quad (10)$$

Wolffersdorff (1996) took the same approach and derived the following formulation for \hat{a} to incorporate the M-N criterion as a predefined limit stress condition:

$$\hat{a}_\theta = \hat{a}_0 \left(\sqrt{\frac{3}{8} \|\hat{\sigma}_d\|^2 + \frac{1 - (3/2) \|\hat{\sigma}_d\|^2}{1 - \sqrt{3/2} \|\hat{\sigma}_d\| \cos(3\theta)}} - \sqrt{\frac{3}{8} \|\hat{\sigma}_d\|} \right). \quad (11)$$

5. Incorporating a predefined limit condition by a simple stress transformation

As a limit stress condition can be incorporated in the model for the critical states through parameter interpolation, the peak stress state is not a predefinition, but comes out as a prediction of the model. With the formulation (11) we found that the predicted peak states are not all satisfactory. The predicted peak strength for stress paths close to triaxial extension (i.e. $\theta = 60^\circ$) is lower than that for triaxial compression, as shown in Fig. 3. This tendency is more significant when a denser initial state is modeled. As a consequence, the limit stress surface predicted for dense sand becomes somewhat concave as shown in Huang et al. (2007).

This prediction does not seem realistic compared with the experimental results from true triaxial tests of sands as summarized in Lade (2006). Many experiment results show higher peak strength for triaxial extension than that for triaxial compression.

The poor model performances in prediction of peak strength may be attributed to the interpolation of parameter \hat{a} , which considered mainly for the description of critical states. From Eq. (11) we can see that \hat{a} keeps constant value for $\theta = 0^\circ$. But for shear tests starting from an isotro-

pic stress state, along stress paths with $\theta > 0^\circ$, \hat{a} decreases before reaching the peak, and then increases to the final critical value (Fig. 2b). This peculiar behaviour leads to a complicated model performance and resulted in an unexpected peak strength prediction.

In this work, we propose to improve model performance by using a stress transformation technique proposed in Matsuoka et al. (1999). This simple stress transformation technique allows a predefined limit stress condition to be implemented in the current hypoplastic model with a constant shear strength parameter. In this approach, Eq. (1) is revised to be written in the following form:

$$\dot{\sigma} = f_s [\hat{a}_0^2 \dot{\varepsilon} + \hat{\sigma}^* (\hat{\sigma}^* : \dot{\varepsilon}) + f_d \hat{a}_0 (\hat{\sigma}^* + \hat{\sigma}_d^*) \|\dot{\varepsilon}\|], \quad (12)$$

where \hat{a}_0 is the constant strength parameter which is related to the critical friction angle ϕ_c by Eq. (9) and $\hat{\sigma}^*$ is the transformed stress also normalized by $\text{tr} \sigma$. The stress transformation is defined by

$$\text{tr} \sigma^* = \text{tr} \sigma, \quad \hat{\sigma}_d^* = r_\theta \hat{\sigma}_d, \quad (13a)$$

or in an more compact form of

$$\hat{\sigma}^* = \left[r_\theta \mathbf{1} + \frac{1}{3} (1 - r_\theta) \mathbf{1} \otimes \mathbf{1} \right] : \hat{\sigma}. \quad (13b)$$

In this transformation, only the deviatoric stress is modified and the mean stress is kept unchanged. On the deviatoric stress plane (refer to Fig. 3a), the current stress point A is mapped to an image point A^* by a multiplier $r_\theta = \hat{\rho}_0 / \hat{\rho}$, where $\hat{\rho}_0 = \|\hat{\sigma}_d\|$ represents the radius of the current normalized stress, and $\hat{\rho} = \hat{\rho}^* = \|\hat{\sigma}_d^*\|$ represents the radius of the image stress, which falls on the circle circumscribing an iso-valued curve passing through point A , at a point on the axis of $\theta = 0^\circ$. The iso-valued curve can be determined based on either the M-N criterion or the Lade criterion. Such mapping is a one-to-one mapping.

If the M-N criterion is applied, an explicit representation for factor r_θ can be obtained (see Appendix for details) as,

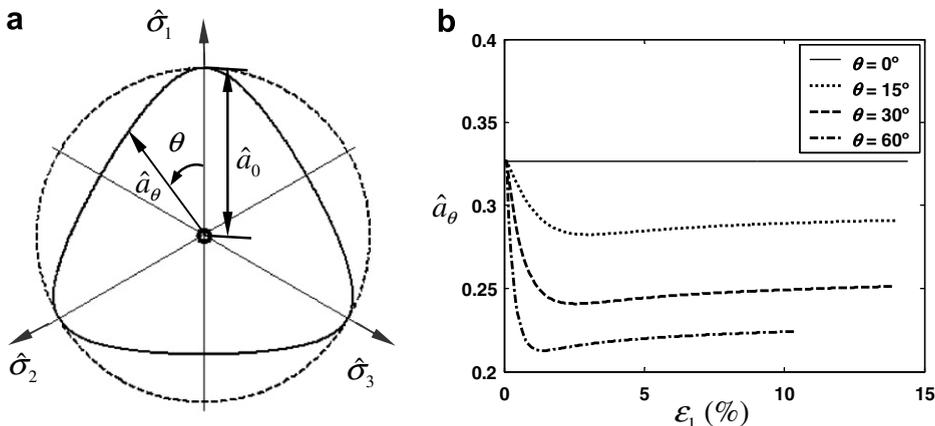


Fig. 2. (a) Schematic illustration of parameter interpolation and (b) variation of parameter \hat{a}_θ in loading.

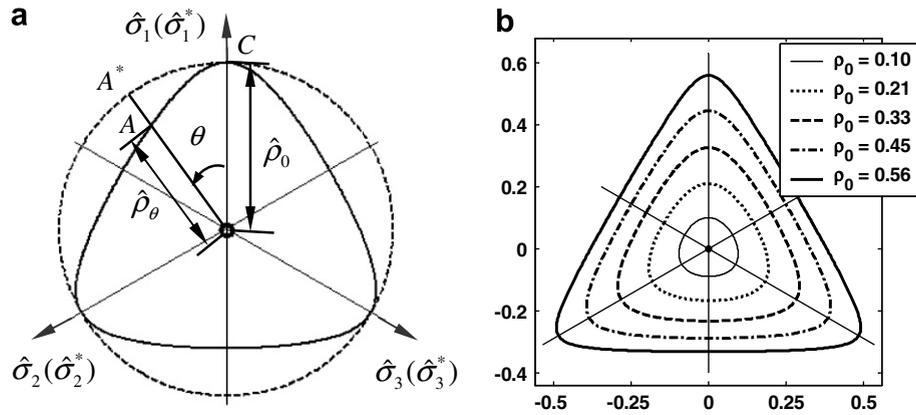


Fig. 3. (a) Schematic illustration of stress transformation in the deviatoric stress plane and (b) iso-valued curves of Matsuoka–Nakai criterion.

$$r_\theta = \sqrt{\frac{\sqrt{3} \left(1 - \frac{\sqrt{3/2} \|\hat{\sigma}_d\| \cos(3\theta)}{1 - 3/2 \|\hat{\sigma}_d\|^2} \right)}{8}} \times \left(\sqrt{\frac{\|\hat{\sigma}_d\|^2 + \frac{8}{3} \frac{1 - 3/2 \|\hat{\sigma}_d\|^2}{1 - \sqrt{3/2} \|\hat{\sigma}_d\| \cos(3\theta)}}{\|\hat{\sigma}_d\|}} + \|\hat{\sigma}_d\| \right). \quad (14)$$

If the Lade criterion is to be used, r_θ can be expressed as (Appendix)

$$r_\theta = \frac{2 \cos \left[\frac{1}{3} \cos^{-1} \left(-\sqrt{27\hat{I}_3} \right) \right] - \sqrt{27\hat{I}_3}}{\sqrt{6} \|\hat{\sigma}_d\| \cos \left[\frac{1}{3} \cos^{-1} \left(-\sqrt{27\hat{I}_3} \right) \right]}. \quad (15)$$

It is worth noting that in the interpolation formulation (11), the M–N criterion is used for the critical state only, i.e., the constant C_{MN} in representation (6) is determined for the critical state only. In the stress transformation with M–N criterion, the criterion is used for all mobilized stress states and C_{MN} varies during loading, as does the multiplier r_θ for a given θ . As shown in Fig. 3b, on the deviatoric stress plane the iso-valued curves of the M–N criterion are nearly round towards the centre and become more angular as ρ_0 increases.

With the M–N criterion, the revised constitutive Eq. (12) predicts the same direction of strain rate at flow and the same critical state as does Eq. (1) with interpolation (11). The direction of strain rate at flow is now governed by $\vec{e} = -\hat{\sigma}_d^*/\hat{a}_0 = -r_\theta \hat{\sigma}_d/\hat{a}_0$ and the critical state is represented by $\|\hat{\sigma}_d^*\| = \hat{a}_0$ or $\|\hat{\sigma}_d\| = \hat{a}_0/r_\theta$. Note that with r_θ given as in (14), \hat{a}_0/r_θ is the same as \hat{a} in Eq. (11).

The predicted peak state, which is achieved at $\dot{\sigma} = 0$ for $f_d < 1$ ($\dot{e}_v > 0$), can be mathematically represented by

$$\psi_p(\hat{\sigma}, e) = \left[\frac{1}{3} \eta^{*2} + (\eta^* + 1)^2 \|\hat{\sigma}_d^*\|^2 \right] - \frac{\hat{a}_0^2}{f_d^2} = 0, \quad (16)$$

where $\eta^* = (\hat{a}_0^2 - \|\hat{\sigma}_d^*\|^2)/(\hat{a}_0^2 + \|\hat{\sigma}_d^*\|^2)$.

In the following, we show that by using the proposed stress transformation, the model predictions of the stress–strain behaviour and the peak shear strength in three-dimensional stress condition are improved. Comparisons of the predicted stress–strain curves and the peak shear strengths are presented in Fig. 4 for the G–B model

with the M–N criterion using the stress transformation and the parameter interpolation equation (11). The predictions are obtained by integration of the constitutive equations for shear tests along different stress paths characterized by the Lode angle θ on the deviatoric stress plane. In all these modelling tests, soil element is initially in an isotropic stress state with $p_0 = 100$ kPa. The initial void ratio e_0 is determined from $I_{d0} = (e_0 - e_d)/(e_c - e_d)$, for which different initial values are set to represent varied initial density. The model parameters used in calculating stress–strain responses are those calibrated for the Karlsruhe sand as given in Bauer (1996): $\phi_c = 30^\circ$, $h_s = 190$ MPa, $n = 0.4$, $e_i = 1.02$, $e_c = 0.82$, $e_d = 0.51$, $\varepsilon = 0.11$, and $\beta = 1.05$. Results show that by using the stress transformation, the peak stress ratio is now achieved earlier at a smaller axial strain for stress paths other than the triaxial compression (Fig. 4a). Stronger shear–dilation behaviour is predicted for stress paths with $\theta > 0$ (Fig. 4b and c). In Fig. 4d, the predicted peak friction angles are presented against the intermediate principal stress parameter $b = (\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_3)$, which can be related to the Lode angle by $\tan \theta = \sqrt{3}b/(2 - b)$. We can see that by the stress transformation, higher peak shear strengths are predicted for stress paths other than triaxial compression (Fig. 4d). In particular, the predicted peak friction angle for triaxial extension is greater than that for triaxial compression. This is qualitatively in agreement with the experimental observations (see, for example, Fig. 6 in Wang and Lade, 2001; Lade, 2006). It is also predicted that the difference of the peak strength between triaxial extension and compression increases with increased initial density of the soil element. Here the influence of the relative density on the peak strength in three-dimensional condition can be noticed. Along stress paths with constant mean pressure and different Lode angle, the peaks are reached at different axial and volumetric strain, resulting in different density at the peak, while the same density will be achieved at critical states.

Finally we note that the proposed stress transformation also allows an easy incorporation of the Lade criterion for the critical state, which is more difficult in the previous approach.

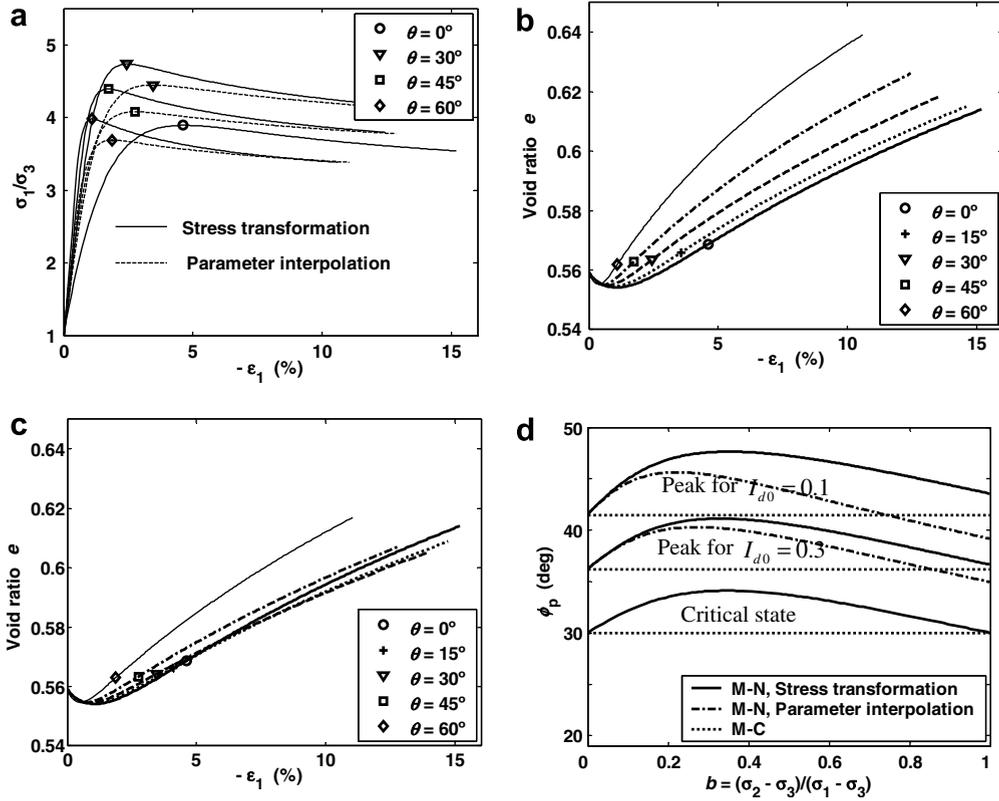


Fig. 4. Comparison of the model predictions using stress transformation and parameter interpolation: (a) stress ratio–strain curves obtained for $I_{d0} = 0.3$, corresponding void ratio (b) by stress transformation and (c) by parameter interpolation; (d) predicted peak friction angle for $I_{d0} = 0.3$ and $I_{d0} = 0.1$.

6. Conclusions

A simple stress transformation technique has been used to incorporate a predefined limit stress condition in the hypoplastic model for sand proposed Gudehus (1996) and Bauer (1996). The simple stress transformation is performed for the deviatoric stress by multiplying a scalar factor related to the particular limit stress criterion used while the mean stress is kept the same. The approach is very easy to implement and it allows a constant shear strength parameter to be used in the constitutive equation. Numerical results show that, compared with the original formulation, the revised model predicts more realistic stress–strain responses and peak strength.

Acknowledgement

The first author appreciates very much the valuable discussions on this topic with Prof. Y.P. Yao of the Beihang University, China. The research is partly supported by the National Basic Research Program of China (2007CB714100/2007CB714103).

Appendix A. Transformation multiplier for the M-N criterion and the Lade criterion

The mathematical expression of the Matsuoka–Nakai criterion and the Lade criterion are given in Eqs. (6) and

(7), respectively. As shown in Fig. 1a, iso-valued curves for both of the M–N criterion and the Lade criterion inscribe the D–P circle at a point on the triaxial compression lines (i.e. in the compression meridian plane). At this point (refer to Fig. 3a), we have

$$\begin{aligned} \hat{\rho}_0 &= \|\hat{\sigma}_d^c\| = \sqrt{\frac{2}{3}}(\hat{\sigma}_1^c - \hat{\sigma}_3^c) = \sqrt{\frac{2}{3}} \frac{\sigma_1^c - \sigma_3^c}{\sigma_1^c + 2\sigma_3^c} = \sqrt{\frac{2}{3}} \frac{R^c - 1}{R^c + 2} \\ &= \sqrt{\frac{8}{3}} \frac{\sin \varphi_m^c}{3 - \sin \varphi_m^c}. \end{aligned}$$

Here $\hat{\sigma}_1^c$ and $\hat{\sigma}_3^c$ are principal components of the normalized stress $\hat{\sigma}^c$ corresponding to point C, $R^c = \sigma_1^c/\sigma_3^c$ is the stress ratio in the triaxial compression, and the mobilized friction angle φ_m^c is defined by $\sin \varphi_m^c = (\sigma_1^c - \sigma_3^c)/(\sigma_1^c + \sigma_3^c)$.

With respect to M–N criterion, the constant C_{MN} can be determined at the point in the compression meridian by

$$\begin{aligned} C_{MN} &= \frac{\hat{I}_2}{\hat{I}_3}_{\text{comp}} = \frac{2\hat{\sigma}_1^c\hat{\sigma}_3^c + \hat{\sigma}_3^{c2}}{\hat{\sigma}_1^c\hat{\sigma}_3^{c2}} = \frac{(2R^c + 1)(R^c + 2)}{R^c} \\ &= \frac{9 - \sin^2 \varphi_m^c}{1 - \sin^2 \varphi_m^c}. \end{aligned}$$

From this equation we can express φ_m^c by the true stress through

$$\sin \varphi_m^c = \sqrt{(C_{MN} - 9)/(C_{MN} - 1)}.$$

Substituting this into Eq. (A3) yields

$$\begin{aligned}\hat{\rho}_0 &= \sqrt{\frac{8}{3}} \frac{\sqrt{C_{MN} - 9}}{3\sqrt{C_{MN} - 1} - \sqrt{C_{MN} - 9}} \\ &= \sqrt{\frac{8}{3}} \frac{\sqrt{(\hat{I}_2 - 9\hat{I}_3)}}{3\sqrt{(\hat{I}_2 - \hat{I}_3)} - \sqrt{(\hat{I}_2 - 9\hat{I}_3)}}.\end{aligned}\quad (\text{A1})$$

The stress transformation multiplier r_θ can be simply calculated from $r_\theta = \hat{\rho}_0 / \|\hat{\sigma}_d\|$. For numerical implementation, we may need to distinguish the isotropic stress state, at which we have $\hat{I}_2 - 9\hat{I}_3 = 0$ and $\hat{\rho}_0 = 0$. Correspondingly, the stress transformation multiplier $r_\theta = \hat{\rho}_0 / \|\hat{\sigma}_d\|$ has a 0/0 form, and in this case we can set $r_\theta = 1$.

For the M-N criterion, it is possible to eliminate $\|\hat{\sigma}_d\|$ from the denominator to obtain an explicit expression for r_θ as a function of the true stress tensor. Note that $\hat{\rho}_0$ can also be expressed as

$$\hat{\rho}_0 = \sqrt{\frac{8}{3}} \frac{(\hat{I}_2 - 9\hat{I}_3)}{8\hat{I}_2} \left(\sqrt{1 + \frac{8\hat{I}_2}{(\hat{I}_2 - 9\hat{I}_3)}} + 1 \right).$$

And with the stress invariants \hat{I}_2 and \hat{I}_3 being expressed as $\hat{I}_2 = \frac{1}{3} - \frac{1}{2} \|\hat{\sigma}_d\|^2$ and $\hat{I}_3 = \frac{1}{27} - \frac{1}{6} \|\hat{\sigma}_d\|^2 + \frac{1}{3\sqrt{6}} \|\hat{\sigma}_d\|^3 \cos(3\theta)$, we can obtain the following expression:

$$\begin{aligned}r_\theta &= \frac{1 - \sqrt{3/2} \|\hat{\sigma}_d\| \cos(3\theta)}{1 - \frac{3}{2} \|\hat{\sigma}_d\|^2} \\ &\quad \times \left(\sqrt{\frac{3}{8} \|\hat{\sigma}_d\|^2 + \frac{1 - \frac{3}{2} \|\hat{\sigma}_d\|^2}{1 - \sqrt{3/2} \|\hat{\sigma}_d\| \cos(3\theta)}} + \sqrt{\frac{3}{8}} \|\hat{\sigma}_d\| \right).\end{aligned}\quad (\text{A2})$$

It can be seen that $r_\theta = 1$ as $\|\hat{\sigma}_d\| \rightarrow 0$.

With respect to the Lade criterion, we have

$$C_L = \frac{1}{\hat{I}_3} \Big|_{\text{comp}} = \frac{1}{\hat{\sigma}_1^c \hat{\sigma}_3^c} = \frac{(\hat{\sigma}_1^c + \hat{\sigma}_3^c)^3}{\hat{\sigma}_1^c \hat{\sigma}_3^c} = \frac{(R^c + 2)^3}{R^c}.$$

From this equation R^c can be solved (Yao and Sun, 2000)

$$R^c = \frac{2}{\sqrt{3}} \sqrt{C_L} \cos \left[\frac{1}{3} \cos^{-1} \left(-\sqrt{27/C_L} \right) \right] - 2.$$

This leads to

$$\hat{\rho}_0 = \frac{2 \cos \left[\frac{1}{3} \cos^{-1} \left(-\sqrt{27\hat{I}_3} \right) \right] - \sqrt{27\hat{I}_3}}{\sqrt{6} \cos \left[\frac{1}{3} \cos^{-1} \left(-\sqrt{27\hat{I}_3} \right) \right]}.\quad (\text{A3})$$

The transformation multiplier therefore has the following representation:

$$r_\theta = \frac{\hat{\rho}_0}{\hat{\rho}_0} = \frac{2 \cos \left[\frac{1}{3} \cos^{-1} \left(-\sqrt{27\hat{I}_3} \right) \right] - \sqrt{27\hat{I}_3}}{\sqrt{6} \|\hat{\sigma}_d\| \cos \left[\frac{1}{3} \cos^{-1} \left(-\sqrt{27\hat{I}_3} \right) \right]}.\quad (\text{A4})$$

References

- Bauer, E., 1996. Calibration of a comprehensive hypoplastic model for granular materials. *Soils Foundations* 36 (1), 13–26.
- Bauer, E., 2000. Conditions for embedding Casagrade's critical state into hypoplasticity. *Mech. Cohesive-Frict. Mater.* 5 (2), 125–148.
- Drucker, D.C., Prager, W., 1952. Soil mechanics and plastic analysis or limit design. *Quart. Appl. Math.* 10 (2), 157–165.
- Gudehus, G., 1996. A comprehensive constitutive equation for granular materials. *Soils Foundations* 36 (1), 1–12.
- Huang, W., Nübel, K., Bauer, E., 2002a. Polar extension of a hypoplastic model for granular materials with shear localization. *Mech. Mater.* 34, 563–576.
- Huang, W., Sloan, S.W., Sheng, D., 2002b. Hypoplastic analysis of cone penetration in sands. In: Pande, G.N., Pietruszczak, S. (Eds.), *Proceedings of Eighth International Symposium of Numerical Models in Geomechanics – NUMOG VIII*. A.A. Balkema, pp. 607–612.
- Huang, W., Sun, D.A., Sloan, S., 2007. Analysis of failure mode and softening behaviour of sands in true triaxial tests. *Int. J. Solids Struct.* 44 (5), 1423–1437.
- Kolymbas, D., 1985. A generalized hypoelastic constitutive law. In: *Proceedings of XI International Conference Soil Mechanics and Foundation Engineering*. A.A. Balkema, San Francisco.
- Kolymbas, D., 2000. *Introduction to Hypoplasticity*. A.A. Balkema, Rotterdam.
- Lade, P.V., 1977. Elasto-plastic stress-strain theory for cohesionless soil with curved yield surface. *Int. J. Solids Struct.* 13, 1019–1035.
- Lade, P.V., 2006. Assessment of test data for selection of 3-D failure criterion for sand. *Int. J. Numer. Anal. Methods Geomech.* 30, 307–333.
- Matsuoka, H., Nakai, T., 1974. Stress-deformation and strength characteristics of soil under three difference principal stress. In: *Proceedings of JSCE*, No. 232, pp. 59–70.
- Matsuoka, H., Yao, Y.-P., Sun, D., 1999. The cam-clay models revised by the SMP criterion. *Soils Foundations* 39 (1), 81–95.
- Roscoe, K.H., Schofield, A.N., Thurairajah, A., 1963. Yielding of clay in states wetter than critical. *Géotechnique* 13 (3), 211–240.
- Schofield, A.N., Wroth, C.P., 1968. *Critical State Soil Mechanics*. McGraw-Hill.
- Sun, D., Huang, W., Yao, Y.-P., 2008. An experimental study of failure and softening in sand under three-dimensional stress condition. *Granul. Mat.* 10 (3), 187–195.
- von Wolfersdorff, P.-A., 1996. A hypoplastic relation for granular materials with a predefined limit state surface. *Mech. Cohesive-Frict. Mater.* 1, 251–271.
- Wang, Q., Lade, P.V., 2001. Shear banding in true triaxial tests and its effect on failure in sand. *J. Eng. Mech., ASCE* 127 (8), 754–761.
- Wu, W., 1992. Hypoplastizität als mathematisches Modell zum mechanischen Verhalten granularer Stoffe. *Veröffentlichungen Heft 129*, Institut für Bodenmechanik und Felsmechanik der Universität Fridericiana in Karlsruhe.
- Wu, W., Bauer, E., 1993. A hypoplastic model for barotropy and pyknotropy of granular soils. In: Kolymbas, D. (Ed.), *Modern Approaches to Plasticity*. Elsevier, pp. 225–245.
- Wu, W., Kolymbas, 1990. Numerical testing of the stability criterion for hypoplastic constitutive equations. *Mech. Mater.* 9, 245–253.
- Yao, Y.-P., Sun, D., 2000. Application of Lade's criterion to cam-clay model. *J. Eng. Mech., ASCE* 126 (1), 112–119.