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WITHDRAWAL OF A COMPRESSIBLE PORE FLUID FROM A POINT SINK IN AN ISOTROPIC ELASTIC HALF SPACE WITH ANISOTROPIC PERMEABILITY

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Abstract—The complete solution is presented for the transient effects of pumping fluid at a constant rate from a point sink embedded in a saturated, porous elastic half space. It is assumed that the medium is homogeneous and isotropic with respect to its elastic properties and homogeneous but anisotropic with respect to the flow of pore fluid. The soil skeleton is modelled as an isotropic linear elastic material obeying Hooke's law while the pore fluid may be compressible with its flow governed by Darcy's law. The solution has been evaluated for a particular value of Poisson's ratio of the solid skeleton, i.e. 0.25, and the results have been presented graphically in the form of isochrones of excess pore pressure and surface profile for the half space. The solutions presented may have application in practical problems such as dewatering operations in compressible soil and rock masses and in the extraction of petroleum products from the crust of the earth.

1. INTRODUCTION

In geotechnical, hydraulic and petroleum engineering it is sometimes necessary to pump water or some other fluid from the ground. This may be for a variety of reasons including:

(a) obtaining supplies of water, oil or gas,
(b) reducing pore water pressures in the ground,
(c) lowering the water table in order to allow construction operations to proceed.

In order to remove pore fluid from the ground it is necessary to reduce the pressure in the fluid in the vicinity of the pump and so there will in general be an increase in the compressive effective stress state. This increase of effective stress will cause consolidation of the ground and may lead to large-scale subsidence. The decrease in pore pressure will not occur immediately. After pumping has commenced the pore pressures will gradually decrease below their initial *in situ* values until a steady state distribution is established. Hence the resultant consolidation and surface subsidence will be time dependent.

Probably the best known examples of this phenomenon occur in Bangkok, Venice and Mexico City where widespread subsidence has been caused by withdrawal of water from aquifiers for industrial and domestic purposes. Recorded settlements in Mexico City have reached rates of 5–6 cm per year[1]. However, the problem is more widespread than this with subsidence due to fluid extraction having been reported in a number of other regions of the world[2–5]. The problem is not exclusively caused by the extraction of groundwater; the withdrawal of petroleum, air and gas can also induce surface subsidence[6].

The purpose of this paper is to provide the complete solution for the transient effects of pumping fluid from a point sink embedded in a saturated porous elastic half space. The problem is defined in Fig. 1. In obtaining this solution proper account has been taken of the coupling of the pore fluid flow with the deformation of the solid skeleton. It has been assumed that the saturated medium is homogeneous with respect to its elastic properties and homogeneous but transversely isotropic with respect to the flow of pore fluid so that one value of permeability has been assumed for flow in any horizontal plane and another value for vertical flow. Furthermore, it has been assumed that the pore fluid may be compressible and that the half space remains saturated.

The point sink problem treated here is of course an extreme idealization of any real situation. Nevertheless, it is considered that investigations of this type have much value in

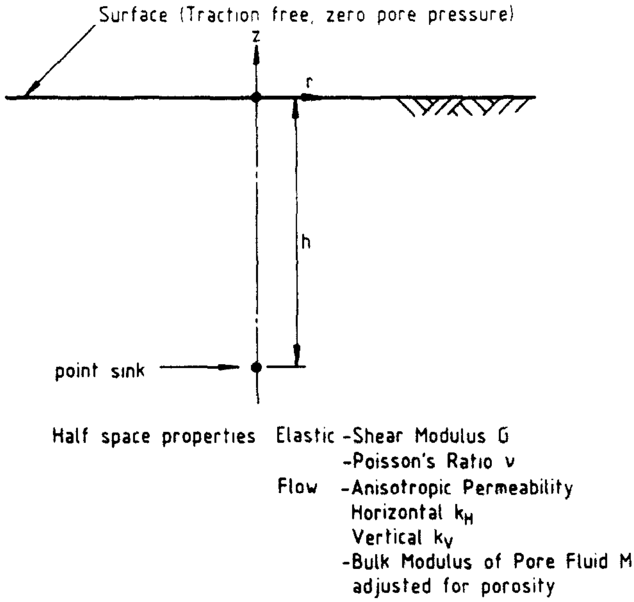

Fig. 1. Problem definition.

that their very lack of complexity allows an uncluttered look at the processes in operation and often allows an assessment of their relative importance. Moreover, for many preliminary investigations this extreme idealization is all that is required in the absence of detailed field data. It also serves to give an idea of the likely severity of various effects.

2. GOVERNING EQUATIONS

The equations governing the consolidation of a poroelastic medium were first developed by Biot[7, 8]. When expressed in terms of a Cartesian coordinate system they take the following forms.

2.1. *Equilibrium*

In the absence of increase in body forces the equations of equilibrium can be written as

$$\partial \boldsymbol{\sigma} = 0 \tag{1}$$

where

$$\partial = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial y & 0 & \partial/\partial z \\ 0 & \partial/\partial y & 0 & \partial/\partial x & \partial/\partial z & 0 \\ 0 & 0 & \partial/\partial z & 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}$$

$$\boldsymbol{\sigma}^{\mathrm{T}} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx})$$

is the vector of total stress components with tensile normal stress regarded as positive (these quantities represent the increase over the initial state of stress).

2.2. *Strain–displacement relations*

The strains are related to the displacement as follows

$$\boldsymbol{\varepsilon} = \partial^{\mathrm{T}}\mathbf{u} \tag{2}$$

where

$$\boldsymbol{\varepsilon}^{\mathrm{T}} = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$$

is the vector of strain components of the soil skeleton, and

$$\mathbf{u}^{\mathrm{T}} = (u_x, u_y, u_z)$$

is the vector of Cartesian displacement components of the skeleton.

2.3. *Effective stress principle*

It is assumed for the saturated soil that the effective stress principle is valid, i.e.

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p\mathbf{a} \tag{3}$$

where

$$\boldsymbol{\sigma}' = (\sigma'_{xx}, \sigma'_{yy}, \sigma'_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx})^{\mathrm{T}}$$

is the vector of effective stress increments (these quantities represent the increase over the initial state of effective stress)

$$\mathbf{a}^{\mathrm{T}} = (1, 1, 1, 0, 0, 0)$$

and p is the excess pore fluid pressure.

2.4. *Hooke's law*

The constitutive behaviour of the solid phase (the skeleton) of the saturated medium is governed by Hooke's law, which is

$$\boldsymbol{\sigma}' = D\boldsymbol{\varepsilon} \tag{4}$$

where

$$D = \begin{bmatrix} \lambda + 2G & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2G & \lambda & 0 & 0 & 0 \\ & & \lambda + 2G & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ & & & & G & 0 \\ \text{symmetric} & & & & & G \end{bmatrix}$$

with λ and G the Lame modulus and shear modulus, of the soil skeleton, respectively.

The moduli λ, G are related to Young's modulus, E and Poisson's ratio ν of the skeleton, i.e.

$$\lambda = \frac{E\nu}{(1 - 2\nu)(1 + \nu)}$$

$$G = \frac{E}{2(1 + \nu)}.$$

2.5. *Darcy's law*

It will be assumed that the flow of pore water is governed by Darcy's law, which for a transversely isotropic soil takes the form:

$$v_x = -\frac{k_H}{\gamma_F}\frac{\partial p}{\partial x}$$

$$v_y = -\frac{k_H}{\gamma_F}\frac{\partial p}{\partial y}$$

$$v_z = -\frac{k_V}{\gamma_F}\frac{\partial p}{\partial z} \tag{5}$$

where k_H, k_V are the horizontal and vertical permeability, respectively, γ_F is the unit weight of pore fluid and the z coordinate direction is aligned vertically and v_x, v_y, v_z are the components of the superficial velocity vector relative to the soil skeleton.

2.6. *Displacement equations*

If Hooke's law, eqn (4), and the equation of equilibrium, eqn (1), are combined it is found that

$$G\nabla^2\mathbf{u} + (\lambda + G)\nabla\varepsilon_v = \nabla p \tag{6}$$

where

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

is the volume strain. This equation can be condensed to give the useful relation

$$(\lambda + 2G)\nabla^2\varepsilon_v = \nabla^2 p. \tag{7}$$

2.7. *The volume constraint equation*

If the skeletal material is incompressible but the pore fluid is compressible then the volume change of any element of soil must balance the difference between the volume of fluid leaving and entering the element by flow across its boundaries plus the volume of fluid extracted from the element by some internal sink mechanism and any change in the volume of pore fluid. Symbolically this continuity condition may be expressed as the volume constraint equation, i.e.

$$\int_0^t \nabla^T\mathbf{v}\,dt + \varepsilon_v + \frac{p}{M} = -\int_0^t q\,dt \tag{8}$$

where q is the volume of fluid extracted per unit volume per unit time from the soil by the sink mechanism, $\mathbf{v}^T = (v_x, v_y, v_z)$ and M is the bulk modulus (adjusted for porosity) of the pore fluid.

If eqn (8) is combined with Darcy's law, eqn (5), and Laplace transforms are taken of the resulting equation, we find that

$$\frac{k_{\mathrm{H}}}{\gamma_{\mathrm{F}}}\left(\frac{\partial^2 \bar{p}}{\partial x^2}+\frac{\partial^2 \bar{p}}{\partial y^2}\right)+\frac{k_{\mathrm{V}}}{\gamma_{\mathrm{F}}}\frac{\partial^2 \bar{p}}{\partial z^2}=s\left(\bar{\varepsilon}_v+\frac{\bar{p}}{M}\right)+\bar{q} \tag{9}$$

or

$$c_{\mathrm{H}}\left(\frac{\partial^2 \bar{p}}{\partial x^2}+\frac{\partial^2 \bar{p}}{\partial y^2}\right)+c_{\mathrm{V}}\frac{\partial^2 \bar{p}}{\partial z^2}=(\lambda+2G)\left[s\left(\bar{\varepsilon}_v+\frac{\bar{p}}{M}\right)+\bar{q}\right] \tag{10}$$

where

$$c_{\mathrm{H}}=k_{\mathrm{H}}(\lambda+2G)/\gamma_{\mathrm{F}}$$
$$c_{\mathrm{V}}=k_{\mathrm{V}}(\lambda+2G)/\gamma_{\mathrm{F}}$$

are the horizontal and vertical coefficients of consolidation of the saturated porous elastic medium.

The superior bar is used here to indicate a Laplace transform, i.e.

$$\bar{f}(s)=\int_0^{\infty} f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t. \tag{11}$$

3. SOLUTION METHOD

In proceeding to the solution of the equations of consolidation for the case of a point sink embedded in a saturated elastic half space, we introduce triple Fourier transforms of the type

$$P^*(\alpha,\beta,\gamma)=(1/2\pi)^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-i(\alpha x+\beta y+\gamma z)}\,p(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{12a}$$

The corresponding inversion formula is

$$p(x,y,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{i(\alpha x+\beta y+\gamma z)}\,P^*(\alpha,\beta,\gamma)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma. \tag{12b}$$

Use will also be made of double Fourier transforms of the type

$$P(\alpha,\beta,z)=(1/2\pi)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-i(\alpha x+\beta y)}\,p(x,y,z)\,\mathrm{d}x\,\mathrm{d}y \tag{13a}$$

and the corresponding inversion formula

$$p(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha x + \beta y)} P(\alpha, \beta, z)\, d\alpha\, d\beta. \tag{13b}$$

If we compare eqns (12) and (13) we see that

$$P^*(\alpha, \beta, \gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\gamma z} P(\alpha, \beta, z)\, dz \tag{14a}$$

and conversely

$$P(\alpha, \beta, z) = \int_{-\infty}^{\infty} e^{i\gamma z} P^*(\alpha, \beta, \gamma)\, d\gamma. \tag{14b}$$

Sometimes it will be convenient to introduce the coordinates (ρ, ε) where

$$\begin{aligned} \alpha &= \rho\cos\varepsilon \\ \beta &= \rho\sin\varepsilon \end{aligned} \tag{15}$$

in which case eqns (13b) become, for polar coordinates (r, θ, z)

$$p(r, \theta, z) = \int_{0}^{\infty}\int_{0}^{2\pi} e^{i\rho r\cos(\theta - \varepsilon)} P\rho\, d\rho\, d\varepsilon. \tag{16}$$

Quite often the transform P will be able to be represented in the form

$$P = \cos n(\theta - \varepsilon) F(\rho, z) \tag{17}$$

and thus

$$p = 2\pi i^n \int_{0}^{\infty} \rho F(\rho, z) J_n(\rho r)\, d\rho \tag{18}$$

where J_n represents the Bessel function of order n.

In the analysis which follows solutions for the equations of consolidation are found in terms of the Laplace transforms of the triple Fourier transforms of the field quantities. Partial inversion of the triple Fourier transforms is then carried out in closed form using eqn (14) or eqn (18) and the inversion is completed using a single numerical integration. This leaves us with the Laplace transforms of the field quantities which in turn are inverted numerically using the technique developed by Talbot[9], giving the time-dependent field quantities.

The complete solution for a point source embedded in a half space is built up by first considering the case of a point sink in an infinite medium and then the case of a half space with no sink. The solutions for these problems are given in the following sections.

4. SOLUTION FOR A POINT SINK

Let us consider a sink of strength F_k located at the point (x_k, y_k, z_k) within an infinite medium, so that

$$q = F_k\delta(x - x_k)\delta(y - y_k)\delta(z - z_k) \tag{19}$$

where δ indicates the Dirac delta function. We introduce triple transforms having the form of eqn (12a) and thus we see, for example, that the transform of q is

$$Q^* = \frac{F_k}{(2\pi)^3}e^{-i(\alpha x_k + \beta y_k + \gamma z_k)}. \tag{20}$$

It will be convenient for our purposes to write this in the form

$$Q^* = \frac{Q}{2\pi}e^{-i\gamma z_k} \tag{21}$$

where

$$Q = \frac{F_k}{(2\pi)^2}e^{-i(\alpha_k + \beta y_k)}.$$

4.1. *Displacement equations*

In terms of triple transforms the displacement eqns (6) become

$$\begin{aligned}
-GD^2U_x^* + (\lambda + G)i\alpha E_v^* &= i\alpha P^* \\
-GD^2U_y^* + (\lambda + G)i\beta E_v^* &= i\beta P^* \\
-GD^2U_z^* + (\lambda + G)i\gamma E_v^* &= i\gamma P^* \\
i\alpha U_x^* + i\beta U_y^* + i\gamma U_z^* &= E_v^*
\end{aligned} \tag{22}$$

where $D^2 = \alpha^2 + \beta^2 + \gamma^2$, and

$$(U_x^*, U_y^*, U_z^*, P^*, E_v^*) = (1/2\pi)^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\alpha x + \beta y + \gamma z)}(u_x, u_y, u_z, p, \varepsilon_v)\,dx\,dy\,dz.$$

Equations (22) have the solution

$$\begin{aligned}
U_x^* &= -\left(\frac{i\alpha}{D^2}\right)E_v^* \\
U_y^* &= -\left(\frac{i\beta}{D^2}\right)E_v^* \\
U_z^* &= -\left(\frac{i\gamma}{D^2}\right)E_v^* \\
P^* &= (\lambda + 2G)E_v^*.
\end{aligned} \tag{23}$$

4.2. *Volume constraint equation*

In terms of the transforms eqn (10) becomes

$$-(\gamma^2 c_V + \rho^2 c_H)\bar{E}_v^* = s\bar{E}_v^*\left[1 + \left(\frac{\lambda + 2G}{M}\right)\right] + \bar{Q}^*. \tag{24}$$

If we now introduce the variables

$$\mu^2 = c_H\rho^2/c_V + s\left[1 + \left(\frac{\lambda + 2G}{M}\right)\right]\Big/c_V$$

$$\rho^2 = \alpha^2 + \beta^2$$

we see that

$$\bar{E}_v^* = \frac{-\bar{Q}^*}{c_V(\gamma^2 + \mu^2)}. \tag{25}$$

4.3. *Stress components*

The stress components may be obtained directly from Hooke's law, eqns (4), and so

$$S_{xx}^* = 2G\left(\frac{\alpha^2}{D^2} - 1\right)E_v^*$$

$$S_{yy}^* = 2G\left(\frac{\beta^2}{D^2} - 1\right)E_v^*$$

$$S_{zz}^* = 2G\left(\frac{\gamma^2}{D^2} - 1\right)E_v^* = -2G\frac{\rho^2}{D^2}E_v^*$$

$$S_{xy}^* = 2G\frac{\alpha\beta}{D^2}E_v^*$$

$$S_{yz}^* = 2G\frac{\beta\gamma}{D^2}E_v^*$$

$$S_{xz}^* = 2G\frac{\alpha\gamma}{D^2}E_v^* \tag{26}$$

where

$$S_{jk}^* = (1/2\pi)^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\alpha x + \beta y + \gamma z)}\sigma_{jk}\,dx\,dy\,dz$$

and j, k denote any of the indices x, y, z.

4.4. *Partial inversion*

All the field quantities determined in this section can be expressed in terms of the three functions $\bar{H}^*$, $\bar{K}^*$, $\bar{L}^*$ where

$$\bar{H}^* = \left(\frac{1}{2\pi}\right)\frac{e^{-i\gamma z_k}}{\gamma^2 + \mu^2} \tag{27a}$$

$$\bar{K}^* = \left(\frac{1}{2\pi}\right)\frac{e^{-i\gamma z_k}}{(\gamma^2+\mu^2)D^2}$$

$$= \frac{1}{2\pi(\mu^2-\rho^2)}\left[-\frac{e^{-i\gamma z_k}}{\gamma^2+\mu^2}+\frac{e^{-i\gamma z_k}}{\gamma^2+\rho^2}\right] \tag{27b}$$

$$\bar{L}^* = \left(\frac{1}{2\pi}\right)\frac{i\gamma\, e^{-i\gamma z_k}}{(\gamma^2+\mu^2)D^2}$$

$$= \frac{1}{2\pi(\mu^2-\rho^2)}\left[-\frac{i\gamma\, e^{-i\gamma z_k}}{\gamma^2+\mu^2}+\frac{i\gamma\, e^{-i\gamma z_k}}{\gamma^2+\rho^2}\right]. \tag{27c}$$

Now for the double Fourier transforms

$$(\bar{H},\bar{K},\bar{L}) = \int_{-\infty}^{\infty} e^{i\gamma z}(\bar{H}^*,\bar{K}^*,\bar{L})^*\,d\gamma$$

and it can be shown (see Appendix) that

$$\bar{H} = \frac{1}{2}\frac{e^{-\mu z}}{\mu} \tag{28a}$$

$$\bar{K} = \frac{1}{2(\mu^2-\rho^2)}\left[\frac{e^{-\rho Z}}{\rho}-\frac{e^{-\mu Z}}{\mu}\right] \tag{28b}$$

$$\bar{L} = \frac{\operatorname{sgn}(z_k-z)}{2(\mu^2-\rho^2)}\left[\frac{e^{-\rho Z}}{\rho}-\frac{e^{-\mu Z}}{\mu}\right] \tag{28c}$$

where $Z = |z - z_k|$.

Thus on combining eqns (13a), (23), (26) and (28) we have

$$\begin{aligned}
i\bar{U}_x &= -\alpha\bar{K}\bar{Q}/c_V\\
i\bar{U}_y &= -\beta\bar{K}\bar{Q}/c_V\\
\bar{U}_z &= \bar{L}\bar{Q}/c_V\\
\bar{P} &= -(\lambda+2G)\bar{H}\bar{Q}/c_V\\
\bar{S}_{xx} &= -2G(\alpha^2\bar{K}-\bar{H})\bar{Q}/c_V\\
\bar{S}_{yy} &= -2G(\beta^2\bar{K}-\bar{H})\bar{Q}/c_V\\
\bar{S}_{zz} &= 2G\rho^2\bar{K}\bar{Q}/c_V\\
\bar{S}_{xy} &= -2G\alpha\beta\bar{K}\bar{Q}/c_V\\
i\bar{S}_{yz} &= -2G\beta\bar{L}\bar{Q}/c_V\\
i\bar{S}_{zx} &= -2G\alpha\bar{L}\bar{Q}/c_V.
\end{aligned} \tag{29}$$

5. SOLUTION FOR A HALF SPACE WITH NO SINK

To analyse this problem we introduce double Fourier transforms leading to representations of the form given by eqn (13b). It will also be useful to introduce auxiliary quantities:

$$\begin{aligned} U_\xi &= \cos\varepsilon\, U_x + \sin\varepsilon\, U_y \\ U_\eta &= -\sin\varepsilon\, U_x + \cos\varepsilon\, U_y \\ S_{\xi z} &= \cos\varepsilon\, S_{xz} + \sin\varepsilon\, S_{yz} \\ S_{\eta z} &= -\sin\varepsilon S_{xz} + \cos\varepsilon S_{yz} \end{aligned} \tag{30}$$

where $\cos\varepsilon = \alpha/\rho$ and $\sin\varepsilon = \beta/\rho$.

5.1. *Displacement equations*

In terms of these double transforms eqns (6) become

$$G\left(\frac{\partial^2 U_\xi}{\partial z^2} - \rho^2 U_\xi\right) + (\lambda + G)i\rho\bar{E}_v = i\rho\bar{P} \tag{31a}$$

$$G\left(\frac{\partial^2 \bar{U}_\xi}{\partial z^2} - \rho^2 U_\eta\right) = 0 \tag{31b}$$

$$G\left(\frac{\partial^2 \bar{U}_z}{\partial z^2} - \rho^2 U_z\right) + (\lambda + 2G)\frac{\partial \bar{E}_v}{\partial z} = \frac{\partial \bar{P}}{\partial z} \tag{31c}$$

where

$$\bar{E}_v = \frac{\partial \bar{U}_z}{\partial z} + i\rho\bar{U}_\xi. \tag{31d}$$

(In the problem considered here it is found that $U_\eta = 0$.)

Equations (31) can be combined to give

$$(\lambda + 2G)\left[\frac{\partial^2 \bar{E}_v}{\partial z^2} - \rho^2 E_v\right] = \left[\frac{\partial^2 \bar{P}}{\partial z^2} - \rho^2 P\right]. \tag{32}$$

5.2. *Volume constraint equation*

Equation (9b) becomes

$$c_V\frac{\partial^2 \bar{P}}{\partial z^2} - c_H\rho^2\bar{P} = (\lambda + 2G)s\left(\bar{E}_v + \frac{\bar{P}}{M}\right) \tag{33a}$$

or

$$c_V\frac{\partial^2 \bar{P}}{\partial z^2} - c_H\rho^2\bar{P} = (\lambda + 2G)s\bar{E}_v + \bar{P}(c_V\mu^2 - c_H\rho^2 - s). \tag{33b}$$

5.3. *Solution*

The solutions of eqns (32) and (33) which remain bounded as $z \to -\infty$ are:

$$\begin{aligned} \bar{E}_v &= A\,e^{\mu z} + \left(\frac{2G}{\lambda + G}\right)\delta B\,e^{\rho z} \\ \bar{P} &= (\lambda + 2G)A\,e^{\mu z} + 2GB\,e^{\rho z} \end{aligned} \tag{34}$$

where

$$\delta = \left(\frac{\lambda + G}{\lambda + 2G}\right)[1 + c_V(\rho^2 - \mu^2)/s].$$

If we substitute eqns (34) into eqn (31c) we find

$$\frac{\partial^2 \bar{U}_z}{\partial z^2} - \rho^2 \bar{U}_z = \mu A \,\mathrm{e}^{\mu z} + 2B\rho(1 - \delta)\,\mathrm{e}^{\rho z}$$

and thus

$$\rho \bar{U}_z = \left(\frac{\mu\rho}{\mu^2 - \rho^2}\right) A \,\mathrm{e}^{\mu z} + \rho z B(1 - \delta)\,\mathrm{e}^{\rho z} + C\,\mathrm{e}^{\rho z}. \tag{35}$$

Furthermore, it is not difficult to show that

$$i\rho \bar{U}_\xi = \left(\frac{-\rho^2}{\mu^2 - \rho^2}\right) A \,\mathrm{e}^{\mu z} + B\left[\left(\frac{2G}{\lambda + G}\right)\delta - (1 + \rho z)(1 - \delta)\right]\mathrm{e}^{\rho z} - C\,\mathrm{e}^{\rho z} \tag{36}$$

$$\frac{\bar{S}_{zz}}{2G} = \left(\frac{\rho^2}{\mu^2 - \rho^2}\right) A \,\mathrm{e}^{\mu z} + B\left[\left(\frac{\lambda}{\lambda + G}\right)\delta - 1 + (1 - \delta)(1 + \rho z)\right]\mathrm{e}^{\rho z} + C\,\mathrm{e}^{\rho z} \tag{37}$$

$$\frac{i\bar{S}_{\xi z}}{2G} = \left(\frac{-\rho\mu}{\mu^2 - \rho^2}\right) A \,\mathrm{e}^{\mu z} + B\left[\left(\frac{G}{\lambda + G}\right)\delta - (1 - \delta)(1 + \rho z)\right]\mathrm{e}^{\rho z} - C\,\mathrm{e}^{\rho z}. \tag{38}$$

6. SOLUTION FOR A SINK IN A HALF SPACE

The solution to this problem can be synthesized by superimposing the solutions found in the previous sections. To do this it is convenient to introduce the following change of notation

$$\begin{aligned} N &= S_{zz}/2G \\ T &= iS_{\xi z}/2G \\ U &= iU_\xi \\ W &= U_z. \end{aligned} \tag{39}$$

The complete solution for the Laplace transforms of the double Fourier transforms can then be written in the form

$$\begin{bmatrix} \bar{N} \\ \bar{T} \\ \bar{P} \\ \bar{U} \\ \bar{W} \end{bmatrix} = -(\bar{Q}/c_V) \begin{bmatrix} \bar{N}_0 \\ \bar{T}_0 \\ \bar{P}_0 \\ \bar{U}_0 \\ \bar{W}_0 \end{bmatrix} + \begin{bmatrix} \bar{N}_1 & \bar{N}_2 & \bar{N}_3 \\ \bar{T}_1 & \bar{T}_2 & \bar{T}_3 \\ \bar{P}_1 & \bar{P}_2 & \bar{P}_3 \\ \bar{U}_1 & \bar{U}_2 & \bar{U}_3 \\ \bar{W}_1 & \bar{W}_2 & \bar{W}_3 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \tag{40}$$

where the functions $\bar{N}_0, \bar{T}_0, \ldots, \bar{W}_3$ are specified in Table 1. The coefficients F_1, F_2, F_3 may be obtained from the boundary conditions, i.e. zero tractions and pore pressure at the surface of the half space, $z = 0$. Thus we have

$$\begin{bmatrix} \bar{N}_1 & \bar{N}_2 & \bar{N}_3 \\ \bar{T}_1 & \bar{T}_2 & \bar{T}_3 \\ \bar{P}_1 & \bar{P}_2 & \bar{P}_3 \end{bmatrix} \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix} = +(\bar{Q}/c_V) \begin{bmatrix} \bar{N}_0 \\ \bar{T}_0 \\ \bar{P}_0 \end{bmatrix} \tag{41}$$

where all of the coefficients in the above equation are evaluated at $z = 0$. Once the unknown coefficients, $\bar{F}_1, \bar{F}_2, \bar{F}_3$ have been found as the solution to eqns (41), any of the transforms of the field quantities may be evaluated from eqns (40). These solutions should be precisely the same independent of which alternative,† specified in Table 1, is used.

7. CALCULATION OF FIELD QUANTITIES

Expressions for $\bar{N}$, $\bar{T}$, $\bar{P}$, $\bar{U}$, $\bar{W}$ were developed in the previous section. It will be observed for a point source that these are all functions of ρ. Thus we see from eqn (19) that

$$(\bar{\sigma}_{zz}, \bar{p}, \bar{u}_z) = 2\pi \int_0^\infty \rho(\bar{N}, \bar{P}, \bar{W}) J_0(\rho r)\,\mathrm{d}\rho. \tag{42}$$

Now we can easily establish that $\bar{U}_\eta = 0$ and thus that

$$\bar{U}_x = \cos\varepsilon \bar{U}_\xi$$
$$\bar{U}_y = \sin\varepsilon \bar{U}_\xi.$$

Thus the expressions for the Laplace transforms of displacement can be written as

$$\bar{u}_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{i(\alpha x + \beta y)} \cos\varepsilon \bar{U}_\xi(\rho)\,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$\bar{u}_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{i(\alpha x + \beta y)} \sin\varepsilon \bar{U}_\xi(\rho)\,\mathrm{d}\alpha\,\mathrm{d}\beta$$

and hence

$$u_r = \int_0^\infty \int_0^{2\pi} \mathrm{e}^{i\rho r \cos(\theta - \varepsilon)} \cos(\theta - \varepsilon) U_\xi(\rho)\rho\,\mathrm{d}\varepsilon\,\mathrm{d}\rho$$
$$= 2\pi \int_0^\infty \rho J_1(\rho r) U(\rho)\,\mathrm{d}\rho. \tag{43}$$

It is not difficult to show that $\bar{u}_\theta = 0$. Similarly we may show for the stresses that

$$\bar{\sigma}_{rz} = 2\pi \int_0^\infty \rho J_1(\rho r) \bar{T}(\rho)\,\mathrm{d}\rho$$
$$\sigma_{\theta z} = 0. \tag{44}$$

† Alternative 1 corresponds to a single sink at $z = -h$ in an unbounded medium while Alternative 2 corresponds to a single sink and an image source placed at $z = h$ in an unbounded medium.

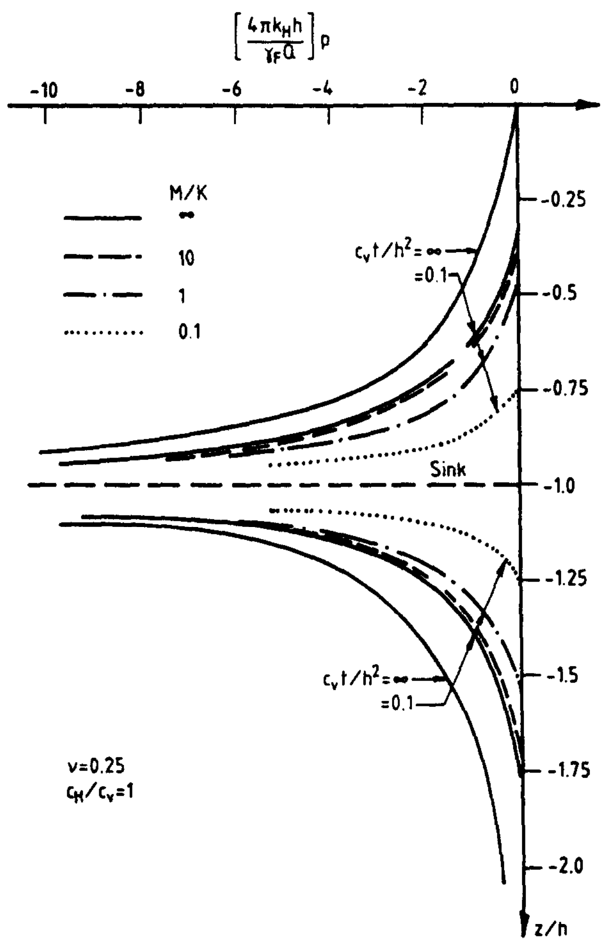

Fig. 2. Isochrones of excess pore pressure on the vertical axis for $c_V t/h^2 = 0.1$ and ∞.

The single infinite integrals contained in eqns (42)–(44) have been evaluated numerically, using Gaussian quadrature.

Evaluation of the field quantities is finally achieved by inversion of the appropriate Laplace transforms. As mentioned earlier, this is also done numerically, using the efficient algorithm developed by Talbot[9].

8. RESULTS

The solutions have been evaluated for the particular case where the soil skeleton has a Poisson's ratio $\nu = 0.25$ and the results have been summarized in Figs 2–4. In discussing the effects of pore fluid compressibility it is convenient to define the relative compressibility as M/K where M is the bulk modulus (adjusted for porosity) of the pore fluid and K is the bulk modulus of the elastic solid skeleton, given by

$$K = \frac{2(1+\nu)}{3(1-2\nu)} G.$$

Figure 2 shows isochrones of excess pore pressure on the vertical axis through the point sink for an isotropic soil ($c_H/c_V = 1$) and for non-dimensional times $c_V t/h^2 = 0.1$ and ∞. The symbol t is used here to represent the elapsed time since the commencement of pumping. In all cases the changes in pore pressure due to pumping are actually suctions and this is indicated by the negative values of p. When the excess pore pressures are

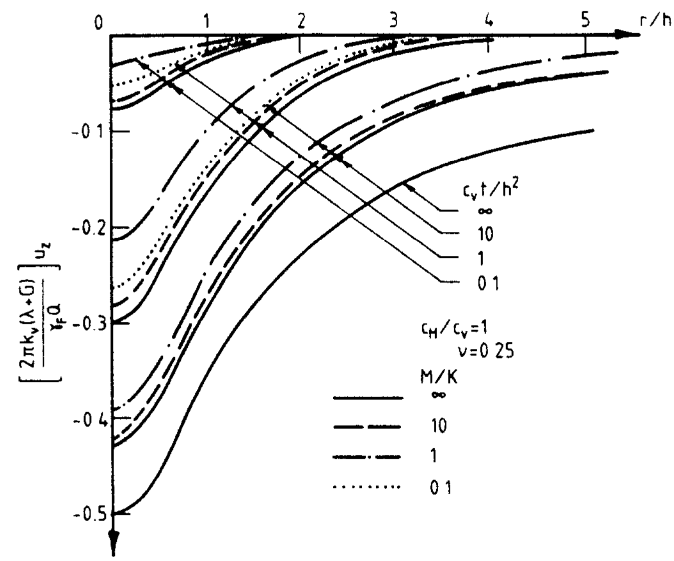

Fig. 3. Isochrones of surface settlement for the isotropic case.

normalized as indicated in Fig. 2 (i.e. using the horizontal permeability k_H) the steady state response ($c_V t/h^2 = \infty$) is independent of both the degree of anisotropy of permeability (i.e. k_H/k_V or c_H/c_V) and also of the degree of compressibility of the pore fluid (i.e. M/K). Indeed it is possible to find a closed form expression for the excess pore pressure distribution at large time and this has been shown by the authors[10] to be

$$p = -\left(\frac{Q\gamma_F}{4\pi k_V \Psi}\right)\left[\frac{1}{\sqrt{[r^2 + \Psi^2(z+h)^2]}} - \frac{1}{\sqrt{[r^2 + \Psi^2(z-h)^2]}}\right] \tag{45}$$

where $\Psi^2 = c_H/c_V = k_H/k_V$.

Along the axis $r = 0$, this of course reduces to

$$p = -\left(\frac{Q\gamma_F}{4\pi k_H}\right)\left[\frac{1}{|z+h|} - \frac{1}{|z-h|}\right] \tag{46}$$

where the dependence on k_H alone is clearly seen.

At intermediate times the normalized excess pore pressures along the axis are a function of the relative compressibility of the pore fluid M/K, as illustrated in Fig. 2 for the time corresponding to $c_V t/h^2 = 0.1$. The results show that the more compressible the pore fluid, i.e. the smaller the value of M/K, then the slower is the development of the excess pore suctions and hence the slower will be the consolidation of the soil around the sink. However, it is interesting to note that even during the transient period the isochrones of normalized excess pore pressure along the axis are practically independent of the degree of anisotropy of permeability for all cases of M/K considered, e.g. the differences between isochrones for $c_H/c_V = 1$ and 10 can hardly be plotted at the scale shown on Fig. 2. Of course, away from the vertical axis the excess pore pressures become more highly dependent on the degree of anisotropy of the soil.

Typical results for displacement are indicated on Figs 3 and 4 where isochrones of surface settlement have been plotted against radial distance from the vertical axis.

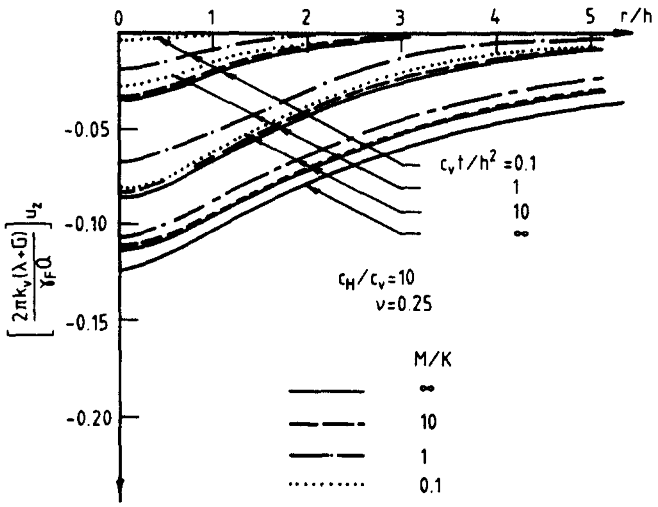

Fig. 4. Isochrone of surface settlement for an anisotropic case: $c_H/c_V = 10$.

The settlement values have been plotted in non-dimensional form and selected cases corresponding to $M/K = 0.1$, 1, 10 and ∞ have been shown. Figure 3 shows the surface profiles for an isotropic soil ($c_H/c_V = 1$) and Fig. 4 for an anisotropic soil ($c_H/c_V = 10$). On each figure the settlements have been normalized using the vertical permeability k_V and thus Figs 3 and 4 allow a comparison of two soils having the same vertical permeability but different horizontal permeabilities. It is obvious that at any given location of the surface and for any given value of M/K the surface settlements at any particular time are smaller in the anisotropic soil than in the isotropic soil, but that the settlements are more uniform in the anisotropic case. The figures also show that the relative compressibility of the pore fluid has a marked influence on the time-dependent surface settlements. Generally, the more compressible the pore fluid (i.e. smaller M/K), the slower is the settlement. The rate of settlement of a soil for which $M/K = 0.1$ is more than 10 times slower than a soil having an incompressible pore fluid ($M/K = \infty$).

The long-term settlements are independent of the compressibility of pore fluid and it is perhaps worth noting their closed form expressions. For the general case the authors reported[10] that the vertical displacement of a point on the surface at large time is given by

$$u_z(r,0) = -\left(\frac{Q\gamma_F}{2\pi k_V(\lambda+G)(\Psi^2-1)}\right)\ln\left[\frac{\Psi h + \sqrt{(\Psi^2 h^2 + r^2)}}{h + \sqrt{(h^2+r^2)}}\right]. \tag{47}$$

For the equivalent isotropic deposit eqn (47) becomes

$$u_z(r,0) = -\left(\frac{Q\gamma_F}{4\pi k_V(\lambda+G)}\right)\frac{1}{\sqrt{(r^2+h^2)}}. \tag{48}$$

9. CONCLUSIONS

A solution has been found for the consolidation of a saturated elastic half space brought about by the commencement of pumping of the pore fluid from a sink embedded within the half space. The medium was assumed to be homogeneous and isotropic with regard to deformation properties but transversely isotropic with regard to flow of pore fluid. The governing equations of the problem have been solved in Laplace transform

Table 1.

Transform	0 Alt. 1	0 Alt. 2	1	2	3
$\bar{N}$	$-\rho^2\bar{K}_b$	$-\rho^2(\bar{K}_b-\bar{K}_a)$	$\dfrac{\rho^2 e^{\mu z}}{\mu^2-\rho^2}$	$e^{\rho z}$	$\left[\left(\dfrac{\lambda}{\lambda+G}\right)\delta-1+(1-\delta)(1+\rho z)\right]e^{\rho z}$
$\bar{T}$	$\rho\bar{L}_b$	$\rho(\bar{L}_b-\bar{L}_a)$	$-\dfrac{\rho\mu e^{\mu z}}{\mu^2-\rho^2}$	$-e^{\rho z}$	$\left[\left(\dfrac{G}{\lambda+G}\right)\delta-(1-\delta)(1+\rho z)\right]e^{\rho z}$
$\bar{P}$	$(\lambda+2G)\bar{H}_b$	$(\lambda+2G)(\bar{H}_b-\bar{H}_a)$	$(\lambda+2G)e^{\mu z}$	0	$2Ge^{\rho z}$
$\bar{U}$	$\rho\bar{K}_b$	$\rho(\bar{K}_b-\bar{K}_a)$	$\dfrac{\rho}{\mu^2-\rho^2}e^{\mu z}$	$-\dfrac{e^{\rho z}}{\rho}$	$\dfrac{1}{\rho}\left[\left(\dfrac{2G}{\lambda+G}\right)\delta-(1-\delta)(1+\rho z)\right]e^{\rho z}$
$\bar{W}$	$-\bar{L}_b$	$-(\bar{L}_b-\bar{L}_a)$	$\dfrac{\mu}{\mu^2-\rho^2}e^{\mu z}$	$\dfrac{e^{\rho z}}{\rho}$	$z(1-\delta)e^{\rho z}$

$$H_b=\frac{1}{2}\frac{e^{-\mu Z_b}}{\mu}, \qquad H_a=\frac{1}{2}\frac{e^{-\mu Z_a}}{\mu}$$

$$K_b=\frac{1}{2(\mu^2-\rho^2)}\left[\frac{e^{-\rho Z_b}}{\rho}-\frac{e^{-\mu Z_b}}{\mu}\right], \qquad K_a=\frac{1}{2(\mu^2-\rho^2)}\left[\frac{e^{-\rho Z_a}}{\rho}-\frac{e^{-\mu Z_a}}{\mu}\right]$$

$$L_b=\frac{\operatorname{sgn}(-h-z)}{2(\mu^2-\rho^2)}[e^{-\rho Z_b}-e^{-\mu Z_b}], \qquad L_a=\frac{\operatorname{sgn}(h-z)}{2(\mu^2-\rho^2)}[e^{-\rho Z_a}-e^{-\mu Z_a}]$$

$$Z_b=|z+h|, \qquad Z_a=|z-h|.$$

space requiring the use of double and triple Fourier transforms. Inversion of some of these transforms has been carried out using numerical integration.

Some particular solutions have been evaluated for an elastic medium having a Poisson's ratio $\nu = 0.25$. These indicate that the major effects of the anisotropy are as follows:

(1) At all comparable non-dimensional times the values of excess pore pressure down the vertical axis containing the sink are virtually independent of the vertical permeability k_V but inversely proportional to the horizontal permeability k_H. (In this context the non-dimensional time factor c_V includes the term k_V.)

(2) At all times the profiles of surface settlement are in the form of an axi-symmetric trough. The deepest part of the trough is centred above the sink and for different soils with the same value of k_V the trough is deepest and the sides steepest in the isotropic case. As the ratio k_H/k_V is increased the profile of surface settlement becomes more uniform, i.e. the settlement trough becomes shallower and more gradual.

It has also been demonstrated that the compressibility of the pore fluid can have a significant influence on the rate of consolidation of the soil surrounding the point sink and thus on the settlement of the surface of the half space. Generally the more compressible the pore fluid then the slower will be the development of excess pore suctions and hence the slower the rate of surface settlement.

The solutions presented may have application in practical problems such as dewatering operations in compressible soils and in the extraction of fluid and gas from petroleum bearing deposits. In such cases the time to reach steady state and the final profile of surface settlement could be of great interest.

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APPENDIX

The aim of this Appendix is to verify the expressions for $\bar{H}, \bar{K}, \bar{L}$ contained in eqns (28). We proceed as follows.

Let

$$\phi = \int_0^\infty e^{-i\gamma z} \frac{e^{-p|z|}}{2p} dz$$

where p has a positive real part. Then

$$\phi = \int_0^\infty \cos \gamma z \frac{e^{-p|z|}}{p} dz = \frac{1}{p^2 + \gamma^2}.$$

Thus using the Fourier inversion theorem

$$\frac{e^{-p|z|}}{2p} = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{i\gamma z}}{\gamma^2 + p^2} d\gamma.$$

Also, let

$$\begin{aligned}\phi &= -\int_{-\infty}^\infty e^{-i\gamma z} \frac{\operatorname{sgn}(z)}{2} e^{-p|z|} dz \\ &= \int_0^\infty i \sin \gamma z \, e^{-p|z|} dz \\ &= \frac{i\gamma}{p^2 + \gamma^2}.\end{aligned}$$

Thus from the Fourier inversion theorem

$$-\frac{\operatorname{sgn}(z)}{2} e^{-p|z|} = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{i\gamma}{p^2 + \gamma^2}.$$

The results of eqns (28) then follow.