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ELASTIC CONSOLIDATION AROUND A POINT SINK EMBEDDED IN A HALF-SPACE WITH ANISOTROPIC PERMEABILITY

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SUMMARY

The complete solution is presented for the transient effects of pumping fluid from a point sink embedded in a saturated, porous elastic half-space. It is assumed that the medium is homogeneous and isotropic with respect to its elastic properties and homogeneous but anisotropic with respect to the flow of pore fluid. The soil skeleton is modelled as a linear elastic material obeying Hooke's law, while the pore fluid is assumed to be incompressible with its flow governed by Darcy's law. The solution has been evaluated for a particular value of Poisson's ratio of the solid skeleton, i.e. 0·25, and the results have been presented graphically in the form of isochrones of excess pore pressure and surface profile for the half-space. The solutions presented may have application in practical problems such as dewatering operations in compressible soil and rock masses.

INTRODUCTION

In geotechnical, hydraulic and petroleum engineering it is sometimes necessary to pump water or some other fluid from the ground. This may be for a variety of reasons including (a) obtaining supplies of water, oil or gas, (b) reducing pore water pressures in the ground, and (c) lowering the water table in order to allow construction operations to proceed.

In order to remove pore fluid from the ground it is necessary to reduce the pressure in the fluid in the vicinity of the pump and so there will in general be an increase in the compressive effective stress state. This increase of effective stress will cause consolidation of the ground and may lead to large-scale subsidence. The decrease in pore pressure will not occur immediately. After pumping has commenced, the pore pressures will gradually decrease below their initial *in situ* values until a steady-state distribution is established. Hence the resultant consolidation and surface subsidence will be time dependent.

Probably the best known examples of this phenomenon occur in Bangkok, Venice and Mexico City where widespread subsidence has been caused by withdrawal of water from aquifers for industrial and domestic purposes. Recorded settlements in Mexico City have reached rates of 5–6 cm per year.[1] However, the problem is more widespread than this, with subsidence due to fluid extraction having been reported in a number of other regions of the world (see, for example, References 2–5). The problem is not exclusively caused by the extraction of groundwater; the withdrawal of air and gas can also induce surface subsidence (see, for example, Reference 6).

The purpose of this paper is to provide the complete solution for the transient effects of pumping fluid from a point sink embedded in a saturated porous elastic half-space. The problem is defined in Figure 1. In obtaining this solution, proper account has been taken of the coupling of the pore fluid

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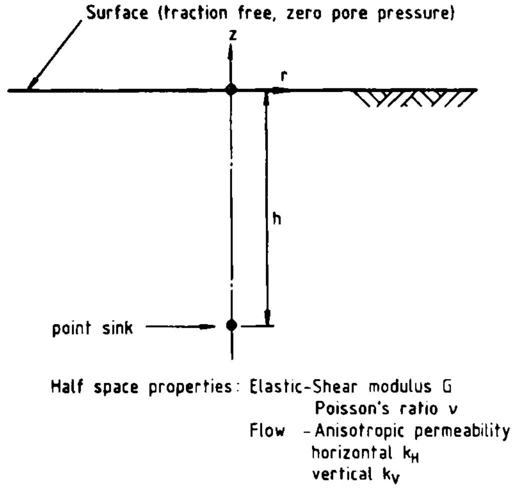

Figure 1. Problem definition

flow with the deformation of the solid skeleton. It has been assumed that the saturated medium is homogeneous with respect to its elastic properties and homogeneous but transversely isotropic with respect to the flow of pore fluid, so that the soil has one value of permeability in any horizontal plane and another value for vertical flow. Furthermore, it has been assumed that the pore fluid is incompressible and that the half-space remains saturated.

The point sink problem treated here is of course an extreme idealization of any real situation. Nevertheless, it is considered that investigations of this type have much value in that their very lack of complexity allows an uncluttered look at the processes in operation and often allows an assessment of their relative importance. Moreover, for many preliminary investigations this extreme idealization is all that is required in the absence of detailed field data. It also serves to give an idea of the likely severity of various effects.

GOVERNING EQUATIONS

The equations governing the consolidation of a poreelastic medium were first developed by Biot.[7,8] When expressed in terms of a cartesian co-ordinate system they take the following forms.

Equilibrium

In the absence of increase in body forces the equations of equilibrium can be written as

$$\partial \boldsymbol{\sigma} = 0 \tag{1}$$

where

$$\partial = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial y & 0 & \partial/\partial z \\ 0 & \partial/\partial y & 0 & \partial/\partial x & \partial/\partial z & 0 \\ 0 & 0 & \partial/\partial z & 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}$$

$$\boldsymbol{\sigma}^{\mathrm{T}} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx})$$

is the vector of total stress components with tensile normal stress regarded as positive (these quantities represent the increase over the initial state of stress).

Strain–displacement relations

The strains are related to the displacement as follows:

$$\boldsymbol{\varepsilon} = \partial^{\mathrm{T}}\mathbf{u} \tag{2}$$

where

$$\boldsymbol{\varepsilon}^{\mathrm{T}} = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$$

is the vector of strain components of the soil skeleton, and $\mathbf{u}^{\mathrm{T}} = (u_x, u_y, u_z)$ is the vector of cartesian displacement components of the skeleton.

Effective stress principle

It is assumed for the saturated soil that the effective stress principle is valid, i.e.

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p\mathbf{a} \tag{3}$$

where

$$\boldsymbol{\sigma}' = (\sigma'_{xx}, \sigma'_{yy}, \sigma'_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx})^{\mathrm{T}}$$

is the vector of effective stress increments (these quantities represent the increase over the initial state of effective stress)

$$\mathbf{a}^{\mathrm{T}} = (1, 1, 1, 0, 0, 0)$$

and p is the excess pore fluid pressure.

Hooke's law

The constitutive behaviour of the solid phase (the skeleton) of the saturated medium is governed by Hooke's law, which is

$$\boldsymbol{\sigma}' = D\boldsymbol{\varepsilon} \tag{4}$$

$$\text{where } D = \begin{bmatrix} \lambda+2G & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2G & \lambda & 0 & 0 & 0 \\ & & \lambda+2G & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ & & & & G & 0 \\ \text{symmetric} & & & & & G \end{bmatrix}$$

with λ and G the Lame modulus and shear modulus of the soil skeleton, respectively.

The moduli λ, G are related to Young's modulus E and Poisson's ratio ν of the skeleton, i.e.

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}$$

$$G = \frac{E}{2(1+\nu)}$$

Darcy's law

It will be assumed that the flow of pore water is governed by Darcy's law, which for a transversely isotropic soil takes the form

$$
\begin{aligned}
v_x &= -\frac{k_H}{\gamma_F}\frac{\partial p}{\partial x} \\
v_y &= -\frac{k_H}{\gamma_F}\frac{\partial p}{\partial y} \\
v_z &= -\frac{k_V}{\gamma_F}\frac{\partial p}{\partial z}
\end{aligned}
\tag{5}
$$

where k_H, k_V are the horizontal and vertical permeability, respectively, γ_F is the unit weight of pore fluid and the z co-ordinate direction is aligned vertically and v_x, v_y, v_z are the components of the superficial velocity vector relative to the soil skeleton.

Displacement equations

If Hooke's law (4) and the equations of equilibrium (1) are combined it is found that

$$G\nabla^2\mathbf{u} + (\lambda + G)\nabla\varepsilon_v = \nabla p \tag{6}$$

where

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

is the volume strain. This equation can be condensed to give the useful relation

$$(\lambda + 2G)\nabla^2\varepsilon_v = \nabla^2 p \tag{7}$$

The volume constraint equation

If the pore fluid and the skeletal material are both incompressible, then the volume change of any element of soil must balance the difference between the volume of fluid leaving and entering the element by flow across its boundaries, plus the volume of fluid extracted from the element by some internal sink mechanism. Symbolically this continuity condition may be expressed as the volume constraint equation, i.e.

$$\int_0^t \nabla^{\mathrm{T}}\mathbf{v}\,\mathrm{d}t + \varepsilon_v = -\int_0^t q\,\mathrm{d}t \tag{8}$$

where q is the volume of fluid extracted per unit volume per unit time of soil by the sink mechanism and $\mathbf{v}^{\mathrm{T}} = (v_x, v_y, v_z)$.

If equation (8) is combined with Darcy's law, equation (5), and Laplace transforms are taken of the resulting equation, we find that

$$\frac{k_H}{\gamma_F}\left(\frac{\partial^2\bar{p}}{\partial x^2} + \frac{\partial^2\bar{p}}{\partial y^2}\right) + \frac{k_V}{\gamma_F}\frac{\partial^2\bar{p}}{\partial z^2} = s\bar{\varepsilon}_V + \bar{q} \tag{9}$$

or

$$c_H\left(\frac{\partial^2\bar{p}}{\partial x^2} + \frac{\partial^2\bar{p}}{\partial y^2}\right) + c_V\frac{\partial^2\bar{p}}{\partial z^2} = (\lambda + 2G)(s\bar{\varepsilon}_V + \bar{q}) \tag{10}$$

where

$$
\begin{aligned}
c_H &= k_H(\lambda + 2G)/\gamma_F \\
c_V &= k_V(\lambda + 2G)/\gamma_F
\end{aligned}
$$

are the horizontal and vertical coefficients of consolidation of the saturated porous elastic medium.

The superior bar is used here to indicate a Laplace transform, i.e.

$$\bar{f}(s) = \int_0^\infty f(t) \mathrm{e}^{-st} \, \mathrm{d}t \tag{11}$$

with s being the Laplace transform variable.

SOLUTION METHOD

In proceeding to the solution of the equations of consolidation for the case of a point sink embedded in a saturated elastic half-space, we introduce triple Fourier transforms of the type

$$P^*(\alpha, \beta, \gamma) = (1/2\pi)^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-i(\alpha x + \beta y + \gamma z)} p(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \tag{12a}$$

The corresponding inversion formula is

$$p(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{+i(\alpha x + \beta y + \gamma z)} P^*(\alpha, \beta, \gamma) \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma \tag{12b}$$

Use will also be made of double Fourier transforms of the type

$$P(\alpha, \beta, z) = (1/2\pi)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-i(\alpha x + \beta y)} p(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \tag{13a}$$

and the corresponding inversion formula

$$p(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{i(\alpha x + \beta y)} P(\alpha, \beta, z) \mathrm{d}\alpha \, \mathrm{d}\beta \tag{13b}$$

If we compare equations (11, 12) we see that

$$P^*(\alpha, \beta, \gamma) = \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{-i\gamma z} P(\alpha, \beta, z) \mathrm{d}z \tag{14a}$$

and conversely

$$P(\alpha, \beta, z) = \int_0^\infty \mathrm{e}^{+i\gamma z} P^*(\alpha, \beta, \gamma) \mathrm{d}\gamma \tag{14b}$$

Sometimes it will be convenient to introduce the co-ordinates (ρ, ε), where

$$\begin{aligned} \alpha &= \rho \cos \varepsilon \\ \beta &= \rho \sin \varepsilon \end{aligned} \tag{15}$$

in which case equations (13b) become, for polar co-ordinates (r, θ, z),

$$p(r, \theta, z) = \int_0^\infty \int_0^{2\pi} \mathrm{e}^{i\rho r \cos(\theta - \varepsilon)} P \rho \, \mathrm{d}\rho \, \mathrm{d}\varepsilon \tag{16}$$

Quite often the transform P will be able to be represented in the form

$$P = \cos n(\theta - \varepsilon) \, F(\rho, z) \tag{17}$$

and thus

$$P = 2\pi i^n \int_0^\infty \rho F(\rho, z) J_n(\rho r) \mathrm{d}\rho \tag{18}$$

where J_n represents the Bessel function of order n.

In the analysis which follows solutions for the equations of consolidation are found in terms of the Laplace transforms of the triple Fourier transforms of the field quantities. Partial inversion of the triple Fourier transforms is then carried out in closed form using equation (14) or (18) and the inversion is completed using a single numerical integration. This leaves us with the Laplace transforms of the field quantities which in turn are inverted numerically using the technique developed by Talbot,[9] giving the time-dependent field quantities.

The complete solution for a point sink embedded in a half-space is built up by first considering the case of a point sink in an infinite medium and then the case of a half-space with no sink. The solutions for these problems are given in the following sections.

SOLUTION FOR A POINT SINK

Let us consider a sink of strength F_k located at the point (x_k, y_k, z_k) within an infinite medium, so that

$$q = F_k \delta(x - x_k)\, \delta(y - y_k)\, \delta(z - z_k) \tag{19}$$

where δ indicates the Dirac delta function. We introduce triple transforms having the form of equation (12a) and thus we see, for example, that the transform of q is

$$Q^* = \frac{F_k}{(2\pi)^3} \mathrm{e}^{-i(\alpha x_k + \beta y_k + \gamma z_k)} \tag{20}$$

It will be convenient for our purposes to write this in the form

$$Q^* = \frac{Q}{2\pi} \mathrm{e}^{-i\gamma z_k} \tag{21}$$

where

$$Q = \frac{F_k}{(2\pi)^2} \mathrm{e}^{-i(\alpha x_k + \beta y_k)}$$

Displacement equations

In terms of triple transforms the displacement equations (6) become

$$\begin{aligned} -GD^2 U_x^* + (\lambda + G) i\alpha E_v^* &= i\alpha P^* \\ -GD^2 U_y^* + (\lambda + G) i\beta E_v^* &= i\beta P^* \\ -GD^2 U_z^* + (\lambda + G) i\gamma E_v^* &= i\gamma P^* \\ i\alpha U_x^* + i\beta U_y^* + i\gamma U_z^* &= E_v^* \end{aligned} \tag{22}$$

where

$$D^2 = \alpha^2 + \beta^2 + \gamma^2$$

$$(U_x^*, U_y^*, U_z^*, P^*, E_v^*) = (1/2\pi)^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\alpha x+\beta y+\gamma z)}(u_x, u_y, u_z, p, \varepsilon_v)\,dx\,dy\,dz$$

Equations (22) have the solution

$$\begin{aligned}
U_x^* &= -\left(\frac{i\alpha}{D^2}\right)E_v^* \\
U_y^* &= -\left(\frac{i\beta}{D^2}\right)E_v^* \\
U_z^* &= -\left(\frac{i\gamma}{D^2}\right)E_v^* \\
P^* &= (\lambda + 2G)E_v^*
\end{aligned} \tag{23}$$

Volume constraint equation

In terms of the transforms equation (10) becomes

$$-(\gamma^2 c_V + \rho^2 c_H)\bar{E}_V^* = s\bar{E}_V^* + \bar{Q}^* \tag{24}$$

If we now introduce the variables

$$\mu^2 = c_H\rho^2/c_V + s/c_V$$
$$\rho^2 = \alpha^2 + \beta^2$$

we see that

$$\bar{E}_v^* = \frac{-\bar{Q}^*}{c_V(\gamma^2 + \mu^2)} \tag{25}$$

Stress components

The stress components may be obtained directly from Hooke's law, equations (4), and so

$$\begin{aligned}
S_{xx}^* &= 2G\left(\frac{\alpha^2}{D^2} - 1\right)E_v^* \\
S_{yy}^* &= 2G\left(\frac{\beta^2}{D^2} - 1\right)E_v^* \\
S_{zz}^* &= 2G\left(\frac{\gamma^2}{D^2} - 1\right)E_v^* = -2G\frac{\rho^2}{D^2}E_v^* \\
S_{xy}^* &= 2G\frac{\alpha\beta}{D^2}E_v^* \\
S_{yz}^* &= 2G\frac{\beta\gamma}{D^2}E_v^* \\
S_{xy}^* &= 2G\frac{\alpha\beta}{D^2}E_v^*
\end{aligned} \tag{26}$$

where

$$S^*_{jk} = (1/2\pi)^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-i(\alpha x + \beta y + \gamma z)} \sigma_{jk} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

and j, k denote any of the indices x, y, z.

Partial inversion

All the field quantities determined in this section can be expressed in terms of the three functions $\bar{H}^*, \bar{K}^*, \bar{L}^*$, where

$$\bar{H}^* = \left(\frac{1}{2\pi}\right) \frac{\mathrm{e}^{-i\gamma z_k}}{\gamma^2 + \mu^2} \tag{27a}$$

$$\bar{K}^* = \left(\frac{1}{2\pi}\right) \frac{\mathrm{e}^{-i\gamma z_k}}{(\gamma^2 + \mu^2) D^2}$$

$$= \frac{1}{2\pi(\mu^2 - \rho^2)} \left[-\frac{\mathrm{e}^{-i\gamma z_k}}{\gamma^2 + \mu^2} + \frac{\mathrm{e}^{-i\gamma z_k}}{\gamma^2 + \rho^2} \right] \tag{27b}$$

$$\bar{L}^* = \left(\frac{1}{2\pi}\right) \frac{i\gamma \mathrm{e}^{-i\gamma z_k}}{(\gamma^2 + \mu^2) D^2}$$

$$= \frac{1}{2\pi(\mu^2 - \rho^2)} \left[-\frac{i\gamma \mathrm{e}^{-i\gamma z_k}}{\gamma^2 + \mu^2} + \frac{i\gamma \mathrm{e}^{-i\gamma z_k}}{\gamma^2 + \rho^2} \right] \tag{27c}$$

Now, for the double Fourier transforms

$$(\bar{H}, \bar{K}, \bar{L}) = \int_{-\infty}^{\infty} \mathrm{e}^{i\gamma z} (\bar{H}^*, \bar{K}^*, \bar{L}^*) \mathrm{d}\gamma$$

it can be shown (see Appendix) that

$$\bar{H} = \frac{1}{2} \frac{\mathrm{e}^{-\mu Z}}{\mu} \tag{28a}$$

$$\bar{K} = \frac{1}{2(\mu^2 - \rho^2)} \left[\frac{\mathrm{e}^{-\rho Z}}{\rho} - \frac{\mathrm{e}^{-\mu Z}}{\mu} \right] \tag{28b}$$

$$\bar{L} = \frac{\operatorname{sgn}(z_k - z)}{2(\mu^2 - \rho^2)} \left[\frac{\mathrm{e}^{-\rho Z}}{\rho} - \frac{\mathrm{e}^{-\mu Z}}{\mu} \right] \tag{28c}$$

where

$$Z = |z - z_k|$$

Thus on combining equations (13a, 23, 26, 28) we have

$$i\bar{U}_x = -\alpha \bar{K} \bar{Q} / c_v$$

$$i\bar{U}_y = -\beta \bar{K} \bar{Q} / c_v$$

$$\bar{U}_z = +\bar{L} \bar{Q} / c_v$$

$$\bar{P} = -(\lambda + 2G) \bar{H} \bar{Q} / c_v$$

$$\begin{aligned}
\bar{S}_{xx} &= -2G(\alpha^2\bar{K} - \bar{H})\bar{Q}/c_v \\
\bar{S}_{yy} &= -2G(\beta^2\bar{K} - \bar{H})\bar{Q}/c_v \\
\bar{S}_{zz} &= +2G\rho^2\bar{K}\bar{Q}/c_v \\
\bar{S}_{xy} &= -2G\alpha\beta\bar{K}\bar{Q}/c_v \\
i\bar{S}_{yz} &= -2G\beta\bar{L}\bar{Q}/c_v \\
i\bar{S}_{zx} &= -2G\alpha\bar{L}\bar{Q}/c_v
\end{aligned} \tag{29}$$

SOLUTION FOR A HALF-SPACE WITH NO SINK

To analyse this problem we introduce double Fourier transforms leading to representations of the form given by equation (13b). It will also be useful to introduce auxiliary quantities:

$$\begin{aligned}
U_\xi &= \cos\varepsilon U_x + \sin\varepsilon U_y \\
U_\eta &= -\sin\varepsilon U_x + \cos\varepsilon U_y \\
S_{\xi z} &= \cos\varepsilon S_{xz} + \sin\varepsilon S_{yz} \\
S_{\eta z} &= -\sin\varepsilon S_{xz} + \cos\varepsilon S_{yz}
\end{aligned} \tag{30}$$

where

$$\cos\varepsilon = \frac{\alpha}{\rho}$$

$$\sin\varepsilon = \frac{\beta}{\rho}$$

Displacement equations

In terms of these double transforms equations (6) become

$$G\left(\frac{\partial^2 U_\xi}{\partial z^2} - \rho^2 U_\xi\right) + (\lambda + G)i\rho\bar{E}_v = i\rho\bar{P} \tag{31a}$$

$$G\left(\frac{\partial^2 \bar{U}_\eta}{\partial z^2} - \rho^2 U_\eta\right) = 0 \tag{31b}$$

$$G\left(\frac{\partial^2 \bar{U}_z}{\partial z^2} - \rho^2 \bar{U}_z\right) + (\lambda + 2G)\frac{\partial \bar{E}_v}{\partial z} = \frac{\partial \bar{P}}{\partial z} \tag{31c}$$

where

$$\bar{E}_v = \frac{\partial \bar{U}_z}{\partial z} + i\rho\bar{U}_\xi \tag{31d}$$

(In the problem considered here it is found that $U_\eta = 0$.) Equations (31) can be combined to give

$$(\lambda + 2G)\left[\frac{\partial^2 \bar{E}_v}{\partial z^2} - \rho^2\bar{E}_v\right] = \left[\frac{\partial^2 \bar{P}}{\partial z^2} - \rho^2\bar{P}\right] \tag{32}$$

Volume constraint equation

Equation 9(b) becomes

$$c_V \frac{\partial^2 \bar{P}}{\partial z^2} - c_H \rho^2 \bar{P} = (\lambda + 2G) s \bar{E}_v \tag{33}$$

Solution

The solutions of equations (32, 33) which remain bounded as $z \to -\infty$ are:

$$\begin{aligned} \bar{E}_v &= A e^{\mu z} + \left(\frac{2G}{\lambda + G}\right) \delta B e^{\rho z} \\ \bar{P} &= (\lambda + 2G) A e^{\mu z} + 2GB e^{\rho z} \end{aligned} \tag{34}$$

where

$$\delta = \left(\frac{\lambda + G}{\lambda + 2G}\right)(c_V - c_H)\frac{\rho^2}{s}$$

If we substitute equations (34) into equation (31c) we find

$$\frac{\partial^2 \bar{U}_z}{\partial z^2} - \rho^2 \bar{U}_z = \mu A e^{\mu z} + 2B\rho(1 - \delta) e^{\rho z}$$

and thus

$$\rho \bar{U}_z = \left(\frac{\mu \rho}{\mu^2 - \rho^2}\right) A e^{\mu z} + \rho z B(1 - \delta) e^{\rho z} + C e^{\rho z} \tag{35}$$

Furthermore, it is not difficult to show that

$$i\rho \bar{U}_\xi = \left(\frac{-\rho^2}{\mu^2 - \rho^2}\right) A e^{\mu z} + B\left[\left(\frac{2G}{\lambda + G}\right)\delta - (1 + \rho z)(1 - \delta)\right] e^{\rho z} - C e^{\rho z} \tag{36}$$

$$\frac{\bar{S}_{zz}}{2G} = \left(\frac{\rho^2}{\mu^2 - \rho^2}\right) A e^{\mu z} + B\left[\left(\frac{\lambda}{\lambda + G}\right)\delta - 1 + (1 - \delta)(1 + \rho z)\right] e^{\rho z} + C e^{\rho z} \tag{37}$$

$$\frac{i\bar{S}_{\xi z}}{2G} = \left(\frac{-\rho\mu}{\mu^2 - \rho^2}\right) A e^{\mu z} + B\left[\left(\frac{G}{\lambda + G}\right)\delta - (1 - \delta)(1 + \rho z)\right] e^{\rho z} - C e^{\rho z} \tag{38}$$

SOLUTION FOR A SINK IN A HALF-SPACE

The solution to this problem can be synthesized by superimposing the solutions found in the previous sections. To do this it is convenient to introduce the following change of notation:

$$\begin{aligned} N &= S_{zz}/2G \\ T &= iS_{\xi z}/2G \\ U &= iU_\xi \\ W &= U_z \end{aligned} \tag{39}$$

Table I

Transform	0 Alt. 1	0 Alt. 2	1	2	3
$\bar{N}$	$-\rho^2 \bar{K}_b$	$-\rho^2(\bar{K}_b - \bar{K}_a)$	$\dfrac{\rho^2 e^{\mu z}}{\mu^2 - \rho^2}$	$e^{\rho z}$	$\left[\left(\dfrac{\lambda}{\lambda + G}\right)\delta - 1 + (1-\delta)(1+\rho z)\right]e^{\rho z}$
$\bar{T}$	$\rho \bar{L}_b$	$\rho(\bar{L}_b - \bar{L}_a)$	$-\dfrac{\rho \mu e^{\mu z}}{\mu^2 - \rho^2}$	$-e^{\rho z}$	$\left[\left(\dfrac{G}{\lambda + G}\right)\delta - (1-\delta)(1+\rho z)\right]e^{\rho z}$
$\bar{P}$	$(\lambda + 2G)\bar{H}_b$	$(\lambda + 2G)(\bar{H}_b - \bar{H}_a)$	$(\lambda + 2G)e^{\mu z}$	0	$2Ge^{\rho z}$
$\bar{U}$	$\rho \bar{K}_b$	$\rho(\bar{K}_b - \bar{K}_a)$	$-\dfrac{\rho e^{\mu z}}{\mu^2 - \rho^2}$	$-\dfrac{e^{\rho z}}{\rho}$	$\dfrac{1}{\rho}\left[\left(\dfrac{2G}{\lambda + G}\right)\delta - (1-\delta)(1+\rho z)\right]e^{\rho z}$
$\bar{W}$	$-\bar{L}_b$	$-(\bar{L}_b - \bar{L}_a)$	$\dfrac{\mu e^{\mu z}}{\mu^2 - \rho^2}$	$\dfrac{e^{\rho z}}{\rho}$	$z(1-\delta)e^{\rho z}$

where

$$\bar{H}_b = \frac{e^{-\mu Z_b}}{2\mu} \qquad \bar{H}_a = \frac{e^{-\mu Z_a}}{2\mu}$$

$$\bar{K}_b = \frac{1}{2(\mu^2 - \rho^2)}\left[\frac{e^{-\rho Z_b}}{\rho} - \frac{e^{-\mu Z_b}}{\mu}\right] \qquad \bar{K}_a = \frac{1}{2(\mu^2 - \rho^2)}\left[\frac{e^{-\rho Z_a}}{\rho} - \frac{e^{-\mu Z_a}}{\mu}\right]$$

$$\bar{L}_b = \frac{\operatorname{sgn}(-h-z)}{2(\mu^2 - \rho^2)}[e^{-\rho Z_b} - e^{-\mu Z_b}] \qquad \bar{L}_a = \frac{\operatorname{sgn}(h-z)}{2(\mu^2 - \rho^2)}[e^{-\rho Z_a} - e^{-\mu Z_a}]$$

$$Z_b = |z + h| \qquad z_a = |z - h|$$

The complete solution for the Laplace transforms of the double Fourier transforms can then be written in the form

$$\begin{bmatrix} \bar{N} \\ \bar{T} \\ \bar{P} \\ \bar{U} \\ \bar{W} \end{bmatrix} = -(\bar{Q}/c_v)\begin{bmatrix} \bar{N}_0 \\ \bar{T}_0 \\ \bar{P}_0 \\ \bar{U}_0 \\ \bar{W}_0 \end{bmatrix} + \begin{bmatrix} \bar{N}_1 & \bar{N}_2 & \bar{N}_3 \\ \bar{T}_1 & \bar{T}_2 & \bar{T}_3 \\ \bar{P}_1 & \bar{P}_2 & \bar{P}_3 \\ \bar{U}_1 & \bar{U}_2 & \bar{U}_3 \\ \bar{W}_1 & \bar{W}_2 & \bar{W}_3 \end{bmatrix}\begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix} \tag{40}$$

where the functions $\bar{N}_0, \bar{T}_0, \ldots, \bar{W}_3$ are specified in Table I. The coefficients $\bar{F}_1, \bar{F}_2, \bar{F}_3$ may be obtained from the boundary conditions, i.e. zero tractions and pore pressure at the surface of the half-space, $z = 0$. Thus we have

$$\begin{bmatrix} \bar{N}_1 & \bar{N}_2 & \bar{N}_3 \\ \bar{T}_1 & \bar{T}_2 & \bar{T}_3 \\ \bar{P}_1 & \bar{P}_2 & \bar{P}_3 \end{bmatrix}\begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \end{bmatrix} = +(\bar{Q}/c_v)\begin{bmatrix} \bar{N}_0 \\ \bar{T}_0 \\ \bar{P}_0 \end{bmatrix} \tag{41}$$

where all of the coefficients in the above equation are evaluated at $z = 0$. Once the unknown coefficients $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_3$ have been found as the solution to equations (41), any of the transforms of the field quantities may be evaluated from equations (40). These solutions should be precisely the same independent of which alternative,† specified in Table I, is used.

CALCULATION OF FIELD QUANTITIES

Expressions for $\bar{N}$, $\bar{T}$, $\bar{P}$, $\bar{U}$, $\bar{W}$ were developed in the previous section. It will be observed for a point source that these are all functions of ρ. Thus we see from equation (18) that

$$(\bar{\sigma}_{zz}, \bar{p}, \bar{u}_z) = 2\pi \int_0^\infty \rho(\bar{N}, \bar{P}, \bar{W}) J_0(\rho r)\,\mathrm{d}\rho \tag{42}$$

Now we can easily establish that $\bar{U}_\eta = 0$ and thus

$$\bar{U}_x = \cos\varepsilon \bar{U}_\xi$$
$$\bar{U}_y = \sin\varepsilon \bar{U}_\xi$$

Thus the expressions for the Laplace transforms of displacement can be written as

$$\bar{u}_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{i(\alpha x + \beta y)} \cos\varepsilon \bar{U}_\xi(\rho)\,\mathrm{d}\alpha\,\mathrm{d}\beta$$

$$\bar{u}_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{i(\alpha x + \beta y)} \sin\varepsilon \bar{U}_\xi(\rho)\,\mathrm{d}\alpha\,\mathrm{d}\beta$$

and hence

$$\begin{aligned} u_r &= \int_0^\infty \int_0^{2\pi} \mathrm{e}^{i\rho r\cos(\theta-\varepsilon)} \cos(\theta - \varepsilon) U_\xi(\rho)\rho\,\mathrm{d}\varepsilon\,\mathrm{d}\rho \\ &= 2\pi \int_0^\infty \rho J_1(\rho r) U(\rho)\,\mathrm{d}\rho \end{aligned} \tag{43}$$

†Alternative 1 corresponds to a single sink at $z = -h$ in an unbounded medium, while alternative 2 corresponds to a single sink and an image source placed at $z = +h$ in an unbounded medium.

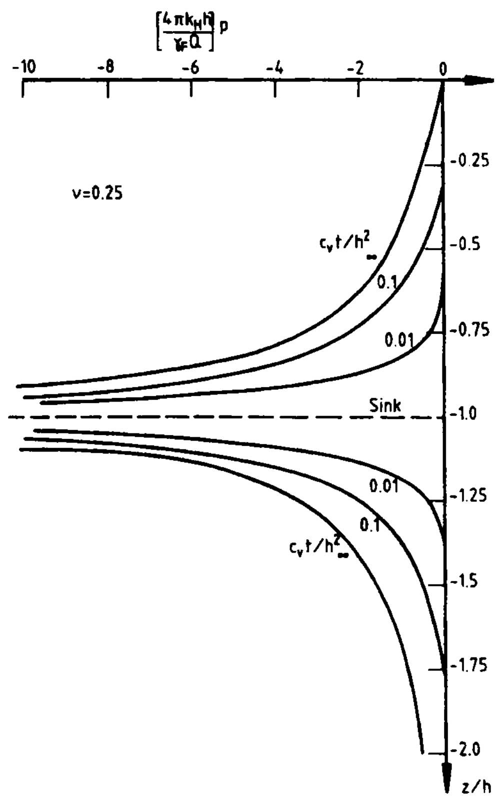

Figure 2. Isochrones of excess pore pressure on the vertical axis containing the sink

It is not difficult to show that $\bar{u}_\theta = 0$. Similarly we may show for the stresses that

$$\bar{\sigma}_{rz} = 2\pi \int_0^\infty \rho J_1(\rho r)\bar{T}(\rho)\mathrm{d}\rho \tag{44}$$

$$\sigma_{\theta z} = 0$$

The single infinite integrals contained in equations (42–44) have been evaluated numerically, using Gaussian quadrature.

Evaluation of the field quantities is finally achieved by inversion of the appropriate Laplace transforms. As mentioned earlier, this is also done numerically, using the algorithm developed by Talbot.[9]

RESULTS

The solution has been evaluated for the particular case where the soil skeleton has a Poisson's ratio $\nu = 0{\cdot}25$ and the results have been summarized in Figures 2–4.

Figure 2 shows the isochrones of excess pore pressure as a function of depth below the surface on a vertical axis passing through the point sink. The isochrones have been plotted for values of the non-dimensional time $c_v t/h^2 = 0{\cdot}01, 0{\cdot}1, \infty$, where the symbol t is used to represent time elapsed since the pumping operation commenced and h indicates the depth of embedment. The curve corresponding to $c_v t/h^2 = \infty$ indicates the steady-state response. At all times greater than zero, the excess pore pressures are negative (suction) because of the pumping operation. The steady-state pore pressure response is virtually achieved at the time given by $c_v t/h^2 = 1$.

It is interesting to note that when the values of excess pore pressure are normalized as indicated in Figure 2 (i.e. using the horizontal permeabiltiy k_H), the pore pressure isochrones along the vertical axis are practically independent of the degree of anisotropy of the permeability. In fact, at the steady-state condition ($t \to \infty$) these normalized values are precisely independent of k_H/k_V, while at intermediate times the differences cannot be plotted at the scale shown in Figure 2.

The steady-state solution has been found previously,[10] but it is perhaps worth repeating it here. The excess pore pressure distribution at large time is given by

$$p = -\left(\frac{Q\gamma_F}{4\pi k_V \Psi}\right)\left[\frac{1}{\sqrt{[r^2 + \Psi^2(z+h)^2]}} - \frac{1}{\sqrt{[r^2 + \Psi^2(z-h)^2]}}\right] \tag{45}$$

where

$$\psi^2 = c_H/c_V = k_H/k_V$$

Along the axis $r = 0$, this of course reduces to

$$p = -\left(\frac{Q\gamma_F}{4\pi k_H}\right)\left[\frac{1}{|z+h|} - \frac{1}{|z-h|}\right] \tag{46}$$

where the dependence on k_H alone is clearly seen.

Away from the vertical axis the excess pore pressures are more highly dependent upon the anisotropy of the soil. This fact is illustrated in Figure 3 where the steady state ($t \to \infty$) distributions of excess pore pressure on the horizontal plane $z = -0{\cdot}5h$ have been plotted for the cases corresponding to $c_H/c_V(=k_H/k_V) = 1, 2, 10$. In Figure 3 the excess pore pressures have been normalized using k_V and this allows a comparison of a number of different soils, each having the same vertical permeability but different horizontal permeabilities. Clearly, larger excess pore pressures are predicted for the isotropic case ($c_H/c_V = 1$), but the distributions of pore pressure tend to be more uniform across the plane as the horizontal permeability is increased. It is reasonable, therefore, to expect that surface settlements induced by the pumping operation might be more uniform in anisotropic soils with $k_H > k_V$, than those soils which have isotropic permeability.

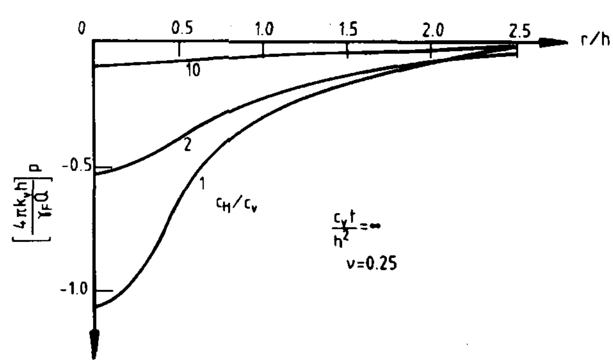

Figure 3. Excess pore pressure distribution on the plane $z = -0{\cdot}5h$ at steady state ($t \to \infty$)

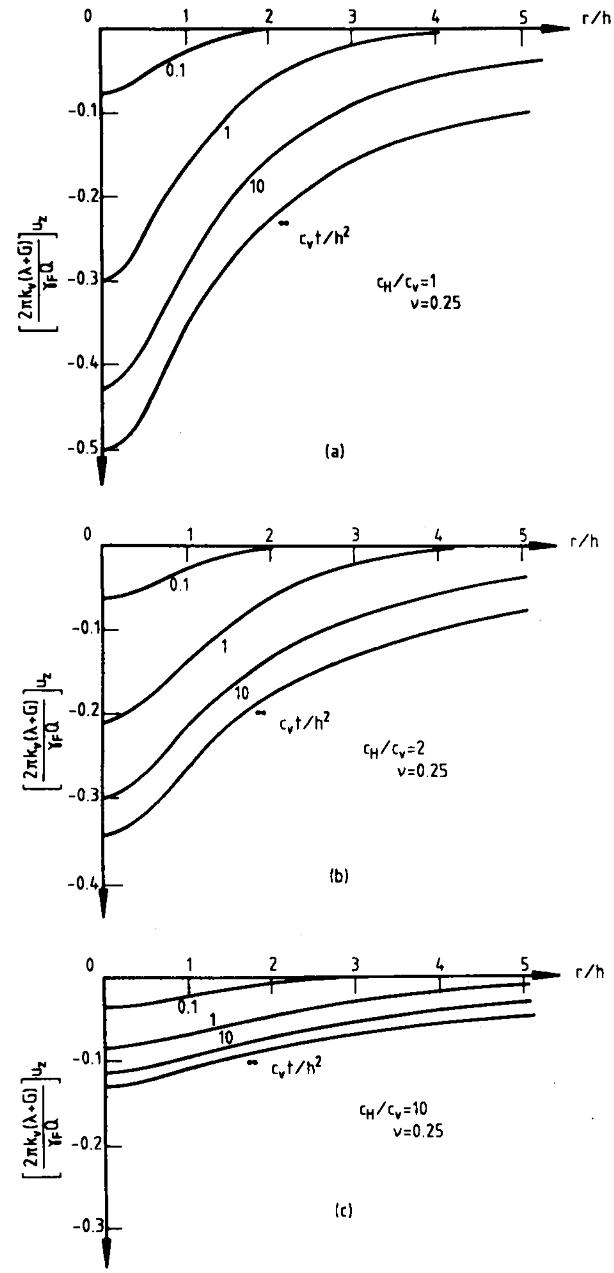

Figure 4. Isochrones of surface settlement

Typical results for displacement are indicated in Figure 4, where isochrones of surface settlement have been plotted against radial distance from the vertical axis. The settlement values have been plotted in non-dimensional form and selected cases corresponding to $c_H/c_V = 1, 2$ and 10 have been shown. In all cases the final surface profile is effectively reached by the time given by $c_V t/h^2 = 100$ and as anticipated the settlement troughs are deepest and steepest for the case of the isotropic soil.

Booker and Carter[10] report that for the general case the vertical displacement of points on the surface at large time is given by

$$u_z(r,o) = -\left(\frac{Q\gamma_F}{2\pi k_V(\lambda+G)(\Psi^2-1)}\right)\ln\left[\frac{\Psi h+\sqrt{(\Psi^2h^2+r^2)}}{h+\sqrt{(h^2+r^2)}}\right] \tag{47}$$

For the equivalent isotropic deposit equation (43) becomes

$$u_z(r,o) = -\left(\frac{Q\gamma_F}{4\pi k_V(\lambda+G)}\right)\frac{1}{\sqrt{(r^2+h^2)}} \tag{48}$$

CONCLUSIONS

A solution has been found for the consolidation of a saturated elastic half-space brought about by the commencement of pumping of the pore fluid from a sink embedded within the half-space. The medium was assumed to be homogeneous and isotropic with regard to deformation properties, but transversely isotropic with regard to flow of pore fluid. The governing equations of the problem have been solved in Laplace transform space requiring the use of double and triple Fourier transforms. Inversion of some of these transforms has been carried out using numerical integration.

Some particular solution have been evaluated for an elastic medium having a Poission's ratio $\nu = 0{\cdot}25$. These indicate that the major effects of the anisotropy are as follows:

1. At all comparable non-dimensional times the values of excess pore pressure down the vertical axis containing the sink are virtually independent of the vertical permeability k_V, but inversely proportional to the horizontal permeability k_H. (In this context the non-dimensional time factor c_V includes the term k_V.)
2. At all times the profiles of surface settlement are in the form of an axisymmetric trough. The deepest part of the trough is centred above the sink, and for different soils with the same value of k_V the trough is deepest and the sides steepest in the isotropic case. As the ratio k_H/k_V is increased, the profile of surface settlement becomes more uniform, i.e. the settlement trough becomes shallower and more gradual.

The solution presented may have application in practical problems such as dewatering operations in compressible soils and rocks. In this case the time to reach steady state and the final surface profile achieved could be of great interest. They may also be used as benchmarks for numerical studies and perhaps more importantly as Green's functions in the boundary element solution of more complicated consolidation problems.

APPENDIX

The aim of this Appendix is to verify the expressions for $\bar{H}, \bar{K}, \bar{L}$ contained in equations (28). We proceed as follows:

Let

$$\phi = \int_0^\infty e^{-i\gamma z}\frac{e^{-p|z|}}{2p}dz$$

where p has a positive real part. Then

$$\phi = \int_0^\infty \cos\gamma z\frac{e^{-p|z|}}{p}dz = \frac{1}{p^2+\gamma^2}$$

Thus using the Fourier inversion theorem

$$\frac{e^{-p|z|}}{2p}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\gamma z}}{\gamma^2+p^2}d\gamma$$

Also, let

$$\psi=-\int_{-\infty}^{\infty}e^{-i\gamma z}\frac{\operatorname{sgn}(z)}{2}e^{-p|z|}dz$$

$$=\int_{0}^{\infty}i\sin\gamma z e^{-p|z|}dz$$

$$=\frac{i\gamma}{p^2+\gamma^2}$$

Thus from the Fourier inversion theorem

$$-\frac{\operatorname{sgn}(z)}{2}e^{-p|z|}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i\gamma}{p^2+\gamma^2}$$

The results of equations (28) then follow.

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