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A variational principle of elastoplasticity and its application to the modeling of frictional materials

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ABSTRACT

Starting from a thermomechanical description of elastoplasticity, a stress-based variational principle is derived. The principle, which generalizes von Mises's principle of maximum plastic dissipation, reproduces the conventional elastic/hardening-plastic framework applicable to metals as a special case and further proves to be suitable for developing constitutive models for frictional materials. Application of the principle to the isotropic and triaxal compression behaviour of sands is considered by means of a non-conventional extension of the modified Cam clay model. The new model allows for the specification of arbitrary stress-dilatancy relations without altering the yield potential or introducing a separate flow potential. Moreover, the elastoplastic tangent modulus is always symmetric, regardless of the degree of apparent nonassociativity.

1. Introduction

The strength and deformation characteristics of frictional materials such as clay, sands and rock are usually modeled within the framework of small-strain, rate-independent elastoplasticity. The governing equations come in the form of an additive decomposition of the total strain into elastic and plastic parts, an elastic law, a yield surface, a flow rule, and a hardening law. The benefits of casting these governing equations in terms of a variational statement have long been recognized and significant efforts have been devoted to developing variational formulations of different classes of elastoplastic models. Representative examples include Washizu (1982), Reddy and Martin (1994), Han and Reddy (2001), and Simo (1998). However, the larger part of this progress has been confined to purely cohesive materials such as metals. The generally accepted validity of associated flow theories for these materials afford the governing equations an obvious normality structure that leads naturally to a variational formulation. The principle of maximum plastic dissipation Lubliner, 1990 is a well known example. For frictional materials the situation is rather different. Here the validity of the associated flow concept is more questionable and the obstacles involved in arriving at useful variational formulations are well recognized. From a mathematical point of view, these obstacles stem essentially from the non-self-adjointness of the governing equations that results from operating with different yield and flow potentials. Thus, although some progress has been made in developing principles akin to those of associated plasticity (Hjiaj et al., 2005; Telega, 1978; Chandler, 1988; De Saxce and Bousshine, 1998), these principles are usually much weaker than the classical ones and typically require knowledge of both the stress and the deformation fields to be effective. On the contrary, classical principles require knowledge of only one set of variables (static/kinematic), after which the other set (kinematic/static) follow as the conjugate, or dual, variables.

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In this paper, the variational basis of elastoplasticity is first reexamined. This is done in the context of a general stress-based principle which derives from basic thermomechanical considerations following primarily the exposition of Simo (1998). The approach also has certain similarities to the work Collins, Houlsby, and coworkers (Collins and Houlsby, 1997; Houlsby and Puzrin, 2000) and to the 'generalized standard materials' approach of Nguyen Quoc Son and coworkers (Nguyen, 1973, 2004; Halphen and Nguyen, 1975). A detailed thermomechanical analysis is not attempted however. Indeed, emphasis is placed on the structure of the governing equations that follow from the variational principle rather than on the structure of the potential from which they derive. As such, the practical application of the principle follows what Collins et al. (2007) refer to as the 'standard extant procedure'. That is, an elastic law, a yield function, and so on are postulated a priori to eventually arrive at a model that can be calibrated against experimental data. The important point, however, is that there is not complete freedom to choose the components comprising the model. For example, it is not possible to postulate a flow potential that is completely disconnected from the yield potential. Certain rules must be obeyed. However, the structure of the governing equations are such that an appropriate degree of deviation from yield surface normality can be achieved while maintaining a normality structure of the governing equations.

The variational principle proposed comes in two different versions. The first one is quite conventional and generalizes von Mises's principle of maximum plastic dissipation. Given the strain rates, the total power of deformation is maximized subject to yield conditions cast in terms of the stresses and a set of stress-like hardening variables. The main point of interest here is that an apparent coupling between the stresses and the stress-like hardening variables follows as an integral part of the constitutive equations. This type of coupling is only rarely employed in conventional elastoplastic formulations although it has been recognized for a long time (see e.g. Collins and Houlsby (1997) and references therein). It is this coupling that allows for prescribing apparently nonassociated flow rules without introducing a separate flow potential explicitly.

Although the above mentioned coupling appears to be of significant relevance to the modeling of frictional materials, the variational principle still imposes too many restrictions on the form of the governing equations to be of practical use. Consequently, a 'relaxation' of the principle is considered. This relaxation comes about by casting the principle in incremental form. That is, instead of operating with one fixed potential, a series of potentials, each with a limited range of validity in time, are specified. This formulation is obviously somewhat weaker than classical variational formulations where the relevant potential would be a true state function of the selected state variables. The principle is, however, believed to be of considerable convenience and practical usefulness, both in terms of developing constitutive models and in terms of numerical implementation.

In the following, the theoretical aspects of the new framework are first described after which its capabilities are demonstrated by the construction of an actual constitutive model. This model is inspired by classical critical state models such as Cam clay but appears to be significantly more versatile despite requiring a similar number of material parameters. In particular, a consistent behaviour in both shear and isotropic compression can be accounted for. Furthermore, all elastoplastic tangent moduli are symmetric, regardless of the degree of apparent nonassociativity.

Standard matrix notation is used throughout the paper with bold upper and lower case letters signifying matrices and vectors respectively, and with the transpose denoted by T. Furthermore, the derivative of a function $f(\boldsymbol{\alpha},\boldsymbol{\beta})$ with respect to $\boldsymbol{\alpha}$ is denoted by $\nabla_{\alpha} f(\boldsymbol{\alpha},\boldsymbol{\beta})$.

2. Conventional elastoplasticity

In this section the governing equations of conventional rate-independent elastoplasticity are briefly summarized for later reference. We make use of a stress-space formulation following standard expositions (Simo, 1998; Lubliner, 1990; Chen and Han, 1988).

The fundamental characteristic of elastoplastic materials is the existence of a yield criterion that effectively limits the magnitude of the stresses:

$$F(\boldsymbol{\sigma},\boldsymbol{\kappa}) \leqslant 0 \tag{1}$$

where F is the yield function, $\boldsymbol{\sigma}$ are the stresses and $\boldsymbol{\kappa}$ are a set of stress-like hardening variables that evolve according to a law that will be specified shortly.

Considering only infinitesimal deformations, a standard assumption is that the total strains can be decomposed additively according to

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \tag{2}$$

where $\boldsymbol{\varepsilon}$ are the total strains, $\boldsymbol{\varepsilon}^e$ are the elastic strains, and $\boldsymbol{\varepsilon}^p$ are the plastic strains.

Following standard elasticity theory, it is be assumed that there exists a complementary elastic energy function, ψ^e, such that the elastic strains are related to the stresses via

$$\boldsymbol{\varepsilon}^e = \nabla_{\sigma}\psi^e(\boldsymbol{\sigma}) \tag{3}$$

The corresponding rate form law is given by

$$\dot{\boldsymbol{\varepsilon}}^e = \nabla^2_{\sigma\sigma}\psi^e(\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} = \mathbf{C}(\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \tag{4}$$

where a superposed dot indicates differentiation with respect to pseudo-time (only rate independent processes are considered).

Similarly, the plastic strain rates are often assumed to be derivable from a flow potential G such that

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\nabla_\sigma G(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \tag{5}$$

where the plastic multiplier $\dot{\lambda}$ satisfies the complementarity conditions:

$$\dot{\lambda} \geqslant 0, \quad \dot{\lambda}F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = 0 \tag{6}$$

Concerning the form of the flow rule it should be noted that more general non-potential forms of course are possible. In fact, the above form of the flow rule has no particular advantages or special properties unless $G = F$.

Finally, the hardening law can be written in the following general form:

$$\dot{\boldsymbol{\kappa}} = \dot{\lambda}\boldsymbol{h}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \tag{7}$$

This law implies that the stress-like hardening variables, $\boldsymbol{\kappa}$, evolve as a result of plastic straining only, i.e. for $\dot{\lambda} > 0$.

The above equations may be summarized in the following compact format:

$$\begin{aligned} &\dot{\boldsymbol{\varepsilon}} = \boldsymbol{C}(\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + \dot{\lambda}\nabla_\sigma G(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \\ &\dot{\boldsymbol{\kappa}} = \dot{\lambda}\boldsymbol{h}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \\ &F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \leqslant 0,\ \dot{\lambda}F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = 0,\ \dot{\lambda} \geqslant 0 \end{aligned} \tag{8}$$

Standard manipulations of these governing equations lead to the incremental stress–strain relations

$$\dot{\boldsymbol{\varepsilon}} = \begin{cases} \boldsymbol{C}\dot{\boldsymbol{\sigma}}, & \dot{\lambda} = 0 \\ \boldsymbol{C}^{ep}\dot{\boldsymbol{\sigma}}, & \dot{\lambda} > 0 \end{cases} \tag{9}$$

where

$$\boldsymbol{C}^{ep} = \boldsymbol{C} + \frac{1}{H}\nabla_\sigma G(\boldsymbol{\sigma}, \boldsymbol{\kappa})\nabla_\sigma F(\boldsymbol{\sigma}, \boldsymbol{\kappa})^{\mathsf{T}} \tag{10}$$

is the elastoplastic compliance modulus and

$$H = -\boldsymbol{h}^{\mathsf{T}}\nabla_\kappa F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \tag{11}$$

is the hardening modulus. We note the well known result that $\boldsymbol{C}^{ep}$ in general is symmetric only for $G = F$.

3. Thermomechanical formulation of elastoplasticity

In this section a thermomechanical formulation of elastoplasticity is briefly summarized following primarily the exposition of Simo (1998).

From the first and second laws of thermodynamics and assuming isothermal conditions, the following central identity can be derived Collins and Houlsby, 1997:

$$\mathcal{P} = \boldsymbol{\sigma}^{\mathsf{T}}\dot{\boldsymbol{\varepsilon}} = \dot{\phi} + \mathcal{D} \tag{12}$$

where $\mathcal{P}$ is the rate of internal work, ϕ is the Helmholtz free energy and $\mathcal{D} \geqslant 0$ is the dissipation. Following Simo (1998) we will assume that the Helmholtz free energy is a function of the elastic strains, $\boldsymbol{\varepsilon}^e$, and a set of strain-like hardening variables, $\boldsymbol{\alpha}$

$$\phi = \phi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}) \tag{13}$$

The time derivative of ϕ is given by:

$$\dot{\phi} = \nabla_{\varepsilon^e}\phi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha})^{\mathsf{T}}\dot{\boldsymbol{\varepsilon}}^e + \nabla_\alpha\phi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha})^{\mathsf{T}}\dot{\boldsymbol{\alpha}} \tag{14}$$

By comparison with (12) for a purely elastic process ($\dot{\boldsymbol{\alpha}} = \boldsymbol{0}, \mathcal{D} = 0$), it can be verified that the stresses are given by:

$$\boldsymbol{\sigma} = \nabla_{\varepsilon^e}\phi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}) \tag{15}$$

The set of variables conjugate to $\boldsymbol{\alpha}$ are referred to as stress-like hardening variables and are denoted by $\boldsymbol{\kappa}$

$$\boldsymbol{\kappa} = \nabla_\alpha\phi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}) \tag{16}$$

Next, the complementary Helmholtz free energy function, $\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})$, is defined via the Legendre transformation

$$\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = -\phi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}) + \boldsymbol{\sigma}^{\mathsf{T}}\boldsymbol{\varepsilon}^e + \boldsymbol{\kappa}^{\mathsf{T}}\boldsymbol{\alpha} \tag{17}$$

from which it follows that the elastic strains and the strain-like hardening variables can be expressed as

$$\boldsymbol{\varepsilon}^e = \nabla_\sigma\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa}),\ \boldsymbol{\alpha} = \nabla_\kappa\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \tag{18}$$

From straightforward manipulations of the above relations the dissipation as follows:

$$\begin{aligned}\mathcal{D} &= \boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}} - \dot{\phi}(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}) \\ &= \boldsymbol{\sigma}^{\mathrm{T}}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^e) - \boldsymbol{\kappa}^{\mathrm{T}}\dot{\boldsymbol{\alpha}}\end{aligned} \tag{19}$$

which is a well-known result [except that a change of sign of $\boldsymbol{\kappa}$ is often made Simo, 1998; Collins and Houlsby, 1997]. Furthermore, the rate of internal work may be expressed in the following form:

$$\begin{aligned}\mathcal{P}(\boldsymbol{\sigma}, \boldsymbol{\kappa}; \boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}) &= \dot{\phi}(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}) + \mathcal{D} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}[-\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa}) + \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}^e + \boldsymbol{\kappa}^{\mathrm{T}}\boldsymbol{\alpha}] + [\boldsymbol{\sigma}^{\mathrm{T}}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^e) - \boldsymbol{\kappa}^{\mathrm{T}}\dot{\boldsymbol{\alpha}}] \\ &= \boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}} - \dot{\psi}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) + \dot{\boldsymbol{\sigma}}^{\mathrm{T}}\boldsymbol{\varepsilon}^e + \dot{\boldsymbol{\kappa}}^{\mathrm{T}}\boldsymbol{\alpha}\end{aligned} \tag{20}$$

This form proves to be useful in what follows.

4. Variational formulation of elastoplasticity

A number of classical variational principles come in the form of saddle-point (min–max) problems where the internal work (or its rate) is maximized with respect to the stresses on the material point level while the total potential energy is minimized with respect to the displacements on the structural level. The Hellinger–Reissner principle of elastostatics and the upper bound theorem of limit analysis are two prominent examples. Inspired by these principles, we seek to derive constitutive models by maximizing the internal work rate subject to yield conditions:

$$\begin{aligned}&\underset{\boldsymbol{\sigma},\boldsymbol{\kappa}}{\text{maximize}} && \mathcal{P}(\boldsymbol{\sigma}, \boldsymbol{\kappa}; \boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}) \\ &\text{subject to} && F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \leqslant 0\end{aligned} \tag{21}$$

From the expression (20) of the internal work rate it follows that

$$\begin{aligned}\sup_{\boldsymbol{\sigma},\boldsymbol{\kappa}} \mathcal{P}(\boldsymbol{\sigma}, \boldsymbol{\kappa}; \boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}) &= \sup_{\boldsymbol{\sigma},\boldsymbol{\kappa}}\{\boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}} - \dot{\psi}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) + \dot{\boldsymbol{\sigma}}^{\mathrm{T}}\boldsymbol{\varepsilon}^e + \dot{\boldsymbol{\kappa}}^{\mathrm{T}}\boldsymbol{\alpha}\} \\ &= \sup_{\boldsymbol{\sigma},\boldsymbol{\kappa}}\{\boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}} - \dot{\psi}(\boldsymbol{\sigma}, \boldsymbol{\kappa})\} + \dot{\boldsymbol{\sigma}}^{\mathrm{T}}\boldsymbol{\varepsilon}^e + \dot{\boldsymbol{\kappa}}^{\mathrm{T}}\boldsymbol{\alpha}\end{aligned} \tag{22}$$

The relevant maximization principle thus reduces to

$$\begin{aligned}&\underset{\boldsymbol{\sigma},\boldsymbol{\kappa}}{\text{maximize}} && \boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}} - \dot{\psi}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \\ &\text{subject to} && F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \leqslant 0\end{aligned} \tag{23}$$

The procedure is now to postulate a relevant potential $\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})$ after which the constitutive equations follow as the first-order Karush–Kuhn–Tucker optimality conditions associated with the above maximization problem. In this connection, we note that von Mises' principle of maximum plastic dissipation (see e.g. Lubliner, 1990) appears for the particular choice of $\psi = 0$.

In order to derive more general constitutive equations, the time derivative of ψ is first expanded as:

$$\dot{\psi}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = \nabla_{\sigma}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})^{\mathrm{T}}\dot{\boldsymbol{\sigma}} + \nabla_{\kappa}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})^{\mathrm{T}}\dot{\boldsymbol{\kappa}} \tag{24}$$

Using this expansion, the constitutive equations associated with (23) follow as (see Appendix A):

$$\begin{aligned}&\dot{\boldsymbol{\varepsilon}} = \nabla^2_{\sigma\sigma}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})\dot{\boldsymbol{\sigma}} + \nabla^2_{\kappa\sigma}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})\dot{\boldsymbol{\kappa}} + \dot{\lambda}\nabla_{\sigma}F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \\ &\mathbf{0} = \nabla^2_{\sigma\kappa}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})\dot{\boldsymbol{\sigma}} + \nabla^2_{\kappa\kappa}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})\dot{\boldsymbol{\kappa}} + \dot{\lambda}\nabla_{\kappa}F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \\ &F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \leqslant 0,\ \dot{\lambda}F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = 0,\ \dot{\lambda} \geqslant 0\end{aligned} \tag{25}$$

It is convenient to introduce the effective moduli:

$$\begin{aligned}&\boldsymbol{C} = \nabla^2_{\sigma\sigma}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa}) - \nabla^2_{\kappa\sigma}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})[\nabla^2_{\kappa\kappa}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})]^{-1}\nabla^2_{\sigma\kappa}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \\ &\boldsymbol{S} = -[\nabla^2_{\kappa\kappa}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})]^{-1}\nabla^2_{\sigma\kappa}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \\ &\boldsymbol{h} = -[\nabla^2_{\kappa\kappa}\psi(\boldsymbol{\sigma}, \boldsymbol{\kappa})]^{-1}\nabla_{\kappa}F(\boldsymbol{\sigma}, \boldsymbol{\kappa})\end{aligned} \tag{26}$$

The constitutive equations can then be expressed in the following format:

$$\begin{aligned}&\dot{\boldsymbol{\varepsilon}} = \boldsymbol{C}\dot{\boldsymbol{\sigma}} + \dot{\lambda}[\nabla_{\sigma}F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) + \boldsymbol{S}^{\mathrm{T}}\nabla_{\kappa}F(\boldsymbol{\sigma}, \boldsymbol{\kappa})] \\ &\dot{\boldsymbol{\kappa}} = \boldsymbol{S}\dot{\boldsymbol{\sigma}} + \dot{\lambda}\boldsymbol{h} \\ &F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \leqslant 0, \dot{\lambda}F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = 0, \dot{\lambda} \geqslant 0\end{aligned} \tag{27}$$

The moduli $\boldsymbol{C}$ and $\boldsymbol{h}$ here have the same physical significance as in the conventional elastoplastic model (8): $\boldsymbol{C}$ is an elastic compliance modulus and $\boldsymbol{h}$ is an array of hardening functions. In addition, the governing equations include a new constitutive modulus, $\boldsymbol{S}$, which in the following will be referred to as a *coupling modulus*. A number of special cases of the above governing equations are considered next.

4.1. Uncoupled models

By direct comparison to (8) we see that $\boldsymbol{S} = \boldsymbol{0}$ reproduces the governing equations of an elastoplastic material where the flow rule and the hardening law are associated, i.e. given in terms of ψ and F following the relations (26). This special case of $\boldsymbol{S} = \boldsymbol{0}$ requires that $\nabla^2_{\sigma\kappa}\psi(\boldsymbol{\sigma},\boldsymbol{\kappa}) = \boldsymbol{0}$ which implies the following decoupling of ψ:

$$\psi(\boldsymbol{\sigma},\boldsymbol{\kappa}) = \psi^e(\boldsymbol{\sigma}) + \psi^p(\boldsymbol{\kappa}) \tag{28}$$

This decoupling of the complementary Helmholtz free energy function is assumed (or implied) in most elastoplastic constitutive models and has further been used extensively as a basis for the development of computational procedures (see e.g. Simo, 1998; Alfano and Rosati, 1998; Ibrahimbegovic et al., 1998; Armero and Perez–Foguet, 2002).

Concerning the relevance of the above potential to the modeling of frictional materials we note that it implies a standard associated flow rule. Although a number of associated plasticity models have been proposed for such materials (with Cam clay being the most prominent) it is generally desirable to be able to adjust the yield and flow potentials independently of each other. The above choice of potential ψ does not allow for such an independent adjustment.

4.2. Coupled models

Next, we consider the case of a fully coupled complementary Helmholtz free energy function. The governing equations here reveal two major differences as compared to the conventional elastoplastic format (8):

(1) The *rate of total strain* can be decomposed into elastic and plastic[1] parts according to:

$$\begin{aligned}
&\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p \\
&\dot{\boldsymbol{\varepsilon}}^e = \boldsymbol{C}\dot{\boldsymbol{\sigma}} \\
&\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}[\nabla_\sigma F(\boldsymbol{\sigma},\boldsymbol{\kappa}) + \boldsymbol{S}^{\mathrm{T}}\nabla_\kappa F(\boldsymbol{\sigma},\boldsymbol{\kappa})]
\end{aligned} \tag{29}$$

In the context of frictional materials the key point is here that by adjusting $\boldsymbol{S}$ appropriately, any direction of plastic strain can in principle be realized while maintaining a variational structure of the constitutive model. The variational principle (23) thus implies associativity in a rather generalized sense that does not necessarily preclude the modeling of materials with *apparently* nonassociated flow rules, i.e. materials where the plastic strain rate vector is not normal to the yield surface. We note, however, that this apparent nonassociativity is possible only for hardening plasticity models, i.e. models that involves at least one hardening variable, κ, such that $\partial F/\partial\kappa \neq 0$. As far as elastic/perfectly plastic models are concerned, the above formulation is thus of limited value – unless, of course, such models are viewed as the limiting cases of hardening models with extreme hardening moduli. An example of such a model is given in Section 6.3.

(2) The *hardening law* contains an elastic as well as a plastic component

$$\begin{aligned}
&\dot{\boldsymbol{\kappa}} = \dot{\boldsymbol{\kappa}}^e + \dot{\boldsymbol{\kappa}}^p \\
&\dot{\boldsymbol{\kappa}}^e = \boldsymbol{S}\dot{\boldsymbol{\sigma}} = \boldsymbol{S}\boldsymbol{C}^{-1}\dot{\boldsymbol{\varepsilon}}^e \\
&\dot{\boldsymbol{\kappa}}^p = \dot{\lambda}\boldsymbol{h}
\end{aligned} \tag{30}$$

This hardening law implies that the yield surface may change in size and shape as a result of elastic straining only. Although somewhat unusual in the context of conventional elastoplasticity, this feature is in fact very convenient as a means of accounting for some of the common features of granular materials. This will be discussed in more detail in Section 6.

Whereas the variational principle (23) thus appears to be of some relevance to frictional materials, especially in terms of the possibility of accounting for apparently nonassociated flow characteristics without introducing a separate flow potential, its usefulness depends crucially on the ability to identify a potential $\psi(\boldsymbol{\sigma},\boldsymbol{\kappa})$ that eventually produces reasonable effective moduli $\boldsymbol{C}, \boldsymbol{h}$, and $\boldsymbol{S}$. In practice the identification of such potentials turns out to be rather problematic, if not impossible. Indeed, even the basic Cam clay models (which would require $\boldsymbol{S} = \boldsymbol{0}$) do not permit a variational formulation since the hardening law is nonassociated (see e.g. Ortiz and Pandolfi, 2004; Hjiaj et al., 2005; Zouain et al., 2007). In the following, therefore, we consider an incremental variational formulation. This formulation requires the specification of a new potential at each time instant. As such, it is considerably weaker than the classical variational formulation discussed so far which requires only that a single, fixed, potential be specified. However, many of the advantages of classical variational formulations are retained. For example, all results concerning existence and uniqueness are still valid, though in a more restricted incremental setting. Similarly, the elastoplastic tangent moduli will still be symmetric, regardless of the degree of apparent nonassociativity.

4.3. Relation to strain-based formulations

The key feature of the above formulation is that the stresses are regarded as the independent variables. Furthermore, in describing hardening, additional stress-like hardening variables are introduced. Finally, a yield function given in

[1] Here and in the following, the term *plastic* is used as being synonymous with *irreversible* or *inelastic*. As such, the plastic strains are the difference between the total and the elastic strains.

terms of the stresses and the stress-like hardening variables is postulated. This approach contrasts formulations where the strains are regarded as the primary variables, i.e. where the stress state is a function of the elastic strain and where hardening typically is described in terms of the plastic strain. Instead of postulating a stress-based yield function, such formulations usually involve a strain-based dissipation function from which the yield function follows implicitly. This type of formulation has been described in detail by Collins and Houlsby (1997) with particular reference to geomaterials.

Compared to strain-based formulations, the above stress-based formulation has both advantages and disadvantages. One of the key advantages is that the governing equations follow in a format that is mathematically transparent and amenable to direct algorithmic treatment using standard methods originally developed for the elastic/perfectly plastic problem. Indeed, algorithmic issues appear to have been the primary motivation of Simo and his co-workers in developing such stress-based formulations (see e.g. Simo et al., 1989). Furthermore, stress-based formulations allow for a direct experimental determination of all components involved whereas the situation is significantly more complicated in strain-based formulations (c.f., for example, the experimental determination of yield versus dissipation functions). On the other hand, strain-based formulations do offer certain insights into the deformation characteristics and mechanisms of energy conversion that stress-based formulations do not provide directly (see e.g. Collins and Kelly, 2002; Collins, 2005).

5. Incremental variational formulation

A relevant incremental variational principle analogous to the general principle (23) can be constructed in the following way. Consider a known state $(\boldsymbol{\sigma}^*, \boldsymbol{\kappa}^*)$ and define the following incremental (or tangent) potential

$$\psi_t(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{C}_t\boldsymbol{\sigma} + \tfrac{1}{2}(\boldsymbol{\kappa} - \boldsymbol{S}_t\boldsymbol{\sigma})^{\mathrm{T}}\boldsymbol{G}_t(\boldsymbol{\kappa} - \boldsymbol{S}_t\boldsymbol{\sigma}) \tag{31}$$

where

$$\boldsymbol{C}_t = \boldsymbol{C}(\boldsymbol{\sigma}^*, \boldsymbol{\kappa}^*), \boldsymbol{G}_t = \boldsymbol{G}(\boldsymbol{\sigma}^*, \boldsymbol{\kappa}^*), \boldsymbol{S}_t = \boldsymbol{S}(\boldsymbol{\sigma}^*, \boldsymbol{\kappa}^*) \tag{32}$$

with subscript t emphasizing the tangent nature of the constitutive moduli. This potential is used in the same maximization context as before

$$\begin{aligned} &\underset{\boldsymbol{\sigma},\boldsymbol{\kappa}}{\text{maximize}} \quad && \boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}} - \dot{\psi}_t(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \\ &\text{subject to} && F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \leqslant 0 \end{aligned} \tag{33}$$

The associated optimality conditions give the following governing equations:

$$\begin{aligned} &\dot{\boldsymbol{\varepsilon}} = \boldsymbol{C}_t\dot{\boldsymbol{\sigma}} + \dot{\lambda}[\nabla_\sigma F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) + \boldsymbol{S}_t^{\mathrm{T}}\nabla_\kappa F(\boldsymbol{\sigma}, \boldsymbol{\kappa})] \\ &\dot{\boldsymbol{\kappa}} = \boldsymbol{S}_t\dot{\boldsymbol{\sigma}} + \dot{\lambda}\boldsymbol{h}_t \\ &F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \leqslant 0, \;\; \dot{\lambda}F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = 0, \;\; \dot{\lambda} \geqslant 0 \end{aligned} \tag{34}$$

where

$$\boldsymbol{h}_t = -\boldsymbol{G}_t^{-1}\nabla_\kappa F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \tag{35}$$

Since $\boldsymbol{G}_t$ can be chosen arbitrarily, any hardening function can be specified in the limit of $||(\boldsymbol{\sigma}, \boldsymbol{\kappa}) - (\boldsymbol{\sigma}^*, \boldsymbol{\kappa}^*)|| \to 0$. Similar conclusions are valid with respect to $\boldsymbol{C}_t$ and $\boldsymbol{S}_t$. A significantly more general set of governing equations, incorporationg arbitrarily varying constitutive moduli, is thus realized. For such models the strain rates are given by:

$$\begin{aligned} &\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p \\ &\dot{\boldsymbol{\varepsilon}}^e = \boldsymbol{C}_t\dot{\boldsymbol{\sigma}} \\ &\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}[\nabla_\sigma F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) + \boldsymbol{S}_t^{\mathrm{T}}\nabla_\kappa F(\boldsymbol{\sigma}, \boldsymbol{\kappa})] \end{aligned} \tag{36}$$

Observations similar to those made in the previous section are here valid, the most important being that the plastic strain rates depend on the modulus $\boldsymbol{S}_t$ which can be chosen arbitrarily to yield any effective flow rule. Similarly, the hardening law can be expressed as

$$\begin{aligned} &\dot{\boldsymbol{\kappa}} = \dot{\boldsymbol{\kappa}}^e + \dot{\boldsymbol{\kappa}}^p \\ &\dot{\boldsymbol{\kappa}}^e = \boldsymbol{S}_t\dot{\boldsymbol{\sigma}} = \boldsymbol{S}_t\boldsymbol{C}_t^{-1}\dot{\boldsymbol{\varepsilon}}^e \\ &\dot{\boldsymbol{\kappa}}^p = \dot{\lambda}\boldsymbol{h}_t \end{aligned} \tag{37}$$

Again, the evolution of $\boldsymbol{\kappa}$ comprises an elastic (reversible) part which depends on the coupling modulus $\boldsymbol{S}_t$ and a plastic (irreversible) part which is specified in terms of a general hardening function $\boldsymbol{h}_t$.

5.1. Elastoplastic tangent modulus

Straightforward manipulations of the relations defining the model (74) give rise to the following incremental stress–strain relations

$$\dot{\boldsymbol{\varepsilon}} = \begin{cases} \boldsymbol{C}_t \dot{\boldsymbol{\sigma}}, & \dot{\lambda} = 0 \\ \boldsymbol{C}_t^{ep} \dot{\boldsymbol{\sigma}}, & \dot{\lambda} > 0 \end{cases} \tag{38}$$

where

$$\boldsymbol{C}_t^{ep} = \boldsymbol{C}_t + \frac{1}{H_t} \boldsymbol{a}_t \boldsymbol{a}_t^{\mathrm{T}} \tag{39}$$

with

$$\begin{aligned} \boldsymbol{a}_t &= \nabla_\sigma F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) + \boldsymbol{S}_t^{\mathrm{T}} \nabla_\kappa F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \\ H_t &= -\boldsymbol{h}_t^{\mathrm{T}} \nabla_\kappa F(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \end{aligned} \tag{40}$$

The elastoplastic tangent compliance modulus $\boldsymbol{C}_t^{ep}$ is thus symmetric provided that the elastic tangent compliance modulus $\boldsymbol{C}_t$ is symmetric. This is a most convenient feature, both from a practical computational point of view as well as in terms of its implications regarding stability. Thus, provided that $H_t > 0$, it is easily shown that $\boldsymbol{C}_t^{ep}$ is positive definite, implying a positive second-order work.

5.2. Finite-step formulation

Finally, a finite-step version of the incremental variational formulation discussed above is considered.

The power of deformation introduced in Section 3 may be approximated in time as

$$\mathcal{P}_{n+1}^n = \dot{\phi} + \mathcal{D}_{n+1}^n \approx \phi_{n+1} - \phi_n + \mathcal{D}_{n+1}^n \tag{41}$$

where the notation X_{n+1}^n is to be understood as the change of a path dependent quantity X over a finite step from n to $n+1$. Using the relations given in Section 3 together with a fully implicit evaluation of the dissipation, the finite-step power of deformation may be expressed in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{\kappa}$ as

$$\mathcal{P}_{n+1}^n = \boldsymbol{\sigma}_{n+1}^{\mathrm{T}} \Delta\boldsymbol{\varepsilon} - (\psi_{n+1} - \psi_n) + \Delta\boldsymbol{\sigma}_{n+1}^{\mathrm{T}} \nabla_\sigma \psi_n + \Delta\boldsymbol{\kappa}_{n+1}^{\mathrm{T}} \nabla_\kappa \psi_n \tag{42}$$

where $\Delta\boldsymbol{\varepsilon}$ is assumed known while $\Delta\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_n$ and $\Delta\boldsymbol{\kappa}_{n+1} = \boldsymbol{\kappa}_{n+1} - \kappa_n$ are unknown. In terms of the incremental potential (31) we then have the following finite-step principle:

$$\begin{aligned} &\underset{(\boldsymbol{\sigma},\boldsymbol{\kappa})_{n+1}}{\text{maximize}} \quad \boldsymbol{\sigma}_{n+1}^{\mathrm{T}} \Delta\varepsilon - \tfrac{1}{2} \Delta\boldsymbol{\sigma}_{n+1}^{\mathrm{T}} \boldsymbol{C}_t \Delta\boldsymbol{\sigma}_{n+1} - \tfrac{1}{2} (\Delta\boldsymbol{\kappa}_{n+1} - \boldsymbol{S}_t \Delta\boldsymbol{\sigma}_{n+1})^{\mathrm{T}} \boldsymbol{G}_t (\Delta\boldsymbol{\kappa}_{n+1} - \boldsymbol{S}_t \Delta\boldsymbol{\sigma}_{n+1}) \\ &\textit{subject to} \quad \mathsf{F}(\boldsymbol{\sigma}_{n+1}, \boldsymbol{\kappa}_{n+1}) \leqslant 0 \end{aligned}$$

Introducing the 'trial stresses':

$$\begin{aligned} \boldsymbol{\sigma}^{\mathrm{tr}} &= \boldsymbol{\sigma}_n + \boldsymbol{C}_t^{-1} \Delta\boldsymbol{\varepsilon} \\ \boldsymbol{\kappa}^{\mathrm{tr}} &= \boldsymbol{\kappa}_n + \boldsymbol{S}_t \boldsymbol{C}_t^{-1} \Delta\boldsymbol{\varepsilon} \end{aligned} \tag{43}$$

this principle may be stated alternatively as

$$\begin{aligned} &\underset{\boldsymbol{\chi}_{n+1}}{\text{minimize}} \quad \tfrac{1}{2} (\boldsymbol{\chi}_{n+1} - \boldsymbol{\chi}^{\mathrm{tr}})^{\mathrm{T}} \boldsymbol{L}_t (\boldsymbol{\chi}_{n+1} - \boldsymbol{\chi}^{\mathrm{tr}}) \\ &\text{subject to} \quad F(\boldsymbol{\chi}_{n+1}) \leqslant 0 \end{aligned} \tag{44}$$

where

$$\boldsymbol{\chi} = \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\kappa} \end{pmatrix}, \quad \boldsymbol{\chi}^{\mathrm{tr}} = \begin{pmatrix} \boldsymbol{\sigma}^{\mathrm{tr}} \\ \boldsymbol{\kappa}^{\mathrm{tr}} \end{pmatrix}, \quad \boldsymbol{L}_t = \begin{bmatrix} \boldsymbol{C}_t + \boldsymbol{S}_t^{\mathrm{T}} \boldsymbol{G}_t \boldsymbol{S}_t & -\boldsymbol{S}_t^{\mathrm{T}} \boldsymbol{G}_t \\ -\boldsymbol{G}_t \boldsymbol{S}_t & \boldsymbol{G}_t \end{bmatrix} \tag{45}$$

This is a standard closest point projection problem of the type that forms the basis of implicit integration procedures (Simo, 1998; Armero and Perez–Foguet, 2002). The numerical implementation of models deriving from the incremental variational principle is thus straightforward and can be carried out with very minor modification of existing routines for more conventional models. In addition, the possibility of applying state-of-the-art optimization methods, for example those specialized to so-called conic programs (Ben-Tal and Nemirovski, 2001; Andersen et al., 2003; Sturm, 2002), is an interesting one. Such methods have recently proved to be remarkably efficient (Krabbenhoft et al., 2007a,b), especially in cases where the yield surface contains singularities as is usually the case for granular and other frictional materials.

6. Constitutive modeling of granular materials

We now consider the application of the framework described in the previous sections to the modeling of granular materials such as sands. The starting point is the well known modified Cam clay model, a detailed description of which can be found in Wood et al. (1990). The limitations of this model in accounting for the behaviour of both clays and sands are well known and well documented (Gens and Potts, 1988; Wood et al., 1990; Yu, 2006). On the other hand, critical state models such as modified Cam clay have been used extensively in the development of more realistic models and the present approach follows this well established modeling paradigm.

The main idea behind the Cam clay models (Schofield and Wroth, 1968, 2007) was to unify isotropic compression and shear behaviour in a single model. In the resulting model, this unification implies that some of the material parameters have dual physical meanings. In particular, the compression index used to quantify the elastoplastic bulk compliance enters into the hardening law which eventually determines the effective shear compliance. While this relationship between elastoplastic bulk and shear moduli to a certain extent can be verified for clays, it appears to have much less experimental justification in the case of granular materials such as sands. In the present model an appropriate decoupling between the properties in isotropic compression and shear is therefore considered. This decoupling is realized by means of an appropriate coupling modulus $\boldsymbol{S}$ which at the same time establishes a relevant stress-dilatancy relation.

As is common practice in critical state soil mechanics Wood et al., 1990, we will work in a two-dimensional space with the stresses and strains given by

$$\boldsymbol{\sigma} = (p, q)^{\mathsf{T}}, \ \boldsymbol{\varepsilon} = (\varepsilon_v, \varepsilon_s)^{\mathsf{T}} \tag{46}$$

where

$$\begin{aligned} p &= \tfrac{1}{3}(\sigma_1 + 2\sigma_3), \qquad & q &= \sigma_1 - \sigma_3 \\ \varepsilon_v &= \varepsilon_1 + 2\varepsilon_3, & \varepsilon_s &= \tfrac{2}{3}(\varepsilon_1 - \varepsilon_3) \end{aligned} \tag{47}$$

with σ_1 and σ_3 being the major and minor principal stresses respectively, and ε_1 and ε_3 being the major and minor principal strains respectively. Compressive stresses are taken as being positive. We will also make frequent use the stress ratio

$$\eta = \frac{q}{p} \tag{48}$$

6.1. Elements of a conventional critical state model

In the following, some of the common elements of a typical critical state type model are briefly summarized and discussed with emphasis on the implied coupling between isotropic compression and pure shear behaviour.

6.1.1. Isotropic compression behaviour

For many soils, sands as well as clays, the isotropic compression ($\dot{p} > 0, q = 0$) behaviour can to a good approximation be described via the following stress–strain relations:

$$\begin{aligned} \dot{\varepsilon}_v &= \frac{C_v^e}{p}\dot{p}, \quad \dot{\lambda} = 0 \\ \dot{\varepsilon}_v &= \frac{C_v^{ep}}{p}\dot{p}, \quad \dot{\lambda} > 0 \end{aligned} \tag{49}$$

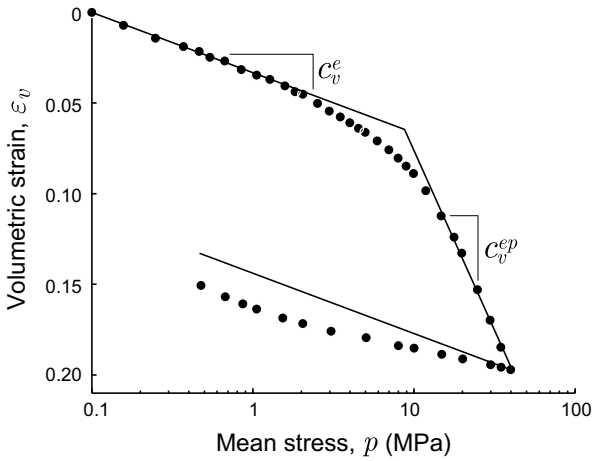

Fig. 1. Isotropic compression behaviour of a silica sand (after McDowell et al., 2002).

where c_v^e and c_v^{ep} will be referred to as the elastic and elastoplastic bulk compliance constants respectively. These represent the slopes of the stress–strain curve in $\varepsilon_v - \ln p$ diagram. Fig. 1 shows a set of experimental results for a silica sand and we see that the response to a good approximation can be represented by means of the classical relations (49).

With reference to the conventional elastoplastic modeling format (8) an elastoplastic model incorporating the elastic law implied by the first part of (49) gives the following stress–strain relations in isotropic compression:

$$\begin{aligned} \dot{\varepsilon}_v &= \frac{c_v^e}{p}\dot{p}, && \dot{\lambda} = 0 \\ \dot{\varepsilon}_v &= \left[\frac{c_v^e}{p} + \frac{1}{H}\frac{\partial G}{\partial p}\frac{\partial F}{\partial p}\right]\dot{p}, && \dot{\lambda} > 0 \end{aligned} \tag{50}$$

where H is the hardening modulus (c.f. Eq. 11), F is the yield function, and G is the flow potential which in general may be different from F. If the stress–strain relations (49) are to be verified we must then necessarily have

$$\frac{1}{H} = \frac{c_v^{ep} - c_v^e}{p}\frac{1}{H'} \tag{51}$$

where H' is such that

$$\frac{1}{H'}\frac{\partial G}{\partial p}\frac{\partial F}{\partial p} = 1 \quad \text{for} \quad q = 0 \tag{52}$$

6.1.2. Pure shear behaviour

The response of a medium-dense sand in pure shear ($p = p_0, \dot{q} \geqslant 0$) at a confining pressure of $p_0 = 100$ kPa is shown in Fig. 2. Assuming isotropic elasticity, the initial shear stiffness is given by the elastic shear modulus $3G^e$. Inelastic strains develop rather early on, however, and the stiffness gradually decreases until some constant level defined by $\eta = M$ is reached and the behaviour becomes perfectly plastic (and remains so at moderate levels of strain after which a softening behaviour usually will be observed).

For a general elastoplastic model, the response in pure shear is given by

$$\begin{aligned} \dot{\varepsilon}_s &= \frac{1}{3G^e}\dot{q}, && \dot{\lambda} = 0 \\ \dot{\varepsilon}_s &= \left[\frac{1}{3G^e} + \frac{1}{H}\frac{\partial G}{\partial q}\frac{\partial F}{\partial q}\right]\dot{q}, && \dot{\lambda} > 0 \end{aligned} \tag{53}$$

or, in terms of the hardening modulus (51), as

$$\begin{aligned} \dot{\varepsilon}_s &= \frac{1}{3G^e}\dot{q}, && \dot{\lambda} = 0 \\ \dot{\varepsilon}_s &= \left[\frac{1}{3G^e} + \frac{c_v^{ep} - c_v^c}{p}\frac{1}{H'}\frac{\partial G}{\partial q}\frac{\partial F}{\partial q}\right]\dot{q}, && \dot{\lambda} > 0 \end{aligned} \tag{54}$$

Comparing the above to the isotropic compression relations (49)–(51), the previously mentioned coupling between the elastoplastic bulk and shear properties is evident. Thus, the constant c_v^{ep} accounts both for the bulk compliance as well as for the magnitude of the effective shear compliance. In the context of sands and other granular materials, this coupling between different physical properties is rather problematic. Thus, as far as the bulk properties are concerned, c_v^{ep} must necessarily have its physical origins in the crushing of grains that takes place above a certain characteristic pressure. From Fig. 1 this

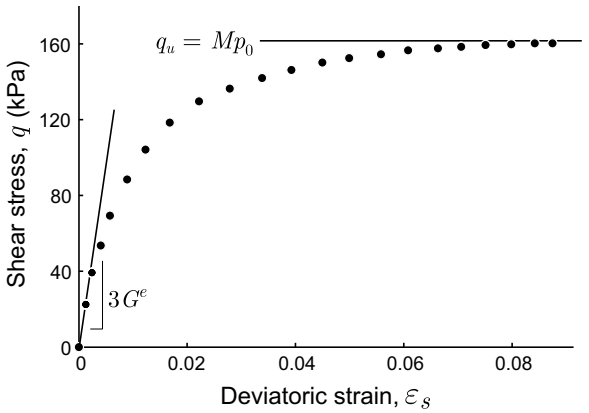

Fig. 2. Pure shear behaviour of a dense sand at a confining pressure of $p_0 = 100$ kPa (Krabbenhoft 2007).

pressure would appear to be of order 10 MPa. On the other hand, the apparent decrease of shear stiffness observed in pure shear at much lower pressures (for example 100 kPa as in Fig. 2) has its origins in frictional particle contact and rearrangement. These processes take place at any pressure and are quite unrelated to grain crushing. This implied link between physically unrelated quantities and properties is a major deficiency of conventional critical state type models and it is not easily rectified within a conventional, uncoupled, elastoplastic modeling framework.

6.2. Coupled model

The coupled model is based on well known concepts of classical Cam plasticity and its subsequent adaptations to granular materials. The only new feature is a suitable coupling modulus $\boldsymbol{S}$ which accounts for a relevant stress-dilatancy rule and which further resolves the complications discussed above. The model applies only to stress ranges below the grain crushing pressure. Softening behaviour, shear banding and related phenomena are not considered either. In fact, in Section 6.3, the model is modified slightly to explicitly preclude any possibility of instabilities.

6.2.1. Elastic law

Following conventional practice Wood et al., 1990, the elastic compliance modulus is taken as

$$\boldsymbol{C}_t^e = \begin{bmatrix} \dfrac{c_v^e}{p} & \\ & \dfrac{1}{3G^e} \end{bmatrix} \tag{55}$$

where the shear modulus G^e is constant, making the effective Poisson's ratio variable. More elaborate elastic laws, incorporating an induced coupling between shear and volumetric modes of deformation have been presented by Einav and Puzrin (2004).

6.2.2. Yield function

The yield function is taken as that associated with the modified Cam clay model (Wood et al., 1990):

$$F(p,q,\kappa) = q^2 - M^2 p(\kappa - p) \tag{56}$$

where κ is a hardening variable that determines the size of the elastic domain. In the context of clays, this hardening variable would be the preconsolidation pressure while its physical meaning for sands is less clear, c.f. the discussion above.

6.2.3. Plastic strains, stress-dilatancy, and coupling modulus

Following (29), the plastic strain rates are given by:

$$\begin{aligned} \dot{\varepsilon}_v^p &= \dot{\lambda}\left[\frac{\partial F}{\partial p} + S_v \frac{\partial F}{\partial \kappa}\right] = \dot{\lambda} p(M^2 - \eta^2 - M^2 S_v) \\ \dot{\varepsilon}_s^p &= \dot{\lambda}\left[\frac{\partial F}{\partial q} + S_s \frac{\partial F}{\partial \kappa}\right] = \dot{\lambda} p(2\eta - M^2 S_s) \end{aligned} \tag{57}$$

where S_v and S_s are the components of the coupling modulus $\boldsymbol{S} = [S_v, S_s]$.

As discussed previously, the primary benefit of the coupling modulus appears to be that of incorporating an arbitrary effective flow rule without introducing a separate flow potential explicitly. Following this line of reasoning, we will in the following assume that $S_s = 0$ so that S_v can be determined uniquely by imposing a relevant stress-dilatancy relation

$$\frac{\dot{\varepsilon}_v^p}{\dot{\varepsilon}_s^p} = \frac{M^2 - \eta^2 - M^2 S_v}{2\eta} = d(\eta) \Longleftrightarrow S_v = S_v(\eta) = \frac{M^2 - \eta^2 - 2d(\eta)\eta}{M^2} \tag{58}$$

where d is the stress-dilatancy function. It is worth noting that $S_v(\eta = 0) = 1$ as long as $d(\eta = 0)$ is finite.

Concerning the choice of stress-dilatancy relation, it has been observed in numerous experiments (Stroud, 1971; Wood et al., 1990; Taylor, 1948; Collins et al., 2007) that the strain rate ratio, especially at larger strains, adhere rather closely to a relation of the type

$$\frac{\dot{\varepsilon}_v}{\dot{\varepsilon}_s} + \frac{q}{p} = N \tag{59}$$

where N is a constant that, for a given sand, usually is found to be roughly independent of density and pressure while potentially being rather different for different types of sands. It should be noted that the above stress-dilatancy relation is the basis of the original Cam clay model (Schofield and Wroth, 1968, 2007), except that this model assumes $N = M$ while N in the present model is an independent parameter.

Motivated by these experimental findings, the following simple stress-dilatancy function is adopted

$$d(\eta) = N - \eta \tag{60}$$

The volumetric coupling modulus it then given by

$$S_v(\eta) = \frac{M^2 + \eta^2 - 2N\eta}{M^2} \tag{61}$$

where we again note that $S_v(\eta = 0) = 1$. This leads to the following plastic strain rates:

$$\begin{aligned} \dot{\varepsilon}_v^p &= \dot{\lambda}\left[\frac{\partial F}{\partial p} + S_v \frac{\partial F}{\partial \kappa}\right] = \dot{\lambda} 2p\eta(N - \eta) \\ \dot{\varepsilon}_s^p &= \dot{\lambda}\frac{\partial F}{\partial q} = \dot{\lambda} 2p\eta \end{aligned} \tag{62}$$

The effect of the dilatancy parameter on the direction of plastic flow is illustrated in Fig. 3. It is here seen that $q/p = N$ defines the line that separates contractive states from dilative states. Several models make use of this concept and the line $q/p = N$ is known variously as the phase transition line (Ishihara, 1993; Collins et al., 2007), the zero-dilatancy line (Mroz, 1998), or the characteristic state line (Lee and Seed, 1967; Luong, 1980; Krenk, 2000).

We further note that no plastic flow takes place for q=0, i.e. isotropic compression is purely elastic. This feature is discussed in detail in Section 6.2.5.

6.2.4. Hardening law

The plastic part of the hardening law is established on the basis of an idea due to Krenk (2000). First of all, the evolution of κ is assumed to depend on the volumetric part of the plastic work, $p\dot{\varepsilon}_v^p$, rather that the quantity $\kappa\dot{\varepsilon}_v^p$ as in standard Cam clay. Secondly, a suitable fraction of the plastic work due to shear, $q\dot{\varepsilon}_s^p$, is included into the hardening law so that the evolution of κ is given by

$$\dot{\kappa}^p = \frac{1}{c_s^p}(p\dot{\varepsilon}_v^p + wq\dot{\varepsilon}_s^p) \tag{63}$$

where w is a weighting factor that determines the contribution of hardening due to shear and c_s^p is a model constant that will be referred to as the plastic shear compliance constant. Following Krenk (2000), the weighting factor, w, is taken as being constant and is determined from the condition that the yield surface should seize hardening at the ultimate limit state, $\eta = M$. Straightforward manipulations show that this condition leads to

$$w = 1 - \frac{N}{M} \tag{64}$$

The resulting hardening law can then be written as:

$$\dot{\kappa} = \dot{\lambda} h(p,q), h(p,q) = \frac{1}{c_s^p}\frac{2p^2\eta N(M - \eta)}{M} \tag{65}$$

Other hardening functions, possibly established solely on the basis of experimental data, may of course also be used. Also, we note that the above function, where $h(\eta = 0) = 0$, may be somewhat inconvenient from a numerical point of view and that a slight modification to amend this feature is entirely possible.

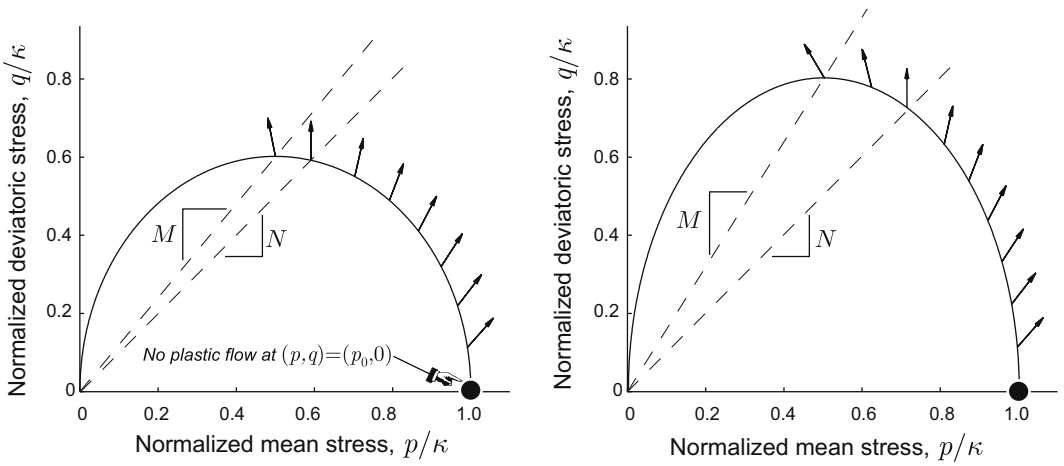

Fig. 3. Yield surfaces and plastic strain rate vectors a loose to medium ($M = 1.2$) and dense ($M = 1.6$) sand. In both cases the dilatancy constant is $N = 1.0$.

6.2.5. Initial state, isotropic compression and pure shear

In conventional critical state type models the identification of the initial state is a somewhat contentious issue. While the basic idea of Cam clay was to choose the initial state, and thereby the initial size of the yield surface, in accordance with the overconsolidation ratio, this concept appears to have much less validity for granular materials such as sands, c.f. the discussion in Section 6.1. Thus, if the initial size of the yield surface is adjusted according to the grain crushing pressure as measured in isotropic compression, the response in pure shear will be much too stiff. Conversely, if the initial size of the yield surface is chosen in accordance with the observation that plastic strains develop under even very moderate shear stresses, the yield limit in isotropic compression, i.e. the grain crushing pressure, would in most cases be unacceptably low.

The introduction of the coupling modulus S_v effectively resolves these issues. For the above model, the stress–strain relations corresponding to isotropic compression are given by

$$\begin{aligned} \dot{\varepsilon}_v &= \frac{c_v^e}{p}\dot{p}; \quad q = 0,\ \dot{\lambda} = 0 \\ \dot{\varepsilon}_v &= \left[\frac{c_v^e}{p} + \frac{c_s^p}{p}\frac{2\eta(N-\eta)^2}{MN(M-\eta)}\right]\dot{p} = \frac{c_v^e}{p}\dot{p}; \quad q = 0,\ \dot{\lambda} > 0 \end{aligned} \tag{66}$$

In this way, the behaviour in isotropic compression and shear are 'decoupled'. That is, the plastic shear compliance constant c_s^p, which in Cam clay would correspond to the difference between the elastoplastic and elastic bulk compliance constants, $c_v^{ep} - c_v^e$, has in the present model no bearing on the response in isotropic compression which is always purely elastic. Furthermore, in isotropic compression the hardening law is given by

$$\dot{\kappa} = \dot{\kappa}^e + \dot{\kappa}^p = \dot{\kappa}^e = \dot{p} \tag{67}$$

Suppose that the initial value of the hardening variable, κ_0, is chosen as being equal to the initial mean stress p_0. In loading the response will then always be purely elastic. In unloading the same considerations are valid: a decrease in p will lead to an identical decrease in κ, thus contracting the yield surface at a rate equal to the rate of change in mean stress. With this choice of initial state, $\kappa_0 = p_0$, the response in pure shear, will be independent of the isotropic compression history. As such, the 'overconsolidation ratio', i.e. the ratio between the initial value of the hardening variable and the initial pressure, concerns only the shear history. Thus, a material that has never been subjected to shear implies a ratio of $\kappa_0/p_0 = 1$ whereas previously sheared materials will imply a ratio greater than unity.

Finally, in pure shear the response is given by

$$\begin{aligned} \dot{\varepsilon}_s &= \frac{1}{3G^e}\dot{q}; \quad p = 0,\ \dot{\lambda} = 0 \\ \dot{\varepsilon}_s &= \left[\frac{1}{3G_t} + \frac{c_s^p}{p}\frac{2\eta}{MN(M-\eta)}\right]\dot{q}; \quad p = 0,\ \dot{\lambda} > 0 \end{aligned} \tag{68}$$

so that the plastic part of the shear strain is given in terms of the plastic shear compliance constant, c_s^p, only. In other words, the elastoplastic bulk compliance, which has its physical origins in the grain crushing properties of the material, is of no relevance to the plastic shear properties below the grain crushing pressure.

In conclusion, the introduction of a relevant coupling modulus $\boldsymbol{S}$ first of all enables the specification of a relevant stress-dilatancy relation. Secondly, the effect of this new term with respect to the 'elastic hardening' law is such that the plastic bulk and shear properties are decoupled, thus resolving one of the major shortcomings of conventional critical state type models.

6.2.6. Elastoplastic tangent modulus

With all constitutive moduli established, the complete incremental stress–strain relations are given by

$$\begin{pmatrix} \dot{\varepsilon}_v \\ \dot{\varepsilon}_s \end{pmatrix} = \left\{ \begin{bmatrix} c_v^e/p & \\ & 1/(3G^e) \end{bmatrix} + \frac{2\eta c_s^p}{pMN(M-\eta)} \begin{bmatrix} (N-\eta)^2 & N-\eta \\ N-\eta & 1 \end{bmatrix} \right\} \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} \tag{69}$$

We note that the elastoplastic tangent modulus is symmetric in accordance with (39). This is a most convenient feature, both from a practical numerical point of view as well in terms its consequences in relation to potential instabilities (Bigoni and Hueckel, 1991). In terms of incorporating an arbitrary stress-dilatancy relation, the above approach is thus in many ways preferable to introducing a nonassociated flow rule explicitly.

6.2.7. Experimental validation – triaxial compression

Next, the extended Cam clay model is sought calibrated to a set of triaxial test data. This type of test proceeds by subjecting the sample to a state of isotropic compression, $(p, q) = (p_0, 0)$, after which a stress path defined by $\dot{q}/\dot{p} = 3$ $(\dot{\sigma}_1 > 0, \dot{\sigma}_3 = 0)$ is imposed. The experimental results of such tests for three identical sands at different relative densities are shown in Fig. 4. The model fits were obtained using the parameters given in Table 1.

Regarding the curve-fitting procedure, the elastic shear modulus was chosen on the basis of the initial q vs. ε_s response. The friction coefficient M was chosen on the basis of the ultimate limit and, as expected, M increases with density. The dilat-

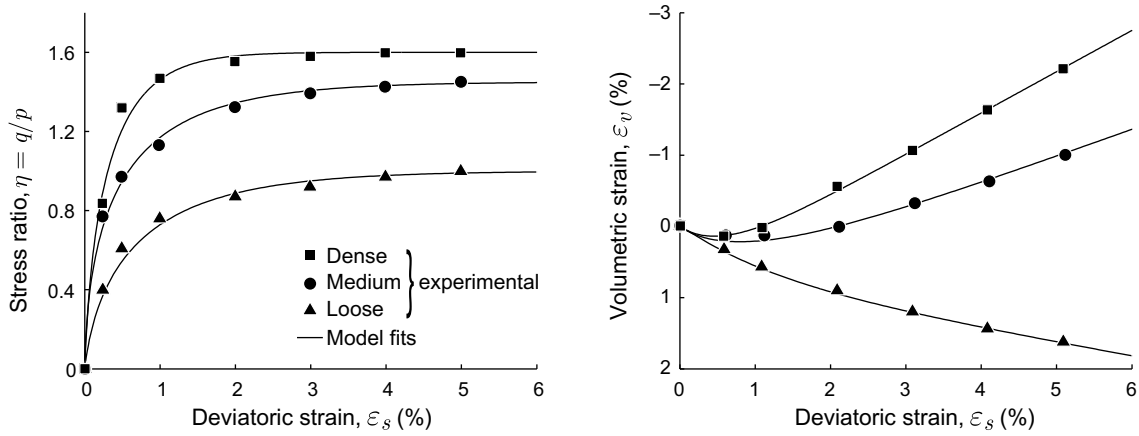

Fig. 4. Triaxial test results and model fits for Erksak sand (model parameters given in Table 1, test data from Yu (2006)).

Table 1
Model parameters for loose, medium and dense Erksak sand.

		Loose	Medium	Dense
Initial void ratio[a]		0.82	0.677	0.59
Initial pressure[b]	p_0 (kPa)	200	60	130
Initial hardening var.[b]	κ_0 (kPa)	200	60	130
Elastic bulk compliance	c_v^e ($\times 10^{-3}$)	4.0	1.0	2.0
Elastic shear modulus	G^e (MPa)	20.0	30.0	40.0
Plastic shear compliance	c_s^p ($\times 10^{-3}$)	18.0	11.0	6.0
Friction coefficient	M	1.0	1.45	1.6
Dilatancy parameter	N	1.19	1.07	1.02

[a] Not a model parameter.
[b] The initial value of the hardening variable, $\kappa_0 = p_0$, corresponds to a sand that has not previously been subjected to shear, c.f. Section 6.2.5.

ancy parameter N was chosen on the basis of the deformations close to the ultimate state. It is noted that N varies only little between the different tests, verifying the trend observed in many other experiments (Taylor, 1948; Stroud, 1971; Collins et al., 2007). Finally, the compliance constants c_v^e and c_s^p were chosen on the basis of the response at small strains and the degree of hardening at larger strains.

6.3. Equivalent unconditionally stable model

One of the major advantages of variational formulations in general is that properties related to existence and uniqueness of solutions may be easily assessed. For models of the kind following from the incremental variational principle (33) it can be shown that existence of solutions is guaranteed as long as $F(\mathbf{0},\mathbf{0}) \leqslant 0$. The state $(\boldsymbol{\sigma},\boldsymbol{\kappa}) = (\mathbf{0},\mathbf{0})$ is then always a solution. Similarly, using well known results from nonlinear programming (see e.g. Boyd and Vandenberghe, 2006), it can be shown that the solution is unique if the quadratic part of the objective function and the yield constraints are both convex. The former condition is satisfied provided that both $\boldsymbol{C}_t$ and $\boldsymbol{G}_t$ are positive definite. While most elastic laws involve a positive definite elastic compliance modulus, the requirement of objective function convexity will be violated if the hardening function, $\boldsymbol{h}$, attains negative values. In the present model such negative values correspond to stress states $q \geqslant Mp$ and the yield condition adopted does not prevent such states from being attained. While this does not compromise the variational structure of the model, it does lead to a nonconvex potential and thereby to a situation where uniqueness no longer can be guaranteed. In the following we therefore endeavor to modify the model so that the behaviour for stress states inside the failure envelope $q \leqslant Mp$ are unaffected, while stresses states outside are explicitly precluded.

The requirement that $q \leqslant Mp$ at all times can be satisfied by adopting the following yield function:

$$F(p,q,\kappa) = q^2 - M^2p^2(1 - p/\kappa) \tag{70}$$

where $\kappa \geqslant 0$ again is a hardening variable. The evolution of this yield function with κ is shown in Fig. 5.

Imposing the stress-dilatancy relation (60) leads to the following coupling parameters:

$$S_v = S_v(\eta) = \frac{M^2(M^2 - \eta^2 - 2N\eta)}{(M^2 - \eta^2)^2};\ S_s = 0 \tag{71}$$

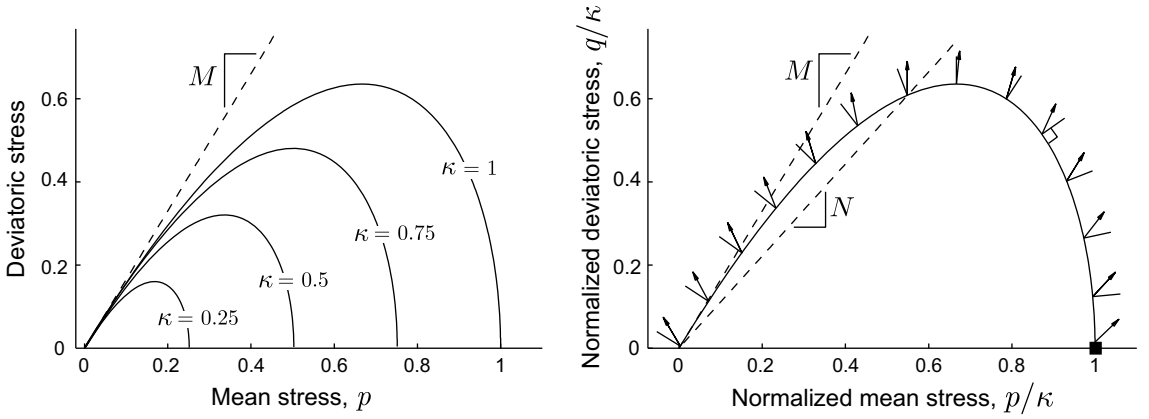

Fig. 5. Evolution of yield surface with κ (left) and plastic strain rate vectors (right). The material parameters, $M = 1.6$ and $N = 1.0$, correspond to a dense sand.

The plastic strain rate vectors $\boldsymbol{a}_t$ (c.f. Eq. 40) of the two models are now identical and it only remains to fix the hardening function of the new model to make the correspondence complete. Straightforward calculations show that the necessary hardening function is given by

$$h(p,q) = \frac{1}{c_s^p}\frac{2\eta p^2 M^3 N}{(M+\eta)(M^2-\eta^2)} \tag{72}$$

This concludes the construction of the new unconditionally stable model whose incremental stress–strain relations again are given by (69).

While various instabilities of course are observed experimentally and eventually should be accounted for, it is an interesting result that it is possible to construct a reasonable model of frictional plasticity that precludes unstable behaviour a priori. From a computational point of view, such a model is also very convenient as it always leads to the particular types of elliptic boundary value problems for which the standard finite element method is directly applicable.

Finally, it is interesting to note that a standard nonassociated elastic/perfectly plastic model of the Drucker-Prager type is achieved in the limit of an initial 'overconsolidation ratio', κ_0/p_0, tending to infinity.

7. Conclusions

Starting from a thermomechanical formulation of elastoplasticity a stress-based variational principle has been derived. This principle suggests the existence of a new modulus, $\boldsymbol{S}$, which alters the governing equations of standard associated elastoplasticity on two counts: 1) An effectively nonassociated flow rule may be specified via $\boldsymbol{S}$ (Eq. 36) without losing the normality structure of the governing equations and 2) The new modulus $\boldsymbol{S}$ implies that the size and shape of the yield surface may change (in a reversible manner) as a result of elastic straining only (Eq. 37).

Both these features appear to have particular relevance to the modeling of granular materials as has been demonstrated by the development of an actual constitutive model. Although this model is far from complete, it does account correctly (qualitatively as well as quantitatively) for the behaviour within its range of validity. The physically consistent decoupling of shear and isotropic compression characteristics is particularly noteworthy.

Immediate possibilities for extension include the incorporation of features accounting for grain crushing and softening behaviour. The former extension should be carried out via the specification of a new yield surface (and an affiliated hardening law) that accounts explicitly for the grain crushing properties without altering the performance of the model below the grain crushing pressure. If a purely phenomenological approach is taken, this would as a minimum require two additional material parameters: a characteristic pressure at which grain crushing commences and an elastoplastic bulk compliance constant similar to the constant c_v^{ep} introduced in Section 6.1.1. More rigorous models would also include information about the grain size distribution as discussed by Einav (2007a,b).

The more challenging question concerns the issue of instabilities. While most models have the tendency to bifurcation built in, either explicitly in terms of a softening yield surface or implicitly in terms of a flow potential separate from the yield potential, the present modeling approach suggests that instabilities should be accounted for in an rather different way. That is, the unconditional stability of the second model presented requires that a separate bifurcation criterion be specified. Once this criterion is satisfied, i.e. once a bifurcation is detected, the stable model loses its validity and it becomes necessary to proceed via a completely different avenue that yet remains to be identified. However, regardless of the exact approach taken, the modeling of laboratory tests would reasonably require that the sample be considered as a system rather than as a point (Desrues and Chambon, 2002). In the context of boundary value problems of engineering interest (foundations, slopes, etc.),

relatively involved modifications of standard finite element procedures would be required (e.g. Leroy and Ortiz, 1989; Cervera and Chiumenti, 2006).

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Appendix 1. Optimality conditions for inequality constrained optimization problems

Consider the inequality constrained optimization problem:

$$\begin{array}{ll} \text{maximize} & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} - w(\boldsymbol{x}) \\ \text{subject to} & f(\boldsymbol{x}) \leqslant 0 \end{array} \tag{73}$$

The associated optimality conditions can be derived in a number of ways. Following common practice in the modern optimization literature Boyd and Vandenberghe, 2006, a positively restricted variable is first added to the inequality constraint thus converting it into an equality:

$$\begin{array}{ll} \text{maximize} & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} - w(\boldsymbol{x}) \\ \text{subject to} & f(\boldsymbol{x}) + s = 0, s \geqslant 0 \end{array} \tag{74}$$

Next, a logarithmic barrier function is added to the objective function:

$$\begin{array}{ll} \text{maximize} & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} - w(\boldsymbol{x}) + \mu \ln s \\ \text{subject to} & f(\boldsymbol{x}) + s = 0, (s > 0) \end{array} \tag{75}$$

where $\mu > 0$ is an arbitrarily small positive parameter. The introduction of the logarithmic barrier function avoids the need to make explicit reference to the sign of s.

The Lagrangian of the above modified problem is given by

$$\mathcal{L}(\boldsymbol{x}, s, \lambda) = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} - w(\boldsymbol{x}) + \mu \ln s - \lambda[f(\boldsymbol{x}) + s] \tag{76}$$

where λ is a Lagrange multiplier.

The first-order Karush-Kuhn-Tucker optimality conditions then follow by rendering the Lagrangian stationary:

$$\begin{array}{rcl} \nabla_x \mathcal{L} & = & \boldsymbol{b} - \nabla_x w(\boldsymbol{x}) - \lambda \nabla_x f(\boldsymbol{x}) = \boldsymbol{0} \\ \nabla_\lambda \mathcal{L} & = & -f(\boldsymbol{x}) - s = 0 \\ \nabla_s \mathcal{L} & = & \mu s^{-1} - \lambda = 0 \Longleftrightarrow s\lambda = \mu \end{array} \tag{77}$$

In the limit of $\mu \to 0$ these conditions are equivalent to

$$\begin{array}{l} \boldsymbol{b} - \nabla_x w(\boldsymbol{x}) - \lambda \nabla_x f(\boldsymbol{x}) = \boldsymbol{0} \\ f(\boldsymbol{x}) \leqslant 0, \lambda f(\boldsymbol{x}) = 0, \lambda \geqslant 0 \end{array} \tag{78}$$

If w and f are both convex, the optimization problem (73) as a whole is convex and the KKT conditions are necessary and sufficient for the attainment of a global optimum (in the above case a maximum).

The logarithmic barrier approach outlined above is discussed in more detail by Krabbenhoft and Damkilde (2003) and Krabbenhoft et al. (2007c) with particular reference to conventional plasticity theory.

References

Alfano, G., Rosati, L.A., 1998. General approach to the evaluation of consistent tangent operators for rate-independent elastoplasticity. Computer Methods in Applied Mechanics and Engineering 167, 75–89.

Andersen, E.D., Roos, C., Terlaky, T., 2003. On implementing a primal-dual interior–point method for conic quadratic optimization. Mathematical Programming 95, 249–277.

Armero, F., Perez–Foguet, A., 2002. On the formulation of closest-point projection algorithms in elastoplasticity – part I: The variational structure. International Journal for Numerical Methods in Engineering 53, 297–329.

Ben-Tal, A., Nemirovski, A., 2001. Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications. MPS-SIAM Series on Optimization.

Bigoni, D., Hueckel, T., 1991. Uniqueness and localization – I. Associative and non-associative elastoplasticity. International Journal of Solids and Structures 2, 197–213.

Boyd, S., Vandenberghe, L., 2006. Convex Optimization. Cambridge University Press.

Cervera, M., Chiumenti, M., 2006. Smeared crack approach : back to the original track. International Journal for Numerical and Analytical Methods in Geomechanics 30, 1173–1199.

Chandler, H.W., 1988. A variational principle for granular materials. International Journal for Numerical and Analytical Methods in Geomechanics 12, 371–378.

Chen, W.F., Han, D.J., 1988. Plasticity for Structural Engineers. Springer-Verlag, New York.

Collins, I.F., 2005. Elastic/plastic models for soils and sands. International Journal of Mechanical Sciences 47, 493–508.

Collins, I.F., Houlsby, G., 1997. Application of thermomechanical principles to the modelling of geotechnical materials. Proceeding of the Royal Society London A 453, 1975–2000.
Collins, I.F., Kelly, P.A., 2002. A thermomechanical analysis of a family of soil models. Geotechnique 52, 507–518.
Collins, I.F., Muhunthan, B., Tai, A.T.T., Pender, M.J., 2007. The concept of a 'Reynolds-Taylor state' and the mechanics of sands. Geotechnique 57, 437–447.
De Saxce, G., Bousshine, L., 1998. Limit analysis theorems for implicit standard materials: application to the unilateral contact with dry friction and the non-associated flow rules in soils and rocks. International Journal of Mechanics Sciences 40, 387–398.
Desrues, J., Chambon, R., 2002. Shear band analysis and shear moduli calibration. International Journal of Solids and Structures 39, 3757–3776.
Einav, I., 2007a. Breakage mechanics-Part I: theory. Journal of the Mechanics and Physics of Solids 55, 1274–1297.
Einav, I., 2007b. Breakage mechanics-Part II: modelling granular materials. Journal of the Mechanics and Physics of Solids 55, 1298–1320.
Einav, I., Puzrin, A., 2004. Pressure-dependent elasticity and energy conservation in elastoplastic models for soils. Journal of Geotechical and Geoenvironmental Engineering, ASCE 130, 81–92.
Gens, A., Potts, D.M., 1988. Critical state models in computational geomechanics. Engineering Computations 5, 178–197.
Halphen, B., Nguyen, Q.S., 1975. Sur les matériaux standards généralisés. Journal de Mécanique 14, 49–63.
Han, W., Reddy, B.D., 2001. Plasticity: The Variational Basis and Numerical Analysis. Springer–Verlag.
Hjiaj, M., Huang, W., Krabbenhoft, K., Sloan, S.W., 2005. Formulation of non-standard dissipative behavior of geomaterials. Journal of Engineering Mathematics 52, 147–165.
Houlsby, G.T., Puzrin, A.M., 2000. A thermomechanical framework for constitutive models for rate-independent dissipative materials. International Journal of Plasticity 16, 1017–1047.
Ibrahimbegovic, A., Gharzedinne, F., Chorfi, L., 1998. Classical plasticity and viscoplasticity models reformulated: Theoretical basis and numerical implementation. International Jounal for Numerical Methods in Engineering 42, 1499–1535.
Ishihara, K., 1993. Liquefaction and flow failure during earthquakes. Geotechnique 43, 351–415.
Krabbenhoft, K., Damkilde, L.A., 2003. general nonlinear optimization algorithm for lower bound limit analysis. International Journal for Numerical Methods in Engineering 56, 165–184.
Krabbenhoft, K., Lyamin, A.V., Sloan, S.W., 2007a. Formulation and solution of some plasticity problems as conic programs. International Jounal of Solids and Structures 44, 1533–1549.
Krabbenhoft, K., Lyamin, A.V., Sloan, S.W., 2007b. Three-dimensional Mohr-Coulomb limit analysis using semidefinite programming. Comunications in Numerical Methods in Engineering. doi:10.1002/cnm.1018.
Krabbenhoft, K., Lyamin, A.V., Sloan, S.W., Wriggers, P., 2007c. An interior-point method for elastoplasticity. International Journal for Numerical Methods in Engineering 69, 592–626.
Krenk, S., 2000. Characteristic state plasticity for granular materials. Part I: Basic theory. International Journal of Solids and Structures 37, 6343–6360.
Lee, K.L., Seed, H.B., 1967. Drained strength characteristics of sands. Journal of the Soil Mechanics and Foundations Division, ASCE 93 (SM6), 117–141.
Leroy, Y., Ortiz, M., 1989. Finite element analysis of strain localization in frictional materials. International Journal for Numerical and Analytical Methods in Geomechanics 13, 53–74.
Lubliner, J., 1990. Plasticity Theory. Macmillan.
Luong, M.P. 1980. Stress–strain aspects of cohesionless soils under cyclic and transient loading. In: Pande, G.N., Zienkiewicz, O.C., (Eds.), International Symposium on Soils under Transient and Cyclic Loading, Balkema 1980, 315–324.
McDowell, G.R., Nakata, Y., Hyodo, M., 2002. On the plastic hardening of sand. Geotechnique 52, 349–358.
Mroz, Z., 1998. Elastoplastic and viscoplastic constitutive models for granular materials. In: Cambou, B., (Ed.) Behaviour of granular materials, Springer–Verlag, pp. 269–337.
Nguyen, Q.S., 1973. Contribution a la Theorie de l'Elasto-plasticite avec ecrouissage, Doctoral Thesis, Universite de Pierre-et-Marie-Curie, Paris.
Nguyen, Q.S., 2004. Stability and Nonlinear Solid Mechanics. Wiley.
Ortiz, M., Pandolfi, A., 2004. A variational Cam clay theory of plasticity. Computer Methods in Applied Mechanics and Engineering 193, 2645–2666.
Reddy, B.D., Martin, J.B., 1994. Internal variable formulations of problems in elastostoplasticity: constitutive and algorithmic aspects. Applied Mechanics Reviews 47, 429–456.
Schofield, A., Wroth, P., 1968. Critical State Soil Mechanics. Cambridge University Press.
Schofield, A., Wroth, P., Critical State Soil Mechanics. Available from: <http://www.geotechnique.info/2007>.
Simo, J.C., 1998. Numerical analysis and simulation in plasticity. In: Ciarlet, P.G., Lions, J.L., (Eds.) Handbook of Numerical Analysis, Elsevier, pp. 179–499.
Simo, J.C., Kennedy, J.G., Taylor, R.L., 1989. Complementary mixed finite element formulations for elastoplasticity. Computer Methods in Applied Mechanics and Engineering 74, 177–206.
Stroud, M.A. (1971). The Behaviour of Sand at Low Stress Levels in the Simple Shear Apparatus. PhD thesis, Cambridge University.
Sturm, J.F., 2002. Implementation of interior point methods for mixed semidefinite and second order cone optimization problems. Optimization Methods and Software 17, 1105–1154.
Taylor, D.W., 1948. Fundamentals of Soil Mechanics. Wiley.
Telega, J.J., 1978. Variational principles for rate boundary-value problems in non-associated plasticity. Zeitschrift fnr Angewandte Mathematik und Mechanik 60, 71–82.
Washizu, K., 1982. Variational Methods of Elasticity and Plasticity. Pergamon Press.
Wood, D.M., 1990. Soil Behaviour and Critical State Soil Mechanics. Cambridge University Press.
Yu, H.S., 2006. Plasticity and Geotechnics. Springer-Verlag.
Zouain, N., Filho, I.P., Borges, L., da Costa, L.M., 2007. Plastic collapse in non-associated hardening materials with application to Cam-clay. International Journal of Solids and Structures 44, 4382–4398.