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Technical Note

Sudden Excavation of a Long Circular Tunnel in Elastic Ground

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INTRODUCTION

When cavities are created underground in rock which is prestressed, some of the stored strain energy is released. The speed at which this energy is released has an important effect on the subsequent behaviour of the rock mass. In many applications, particularly in metalliferous mining but also in some civil engineering works, the surface generated by excavation will be created almost instantaneously. The development of an underground opening by the removal or rock which is initially stressed is mechanically equivalent to the application of a set of tractions over the boundaries of the excavation. Thus, in many instances, it is appropriate to model the effects of the excavation process by the impulsive application of these surface tractions within the rock mass.

It is well known that the impulsive loading of an elastic material will cause elastic waves to pass through the medium and that these waves may give rise to transient stresses greater in magnitude than the final, static stresses. In some cases, knowledge of these transient stress changes will be important, particularly if they are large enough to cause failure in the surrounding rock mass.

Formal solutions to some problems of this kind are available in the literature. The classical problem of a spherical opening has been dealt with by Sharpe [1], Eringen [2] and Selberg [3], and a useful treatment in the mining context has been given by Brady and Brown [4], who also considered thin tabular excavations. However, the problem of a long cylindrical cavity has received much less attention, presumably because of the difficulty of obtaining solutions in closed-form. A formal solution procedure for this case has been presented by Selberg [3], but it would appear that the evaluation of the quantities of greatest interest, i.e. the radial and circumferential stress components, has not been pursued in detail.

The aim of this paper is to derive a solution for the problem of a cylindrical cavity excavated rapidly in elastic ground and to evaluate that solution for a practically significant case. The solution is developed in terms of Laplace transforms of the field quantities, and inversion of the transforms is carried out numerically. An examination is also made of the influence of the rate of excavation on the transient stresses around the opening.

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GOVERNING EQUATIONS

The equations governing the dynamic behaviour of a linear elastic body, deforming under conditions of plane strain and axial symmetry are set out below.

Equilibrium

Using a cylindrical coordinate description, the equilibrium of an axisymmetric body in the radial direction, including dynamic effects but excluding bodyforces, can be written as:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u}{\partial t^2}, \tag{1}$$

in which

σ_r, σ_θ = the cylindrical components of the stress change induced by the applied loading, with tension regarded as positive,
u = the radial displacement,
r = the radial coordinate,
t = time,
ρ = the mass density of the elastic material.

Strain–displacement relations

The small strain components in the cylindrical coordinate description are defined as:

$$\epsilon_r = \frac{\partial u}{\partial r}, \tag{2a}$$

$$\epsilon_\theta = \frac{u}{r}. \tag{2b}$$

Hooke's law

For a linear elastic material deforming in plane strain, Hooke's law can be written as:

$$\sigma_r = (\lambda + 2G)\frac{\partial u}{\partial r} + \lambda \frac{u}{r}, \tag{3a}$$

$$\sigma_\theta = \lambda \frac{\partial u}{\partial r} + (\lambda + 2G)\frac{u}{r}, \tag{3b}$$

in which λ and G are the Lamé modulus and the shear modulus of the elastic medium, respectively.

Equation of motion

Equations (3) can be substituted into (1) to give the equation of planar radial motion for the elastic body, viz.

$$\frac{\partial^2 u}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u}{r}\right)=\left(\frac{1}{c_p}\right)^2\frac{\partial^2 u}{\partial t^2}, \tag{4}$$

where

$$c_p=\left(\frac{\lambda+2G}{\rho}\right).$$

The quantity c_p is the speed at which dilational (or compression) waves propagate through the elastic medium.

GENERAL SOLUTION

To obtain the complete solution to this problem it is convenient to take Laplace transforms of the governing equation (4), and assuming an initial quiescent state:

$$r^2\frac{\partial^2\bar{u}}{\partial r^2}+r\frac{\partial\bar{u}}{\partial r}-\left[1+\left(\frac{sr}{c_p}\right)^2\right]\bar{u}=0, \tag{5}$$

where the superior bar indicates a Laplace transform, i.e.

$$\bar{u}=\int_0^\infty u\,e^{-st}\,dt. \tag{6}$$

The general solution to equation (5) is well known and can be written as:

$$\bar{u}=AI_1\left(\frac{sr}{c_p}\right)+BK_1\left(\frac{sr}{c_p}\right), \tag{7}$$

where I_1 and K_1 are modified Bessel functions.

BOUNDARY CONDITIONS

It is obvious that if the solution is to remain bounded as r approaches infinity, then $A=0$. An additional boundary condition is required to determine the constant B, i.e. an assumption about the nature of the imposed loading at the cavity wall, $r=a$, is required. Consider first the case of a long circular tunnel excavated instantaneously in an elastic medium which initially (i.e. prior to excavation) experiences an isotropic compression of magnitude p_0. For purposes of determining the resulting stresses around the opening, excavation of the material from within the tunnel can be regarded as the application of a tensile radial traction of magnitude p_0 to the internal boundary $r=a$, i.e.

$$\sigma_r=p_0 \quad \text{at} \quad r=a \quad \text{and} \quad t\geqslant 0, \tag{8}$$

Laplace transforms may be taken of equation (8) so that:

$$\bar{\sigma}_r=\frac{p_0}{s}. \tag{9}$$

Substitution of this boundary condition into equations (3a) and (7) gives the constant B as:

$$B=\frac{\left(\frac{p_0 a}{s}\right)}{\left[(\lambda+2G)\left(\frac{sa}{c_p}\right)K_1'\left(\frac{sa}{c_p}\right)+\lambda K_1\left(\frac{sa}{c_p}\right)\right]} \tag{10}$$

and from this follows the solutions for the Laplace transforms $\bar{u}$, $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$.

It is also of interest to consider the case where the removal of the radial tractions from the cavity surface is more gradual. Consider the case where the stress reduction occurs at a constant rate over the time interval $0\leqslant t\leqslant t_0$, so that:

$$\sigma_r=\left(\frac{p_0}{t_0}\right)t \quad \text{at} \quad r=a \quad \text{and} \quad 0\leqslant t\leqslant t_0 \tag{11}$$

and

$$\sigma_r=p_0 \quad \text{at} \quad r=a \quad \text{and} \quad t>t_0. \tag{12}$$

The direct determination of the Laplace transforms for this case is straightfoward, however, for purposes of numerically inverting these transforms it is more convenient to consider the loading as the superposition of two components, viz.

Component 1:

$$\sigma_r=\left(\frac{p_0}{t_0}\right)t \quad \text{at} \quad r=a \quad \text{and} \quad t>0; \tag{13}$$

Component 2:

$$\sigma_r=-\left(\frac{p_0}{t_0}\right)(t-t_0) \quad \text{at} \quad r=a \quad \text{and} \quad t>t_0. \tag{14}$$

We notice that Component 2 is merely the negative of Component 1 evaluated at the time $(t-t_0)$ and that the transform of Component 1 is:

$$\bar{\sigma}_r=-\left(\frac{p_0}{t_0}\right)\frac{1}{s^2}. \tag{15}$$

The integration constant B for this case is given by:

$$B=\frac{\left(\frac{p_0 a}{t_0 s^2}\right)}{\left[(\lambda+2G)\left(\frac{sa}{c_p}\right)K_1'\left(\frac{sa}{c_p}\right)+\lambda K_1\left(\frac{sa}{c_p}\right)\right]}. \tag{16}$$

Thus the Laplace transforms corresponding to both boundary conditions involve terms in either $1/s^2$ or $1/s$ and modified Bessel functions of s, and these present no special difficulties for the numerical inversion procedure described below.

INVERSION OF TRANSFORMS

It has been shown above that the solution for the Laplace transforms of the field quantities u, σ_r and σ_θ may be obtained in closed form, i.e. in terms of modified Bessel functions K_1 and their derivatives. To recover the actual quantities these transforms must be inverted. This can be achieved efficiently using the numerical contour integration developed by Talbot [5].

It should be noted, however, that the integration constant B [equations (10) and (16)] has a branch line along the negative real axis and singularities in the complex plane at the origin and at the point where $(sa/c_p) \approx -0.442 \pm 0.448i$. Thus of course the curve along which the numerical integration is performed must enclose these singularities and the branch line.

TYPICAL SOLUTIONS

The solutions presented above have been evaluated for the case of a cylindrical opening excavated in a rock mass with Poisson's ratio $\nu = 0.25$, initially under a hydrostatic stress of $-p_0$. The results for the case of instantaneous excavation have been presented in Figs 1–3, from which it is possible to compare the relative magnitude of the static and dynamic stresses associated with cavity creation. On all figures the cumulative stresses, i.e. the ambient stress plus the stress change, have been plotted using the convention of tension positive. Figure 1 shows the temporal variation of the circumferential stress at the boundary $r = a$, which indicates that σ_θ/p_0 decreases in magnitude immediately after creating the opening from its ambient compressive value of -1 to a value of $-[1 - \nu/(1 - \nu)]$. σ_θ then increases in magnitude with time and attains a maximum compressive value given approximately by $\sigma_\theta/p_0 = -2.25$ at a non-dimensional time of approximately 4. This circumferential stress ratio subsequently reduces in magnitude to achieve its final static value of -2. The maximum magnitude of the transient compressive stress at the boundary is about 12.5% greater than the final static value.

The radial variation of the radial and circumferential stress ratios, σ_r/p_0 and σ_θ/p_0, at non-dimensional times $c_p t/a = 2$ and 5 are plotted in Figs 2 and 3, respectively.

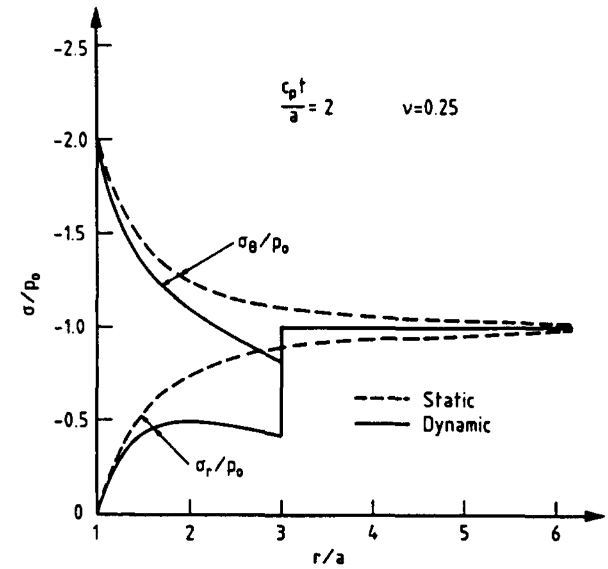

Fig. 2. Stress distribution around a cylindrical cavity suddenly excavated in a hydrostatic stress field—$c_p t/a = 2$.

These confirm that the excavation process initiates a stress wave at the cavity surface which radiates through the medium at the dilational wave velocity c_p before subsequent achievement of the final, static stress distribution around the opening. Thus the strain energy released as the surface tractions at $r = a$ are reduced suddenly by the excavation is propagated to the far field, and in doing so causes transient variations in the stress field. The location of the wavefront can be clearly identified in these figures, as it corresponds to a discontinuity in the dynamic stress field.

The influence of the rate of excavation has been investigated by evaluating solutions for σ_θ at $r = a$ for the case where the *in situ* radial pressure at the cavity surface is reduced linearly with time to zero. These solutions are plotted in Fig. 4, where curves corresponding to $c_p t_0/a = 0$, 1, 10 and 100 have been plotted. t_0 is the time required for the boundary traction to be

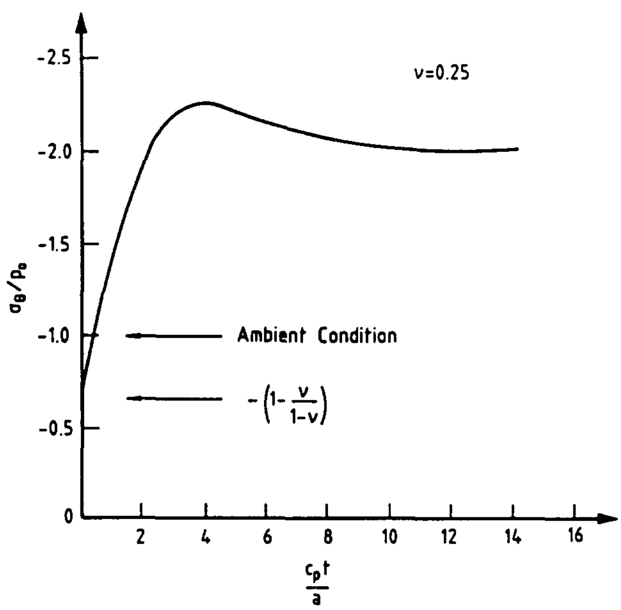

Fig. 1. Temporal variation of boundary stress around a cylindrical cavity suddenly excavated in a hydrostatic stress field.

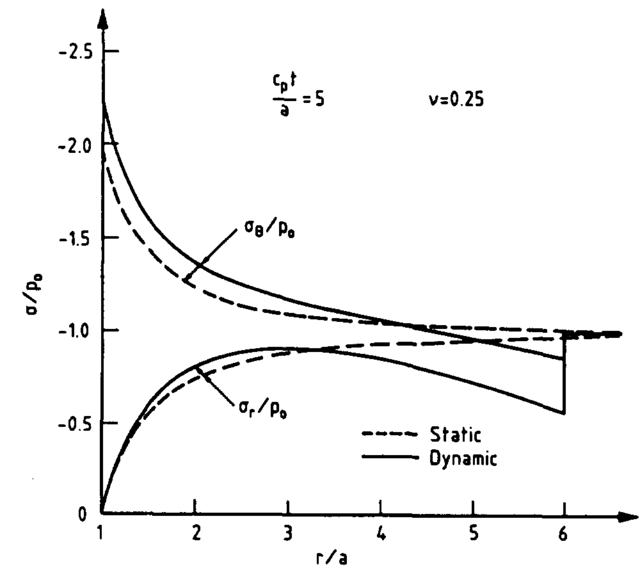

Fig. 3. Stress distribution around a cylindrical cavity suddenly excavated in a hydrostatic stress field—$c_p t/a = 5$.

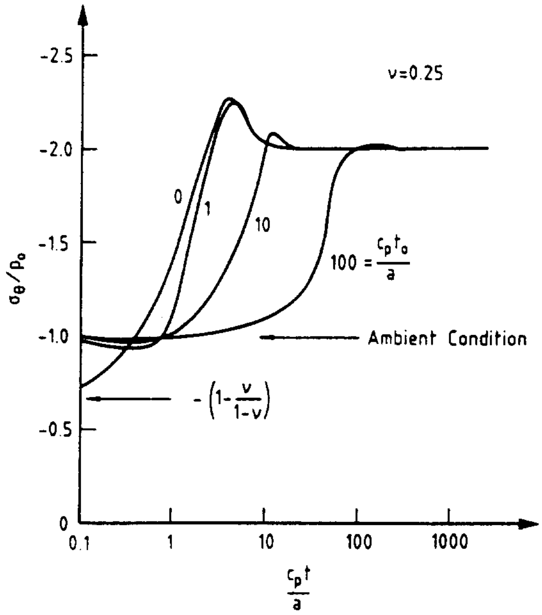

Fig. 4. Influence of rate of excavation on boundary stress around a cylindrical cavity excavated in a hydrostatic stress field.

reduced to zero and hence $c_p t_0/a = 0$ corresponds to the case of instantaneous excavation. The results in Fig. 4 indicate that more gradual removal of the boundary stress results in smaller peaks in the transient values of σ_θ at the boundary, as expected. Excavation periods corresponding to $c_p t_0/a = 100$ or greater result in almost no dynamic magnification of σ_θ.

It is also of interest to assess the physical significance of this last result. Values of the wave velocity c_p for rock masses generally lie within the range from 10^3 to 10^4 m/sec. Thus for a tunnel of 1 m radius, the dynamic effects will be insignificant if the time interval required for stress removal is greater than about 0.01–0.1 sec. It is clear that very rapid rates of stress removal will be required for the dynamic magnification to be important. This is only likely to occur in practice when excavation is carried out by blasting. It should be noted, however, that blasting will induce additional dynamic effects, and these have not been considered here.

CONCLUSIONS

The solution for the transient stress distribution around a circular tunnel excavated in an elastic rock mass has been presented. It has been demonstrated that for rapid excavation the dynamic effects can result in significant differences between the short- and long-term stress distributions. It was also demonstrated that more gradual excavation reduces the differences between the maximum transient and the long-term stresses. For most rock masses the stress removal at the tunnel walls must occur over a period of about 0.1 sec or less for the dynamic effects to have a significant influence.

The solutions presented here should also be of use as benchmarks for other numerical procedures, such as the finite-element method, for the solution of dynamics problems in rock mechanics.

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