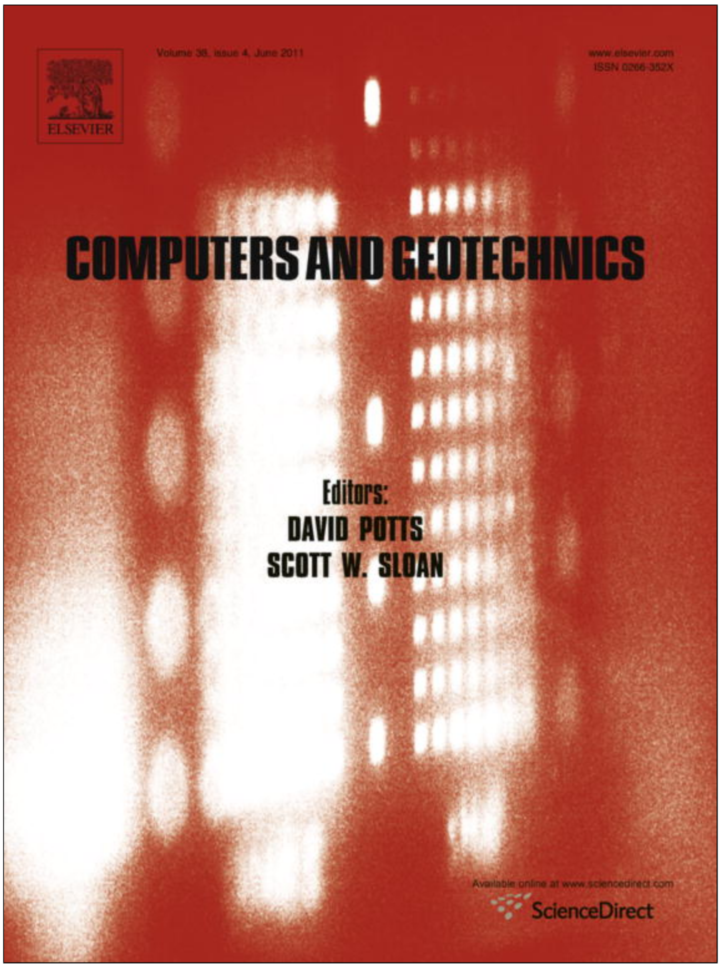
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A fast algorithm for finding the first intersection with a non-convex yield surface

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ABSTRACT

A major task in the numerical modelling of soils using complex elastoplastic material models is stress updating. This paper proposes a fast and robust numerical algorithm for locating the first intersection between a non-convex yield surface and an elastic trial stress path. The intersection problem is cast into a problem of finding the smallest positive root of a nonlinear function. Such a function may have multiple roots within the interval of interest. The method is based on the modified Steffensen method, with important modifications to address the issues arising from the non-convexity. Numerical examples demonstrate that the proposed M^2 Steffensen method is indeed computationally efficient and robust.

1. Introduction

Implementation of complex elastoplastic constitutive models for soils into finite element codes requires development of robust procedures for stress updating, the integration of the constitutive model. While many constitutive models lead to convex yield surfaces there are certain cases where the yield surface of a soil model is non-convex. For example, the yield surface for an unsaturated soil model is non-convex, if both saturated and unsaturated states of the soil are considered. Because both partial and full saturation are only different states of a soil, a single constitutive model should be expected to work for both states. As such, the non-convexity becomes inevitable. Wheeler et al. [13] pointed out the possible non-convexity of the most widely used model for unsaturated soils, the Barcelona Basic Model (BBM [1]) in the unsaturated zone. Sheng [7] argued that this non-convexity in BBM is inevitable if the pore water pressure is allowed to vary between positive and negative values. The non-convexity exists irrespective of the stress state variable used to formulate the constitutive model, as illustrated in Fig. 1. Some researchers have also argued that suction should be treated as a hardening (internal) variable instead of an additional variable in the stress space. However, this argument does not alter the fact that the size of the yield surface in stress space varies with suction and this variation does not necessarily result in plastic deformation. Another example of non-convexity appears when the Hvorslev envelope is added to the Mohr–Coulomb envelope in a Mohr–Coulomb type model. The Hvorslev envelope is usually used for low stress levels and is flatter than the Mohr–Coulomb envelope in the space of normal (or mean) stress versus shear (or deviator) stress, leading to a non-convex elastic zone around the transition.

A closely related problem arises when the elastic behaviour inside the yield surface is nonlinear. In this case, even if the yield surface is convex, the elastic trial stress path may cross the yield surface more than once. In other words, the elastic trial stress path is a curve instead of a straight line in the stress space. As such, a stress path that starts and ends inside the yield surface can still intersect with the yield surface (Fig. 2). In general, a convex elastic zone enclosing nonlinear behaviour is mathematically equivalent to a non-convex zone enclosing linear behaviour.

One of the main challenges in integrating constitutive models with non-convex yield surfaces or nonlinear elasticity is that the elastic trial stress path may cross the initial yield surface more than once, as illustrated in Fig. 1 for unsaturated soil models. Furthermore, the number of times the yield surface is crossed remains unknown, for a given strain increment and an initial stress state. In such circumstances, only the first intersection is of interest, as the stress path is likely to cause an initial elastoplastic loading followed by an elastic unloading. This leads to the key issue in solving non-convex models: to find the first intersection between the elastic trial stress path and the initial yield surface.

Integration of rate-type constitutive equations can be carried out in an implicit or explicit manner. In implicit schemes, all gradients and functions are evaluated at advanced unknown stress states and the solution is achieved by iteration. These methods do not usually involve a procedure to find the intersection between the elastic trial stress path and the yield surface. However, when the yield surface is non-convex, the knowledge of the first

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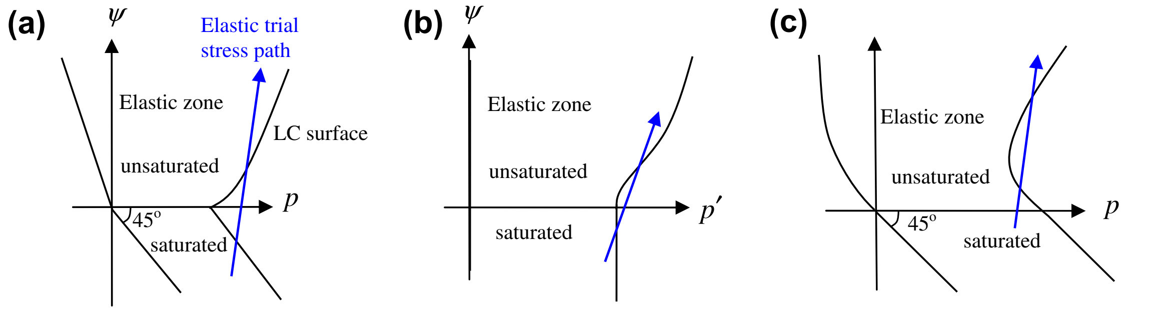

Fig. 1. Non-convexity of yield surfaces in unsaturated soil models in the suction–stress space (ψ: suction – pore air pressure in excess of pore water pressure, p and p': net and effective mean stress respectively; (a) net stress [1]; (b) effective stress [8]; and (c) net stress [9].

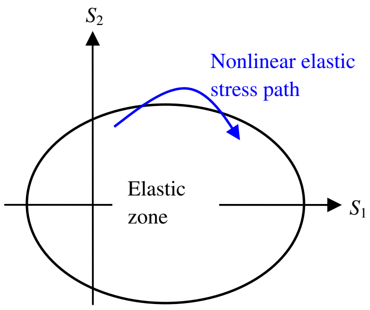

Fig. 2. Multiple intersection problem due to nonlinear elasticity (S_1 and S_2 are two stress variables).

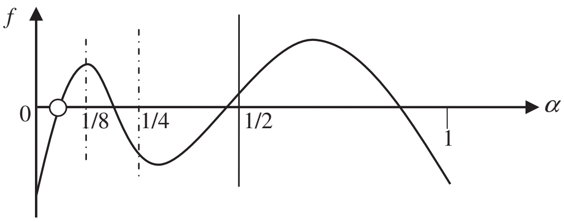

Fig. 3. Bracketing the roots for nonlinear function according to Pedroso et al. [6].

intersection seems to be a prerequisite for making these schemes work. Otherwise, a trial stress path that starts and ends in the initial elastic zone would be assumed to cause only elastic deformation. For unsaturated soil models, where the location of non-convexity is known, implicit schemes may work if the strain increments are kept sufficiently small [2]. The difficult question is: what is sufficiently small? On the other hand, explicit schemes estimate the gradients and functions at the current known stress states and proceed in an incremental fashion. These schemes invariably need to determine the intersection and substepping methods have been developed to control the integration error [11,12]. Therefore, it seems necessary for both implicit and explicit methods to find the first intersection between the current yield surface and the elastic trial stress path.

There is very little discussion in the literature about integrating non-convex soil models. Pedroso et al. [6] proposed a novel method for bracketing the intersection between the elastic trial stress path and a non-convex yield surface. Finding this intersection can be cast into a problem of finding the multiple roots of a nonlinear equation:

$$f(\alpha) = f(\mathbf{s}_\alpha, \mathbf{h}) = 0 \tag{1}$$

where $0 \leqslant \alpha \leqslant 1$, $f(\mathbf{s}, \mathbf{h})$ is the yield function, s is a set of external variables such as stress and suction, $\mathbf{h}$ is a set of internal variables, typically plastic strain or hardening parameters, subscript α indicates that the quantity is evaluated at strain increment $\alpha\Delta\boldsymbol{\varepsilon}$, the strain increment $\Delta\boldsymbol{\varepsilon}$ is assumed to be known, and the variable $\mathbf{s}_\alpha$ is assumed to be fully determined for given strain increment $\alpha\Delta\boldsymbol{\varepsilon}$, for example using elasticity theory.

The method proposed by Pedroso et al. [6] for bracketing the roots (α) is illustrated in Fig. 3. For a given increment ($\alpha = 1$), the number of roots of $f(\alpha)$ is first computed. If there is more than one root, the increment is divided into two equal sub-increments. The number of roots of each sub-increment is then computed. If the first sub-increment contains more than one root, it is further divided into two sub-increments. This process is repeated until the first sub-increment contains at most one root (Fig. 3). Once the roots are bracketed, the solution of the first root can be found by using numerical methods such as the Pegasus method [12].

Sheng et al. [10] applied the method by Pedroso et al. [6] to integrate an unsaturated soil model described by Sheng et al. [9] and found that the method can provide an accurate solution of the intersection problem. However, this approach was found to be computationally extremely expensive. It should be realised that the root-finding procedure must be applied for all strain increments at all Gauss points, irrespective of the starting and ending stress states. Pedroso's formula requires both the first and second orders of gradients of the yield function and also complex numerical integration to compute the number of roots, leading to high computational expense.

The objective of this paper is to propose a fast and robust numerical algorithm that can be used to find the first intersection between a non-convex yield surface and an elastic trial stress path. This problem is further cast into a problem of finding the smallest positive root of a nonlinear function that has multiple roots.

2. M^2 Steffensen's method

The problem of finding the first intersection between a non-convex yield surface and an elastic trial stress path can be reformulated into a problem of finding the smallest positive root of a nonlinear function that has multiple roots:

$$f(x) = 0 \quad 0 \leqslant x \leqslant 1 \tag{2}$$

Furthermore, for elastoplastic problems, the initial stress state must be inside or on the initial yield surface, i.e. $f(0) \leqslant 0$. If the initial stress is on the yield surface, i.e. $f(0) = 0$, the root $x = 0$ is either the desired solution if the angle between the trial stress path and the norm of the yield surface is less than 90°, or otherwise irrelevant [12]. Hence we are only interested in the following situations:

$$f(0) < 0 \tag{3}$$

Due to the non-convex yield surface, function $f(x)$ does not necessarily vary monotonically between $x = 0$ and the smallest positive root. A typical case is shown in Fig. 4. For the nonlinear equation shown in Fig. 4b, most existing algorithms, including the bisection-type methods, secant-type methods and Newton-type methods, will not work. One would argue that the Newton-type methods would work if a better starting point was located. However, finding a better starting stress state can not be achieved in a general manner and is indeed part of the solution of the problem we are seeking.

The method proposed to solve the nonlinear equation illustrated in Fig. 4b is based on the modified Steffensen method ([5] in [3,4]) which solves a nonlinear equation via the following iteration:

$$x_{n+1} = x_n - j\frac{(f(x_n))^2}{z} \tag{4}$$

with

$$z = f(x_n) - f(x_n - f(x_n)) \tag{5}$$

and

$$j = \frac{z^2}{z^2 + f(x_n)(z + f(x_n) - f(x_n + f(x_n)))} \tag{6}$$

The parameter j converges to the multiplicity of the root. Compared with Newton's method, the modified Steffensen method approximates the derivative of the function by

$$f' \approx \frac{f(x) - f(x - f(x))}{f(x)} \tag{7}$$

for simple roots. It is thus clear that the function itself is used as an increment in the method.

The modified Steffensen method has to be further modified to be useful for the nonlinear equation shown in Fig. 4b. In this paper the following M^2 Steffensen method is proposed:

$$x_{n+1} = x_n + \zeta|j|\frac{(f(x_n))^2}{|z|} \tag{8}$$

with

$$z = f(x_n) - f(x_n - \zeta f(x_n)) \tag{9}$$

and

$$j = \frac{z^2}{z^2 + f(x_n)(z + f(x_n) - f(x_n + \zeta f(x_n)))} \tag{10}$$

where ζ is a constant used to adjust the increment size in the approximation (7) which depends on the relative magnitude of function $f(x)$ compared with unknown x. This parameter is further discussed below, in conjunction with numerical examples.

Eq. (7) is designed for functions in the form of Eqs. (2) and (3). It may overshoot the solution near the desired root so that $f(x_{n-1})f(x_n) < 0$. In this case, the iteration can be carried out in the following form:

$$x_{n+1} = x_n - \zeta j\frac{(f(x_n))^2}{z}f(x_{n-1})f(x_n) < 0 \tag{11}$$

The M^2 Steffensen method involves the evaluation of the function only, not its derivatives. It needs only one case of exception handling, i.e. Eq. (11). The method can be implemented with a few lines of coding.

3. Numerical examples

To assess the performance of the proposed method a number of numerical examples are now presented. In the first two cases, simple functions are used which nevertheless possess features which are challenging for root-finding, and which represent different features of non-convex yield surfaces. The final example presented here is for a yield surface from an unsaturated soil model.

3.1. The first root of $f(x) = -\sin(8x) = 0$

In the first example, the M^2 Steffensen method is used to find the root of the following cyclic function:

$$f(x) = -\sin(8x) \quad 0 \leqslant x \leqslant 1$$

The function is illustrated in Fig. 5. The function has three roots within the interval $[0, 1]$. In the context of elastoplastic models, the root ($x = 0$) means that the starting stress point is on the yield surface and hence is not of interest here. The objective is to find its smallest positive root (x_s) for an arbitrary starting point between $0 < x < x_s$.

The performance of the proposed M^2 Steffensen method to find the smallest positive root (x_s) for the function is shown in Table 1 which shows the convergence for two starting points: $x_0 = 0.3$ and $x_0 = 0.0001$. The method is able to locate the correct root for these arbitrary starting points between $0 < x < x_s$. Indeed, the starting point can be very close to zero, i.e. $x_0 = 0.0001$, however as the table shows it generally takes more iterations to find the correct root when the starting point is close to zero. In both cases, the solution converges quadratically towards the desired root (at least asymptotically). The parameter ζ also affects the convergence rate, as shown in Table 2. As mentioned earlier, ζ is used to control the increment size ($\zeta f(x)$) and hence depends on the relative magnitude of the function. The maximum value of ζ can be approximated by:

$$\zeta \leqslant \frac{|x_{max} - x_{min}|}{|f_{max} - f_{min}|} \tag{12}$$

In this example, the above condition leads to $\zeta \leqslant 0.5$.

The cyclic function in the first example has constant amplitude and frequency. In this case, the desired root is approached monotonically from the left on the x axis and Eq. (11) is not evoked.

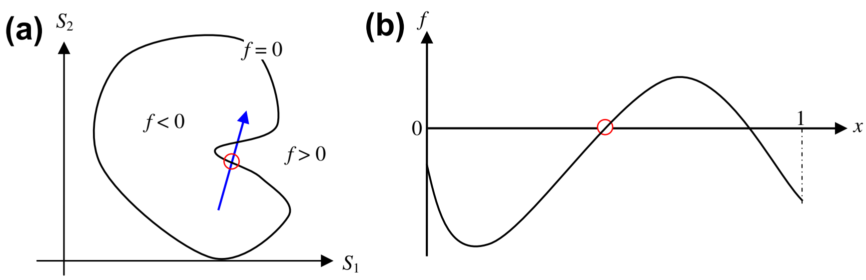

Fig. 4. Casting the stress integration into a root-finding problem. (a) Non-convex yield surface. (b) Yield function along the stress path.

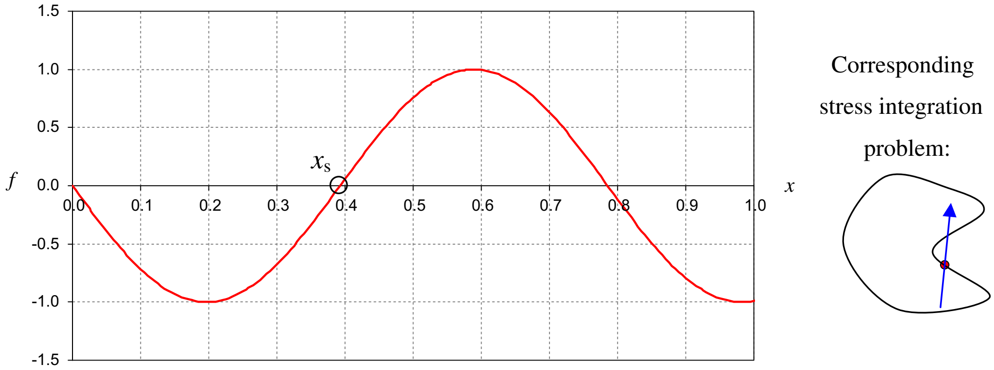

Fig. 5. Equation $-\sin(8x) = 0$ and its roots between [0, 1].

Table 1
Convergence of M^2 Steffensen method for $-\sin(8x) = 0$.

x	ζ	$f(x)$	$f(x+\zeta f(x))$	$f(x-\zeta f(x))$	z	j
0.3	0.3	−0.675463181	−0.70249	0.770432	−1.4459	0.68567
0.364908703	0.3	−0.220496073	−0.68275	0.302074	−0.52257	0.953557
0.391523606	0.3	−0.009403667	−0.03197	0.013165	−0.02257	0.999912
0.392698987	0.3	−7.53898E−07	−2.6E−06	1.06E−06	−1.8E−06	1
0.392699082	0.3	−1.22515E−16	−5.7E−16	3.22E−16	−4.4E−16	1
0.0001	0.3	−0.0008	0.00112	−0.00272	0.00192	0.999999
0.0002	0.3	−0.0016	0.00224	−0.00544	0.00384	0.999997
0.000400001	0.3	−0.003200002	0.00448	−0.01088	0.00768	0.99999
0.000800008	0.3	−0.006400021	0.00896	−0.02176	0.015358	0.999959
0.001600065	0.3	−0.01280017	0.017919	−0.04351	0.030707	0.999836
0.003200521	0.3	−0.025601374	0.035831	−0.08694	0.061336	0.999343
0.006404174	0.3	−0.051210983	0.071612	−0.17326	0.12205	0.997351
0.012833386	0.3	−0.102486822	0.142811	−0.34162	0.239129	0.989064
0.025866544	0.3	−0.205458667	0.282279	−0.64424	0.438784	0.950354
0.05329519	0.3	−0.413560793	0.536416	−0.98849	0.574926	0.680619
0.114037602	0.3	−0.790913762	0.833765	−0.32508	−0.46583	0.116017
0.160775692	0.3	−0.959776624	0.85067	0.433233	−1.39301	0.386928
0.237536228	0.3	−0.946206351	0.36218	0.857089	−1.8033	0.52482
0.315705617	0.3	-0.577732332	−0.90825	0.696573	−1.27431	0.748627
0.374531252	0.3	−0.144831461	−0.47322	0.200877	−0.34571	0.979439
0.39235974	0.3	−0.002714729	−0.00923	0.003801	−0.00652	0.999993
0.392699079	0.3	−1.81395E−08	−6.2E−08	2.54E−08	−4.4E−08	1
0.392699082	0.3	−1.22515E−16	−5.7E−16	3.22E−16	−4.4E−16	1

Table 2
Performance of M^2 Steffensen method in solving $-\sin(8x) = 0$.

ζ	Starting point	Number of iterations	x_s	$f(x_s)$
0.1	0.0001	21	0.392699082	-1.22515×10^{-16}
	0.1	10	0.392699082	-1.22515×10^{-16}
	0.3	5	0.392699082	-1.22515×10^{-16}
0.3	0.0001	18	0.392699082	-1.22515×10^{-16}
	0.1	8	0.392699082	-1.22515×10^{-16}
	0.3	5	0.392699082	-1.22515×10^{-16}
0.5	0.0001	16	0.392699082	-1.22515×10^{-16}
	0.1	5	0.392699082	-1.22515×10^{-16}
	0.3	3	0.392699082	-1.22515×10^{-16}

3.2. The first root of $f(x) = -\frac{\cos(10x-1.5)}{1+10x} = 0$

In the second example, the M^2 Steffensen method is used to find the root for the following function:

$$f(x) = -\frac{\cos(10x - 1.5)}{1 + 10x} \quad 0 \leqslant x \leqslant 1$$

The function is illustrated in Fig. 6. The function has three roots in the interval [0, 1], but only the smallest root is of interest. Again, the starting stress point can be arbitrary, within the interval $0 < x < x_s$.

It is found that $\zeta \leqslant 1.6$ for this example, using Eq. (12). The performance of the method is shown in Table 3. The method again has an asymptotically quadratic convergence rate. For $x_0 = 0.0001$ or $x_0 = 0.1$, only seven iterations are required to find the root. It should be noted that the function changes signs near the desired root and hence Eq. (11) has to be evoked. Table 3 also shows that the three function values at the root become equal, z becomes zero and j becomes undefined.

The function in this example has a decreasing amplitude with increasing x. The root is approached from both right and left (Table 3), and Eq. (11) has been evoked. The M^2 Stephensen method has also been tested for cyclic functions with increasing amplitude and varying frequency and similar convergence performance has been observed.

4. The intersection of SFG model

In the last example, we study the performance of the proposed M^2 Steffensen method in integrating a yield surface for an unsaturated soil model, the SFG model [9]. The SFG model features a non-convex elastic zone in the space of suction versus net mean stress. The yield function is given as follows:

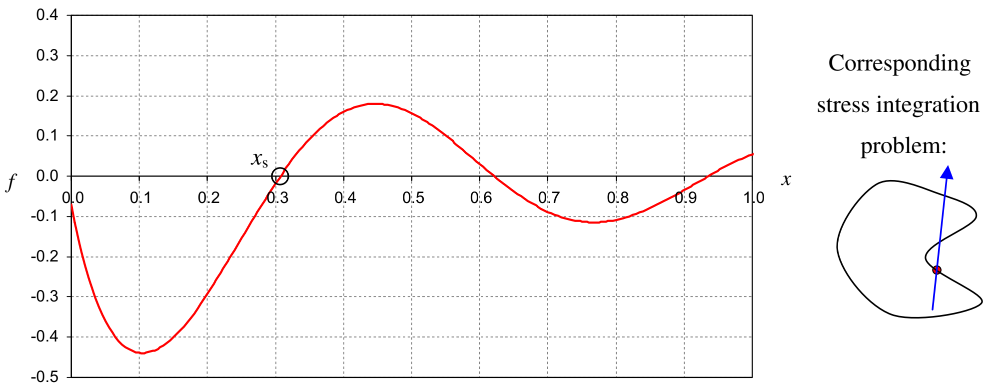

Fig. 6. Equation $-\frac{\cos(10x-1.5)}{1+10x} = 0$ and its roots between [0, 1].

Table 3
Convergence of M^2 Steffensen method for $-\frac{\cos(10x-1.5)}{1+10x} = 0$.

x	ζ	$f(x)$	$f(x+\zeta f(x))$	$f(x-\zeta f(x))$	z	j
0.0001	1.5	−0.071662998	−11.40264186	−0.439042446	0.367379448	−0.191886658
0.004123566	1.5	−0.107370293	−1.751899777	−0.372767816	0.265397523	−0.523166999
0.038211683	1.5	−0.316606253	0.271646783	0.143937738	−0.460543991	0.389779009
0.165467476	1.5	−0.372196889	0.224121055	−0.103772382	−0.268424507	0.182915938
0.307068412	1.5	−2.75657E−05	−0.000129155	7.40031E−05	−0.000101569	1.000055131
0.307079634	1.5	3.56011E−09	1.66783E−08	−9.55813E−09	1.31182E−08	0.999999993
0.307079633	1.5	−1.5048E−17	−1.5048E−17	−1.5048E−17	0	#DIV/0!
0.1	1.5	−0.438791281	0.152263232	−0.114171428	−0.324619853	0.207777411
0.284854697	1.5	−0.057274657	−0.295241294	0.126322101	−0.183596758	1.101786011
0.314383681	1.5	0.017610625	0.07505586	−0.048962198	0.066572823	0.9650003
0.3076404	1.5	0.001375634	0.006404424	−0.003704914	0.005080548	0.997249145
0.307083226	1.5	8.82801E−06	4.13557E−05	−2.37018E−05	3.25298E−05	0.999982344
0.307079633	1.5	3.65093E−10	1.71038E−09	−9.80195E−10	1.34529E−09	0.999999999
0.307079633	1.5	−1.5048E−17	−1.5048E−17	−1.5048E−17	0	#DIV/0!

$$f = q^2 - M^2(p - p_0)(p_c - p) \tag{13}$$

where q is the deviator stress, p is the net mean stress, M is the slope of the critical state line in the $q-p$ space, and p_0 and p_c are two yield stresses when q is zero. Both yield stresses are functions of suction (ψ):

$$p_0 = \begin{cases} -\psi & \psi \leqslant \psi_{sa} \\ -\psi_{sa} - \psi_{sa}\ln\frac{\psi}{\psi_{sa}} & \psi > \psi_{sa} \end{cases} \tag{14}$$

$$p_c = \begin{cases} p_{c0} - \psi & \psi \leqslant \psi_{sa} \\ p_{c0} - \psi + \frac{p_{c0}}{\rho}\left(\psi - \psi_{sa} - \psi_{sa}\ln\frac{\psi}{\psi_{sa}}\right) & \psi > \psi_{sa} \end{cases} \tag{15}$$

where ψ_{sa} is the transition suction between saturated and unsaturated states and is assumed to be a constant in this paper, p_{c0} is the yield stress when $q = 0$ and $\psi = 0$, and ρ is a material constant depending on the stress history of the soil and can not be greater than p_{c0}. The yield stress (p_0) usually remains stationary in the stress space. On the other hand, the yield stress (p_c) can evolve in the stress space due to plastic strain and as such hardening and softening occur. Two examples of the SFG yield surface are shown in Fig. 7: one for an air-dry unsaturated soil and the other for a compacted soil. In both cases, the yield surfaces are non-convex.

The M^2 Steffensen method is now used to find the intersection between a known initial yield surface (p_{c0}) and a given trial stress increment ($dq, dp, d\psi$). The initial stress state (q, p, ψ) is also assumed to be known. Furthermore, without losing generality, the deviator stress q is assumed to remain zero. The problem is illustrated in Fig. 8, where ρ is set to 300 kPa and ψ_{sa} to 100 kPa. The stress increment spans from the initial stress state ($p = 300$ kPa, $\psi = -200$ kPa) to the trial stress state ($p = 500$ kPa, $\psi = 900$ kPa). The problem can be cast into a specific root-finding problem as follows:

$$f(\alpha) = -(300 + 200\alpha - p_0)(p_c - 300 - 200\alpha) = 0$$

with

$$p_0(\alpha) = \begin{cases} -200 - 1100\alpha & 0 \leqslant \alpha \leqslant \frac{3}{11} \\ -100 - 100\ln\frac{1100\alpha - 200}{100} & \frac{3}{11} < \alpha \leqslant 1 \end{cases}$$

$$p_c(\alpha) = \begin{cases} 700 - 1100\alpha & 0 \leqslant \alpha \leqslant \frac{3}{11} \\ 700 - 1100\alpha + \frac{5}{3}\left(1100\alpha - 300 - 100\ln\frac{1100\alpha - 200}{100}\right) & \frac{3}{11} < \alpha \leqslant 1 \end{cases}$$

and α is a fraction of the stress increment and varies between 0 and 1. Fig. 8 shows that the given elastic stress path crosses the yield surface (p_c) twice. Therefore, plastic hardening occurs and the yield surface (p_c) evolves. In this case, it is necessary to find the first intersection (x_s) to make a stress integration scheme work, no matter if it is implicit or explicit.

Parameter ζ is set to 0.000002, according to expression (12) and Fig. 9. Table 4 lists the convergence performance of the M^2 Steffensen method from which it can be seen that only five iterations are required to locate the first root ($s = 153.45$ kPa, $p = 364.26$ kPa) to a yield function accuracy of 10^{-9}. The method again provides a quadratic convergence rate near the root. Fig. 9 shows the yield function values along the stress path shown in Fig. 8. Due to the cyclic behaviour of the function, any Newton-type or bisection-type iterations would not be able to locate the desired root or the correct intersection for this problem. Newton-type iterations (e.g. Newton method, modified Newton method, Steffensen method, modified Steffensen method) would work only if the starting stress point was sufficiently close the desired root (x_s). On the other hand,

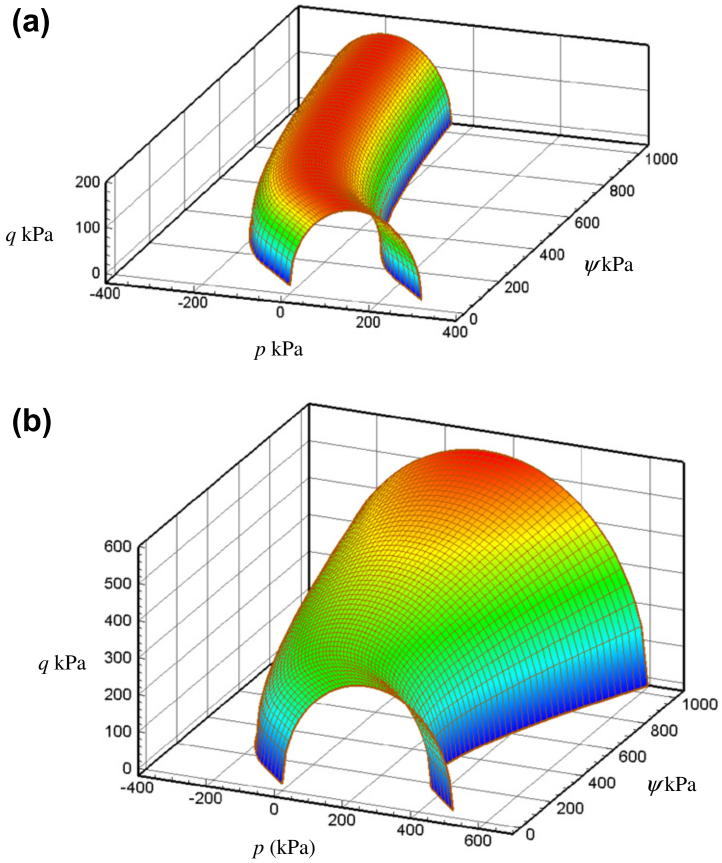

Fig. 7. SFG yield surfaces for different soil types ($M = 1.2$). (a) Yield surface for unsaturated soil air-dried from slurry ($p_{c0} = 30$ kPa, $\rho = p_{c0}$, $s_{sa} = 100$ kPa). (b) Yield surface for compacted soil ($p_{c0} = 500$ kPa, $\rho = 300$ kPa, $\psi_{sa} = 100$ kPa).

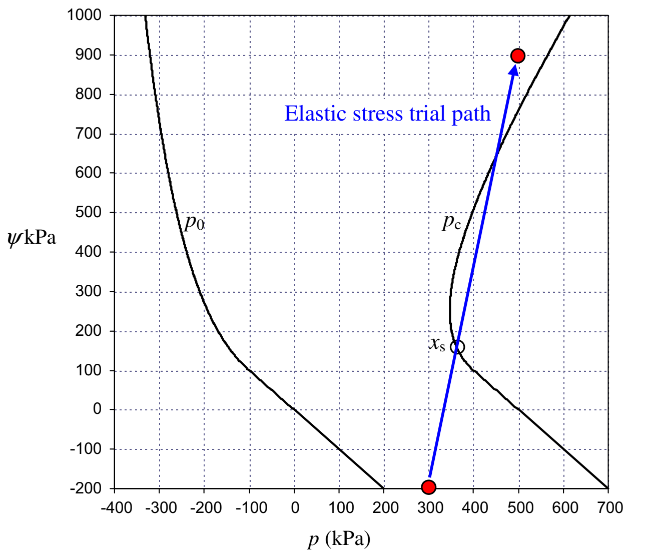

Fig. 8. Yield surfaces and trial stress path ($p_{c0} = 500$ kPa, $\rho = 300$ kPa, $\psi_{sa} = 100$ kPa).

bisection-type methods (e.g. bisection method, Regula Falsi method and Pegasus method) require that the starting and ending points are on different sides of the yield surface and there is only one root within the interval. The M^2 Steffensen method presented here will always locate the desired root, irrespective of the starting and ending stress states. Furthermore, no derivatives of the yield function are required and the method is computationally very robust and efficient.

Application of the proposed method to general three-dimensional stress–strain equations in boundary-value problems requires a complete implementation of the method into the finite element method. The above examples only demonstrate the efficiency of the method in finding the first positive root of cyclic functions or to locating the intersection between the elastic trial stress path and the yield surface. This intersection-finding method can be incorporated into both explicit and implicit stress integration methods.

5. Conclusions

Certain constitutive models for geo-materials feature non-convex elastic zones. A typical example is an unsaturated soil model, where the non-convexity occurs along the suction axis. As a consequence, the elastic trial stress path may cross the yield surface more than once. The same problem can happen when nonlinear elastic behaviour is considered. To integrate these models in the finite element method, an essential step is to find the first intersection between the elastic trial stress path and the current yield surface. This paper presents a simple numerical method for locating this intersection. The method is based on the modified Steffensen method, with important additional modifications to address the issues arising from the non-convexity. The numerical method is then used to find first roots for a number of nonlinear functions that have multiple roots within the interval of interest. The numerical examples demonstrate that the proposed M^2 Steffensen method is indeed computationally efficient and robust. Application of the proposed method for integrating general three-dimensional stress–strain

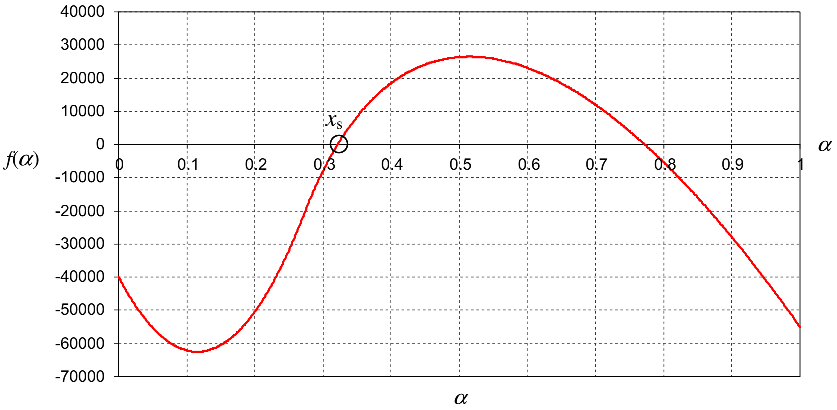

Fig. 9. The SFG yield function along the elastic trial stress path.

Table 4
Convergence of M^2 Steffensen method for the SFG model.

α	$f(\alpha)$	z	j	ψ	p
0.0000000000000000	−40000	20384.000000000	0.324415620	−200.000000000	300.000000000
0.1569858712715860	−59575.17701	−40696.950026594	0.367073967	−27.315541601	331.397174254
0.3314068887386640	3227.894546	2031.035361293	0.914572041	164.547577613	366.281377748
0.3211467983473340	−58.67427575	−39.398755012	1.001634482	153.261478182	364.229359669
0.3213215587240660	−0.015713086	−0.010539637	1.000000450	153.453714596	364.264311745
0.3213216055759780	−1.09533E−09	−7.49439E−10	0.94677871	153.453766134	364.264321115
0.3213216055759810	2.88246E−11	2.88246E−11	0.5	153.453766134	364.264321115
0.3213216055759810	0	0	#DIV/0!	153.453766134	364.264321115

equations in boundary-value problems requires a complete implementation of the method into the finite element method.

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