MATHS AND STATS

Solving equations with a single exponent

Solving equations such as $x^3 = 125$, we need to know how to 'undo' something raised to a power. A power is 'undone' or the inverse of a power is to take the root. This is simpler if the power is odd but requires a little more consideration if the power is even!

Odd exponents

When n is odd, the inverse of raising something to the power of n, is raising it to the power of $\frac{1}{n}$, or taking the **nth root**.

To make sense of this, recall the index law $(a^b)^c = a^{bc}$ and observe that applying a power of $\frac{1}{n}$ will undo a power of n, that is, $(x^n)^{\frac{1}{n}} = x^{\frac{n}{n}} = x^1 = x$. Also, recall that powers and roots are related by $a^{\frac{1}{c}} = \sqrt[c]{a}$.

In other words, when n is odd:

$$\boldsymbol{(x^n)^{\frac{1}{n}} = x} \quad \text{or denoted another way} \quad \boldsymbol{\sqrt[n]{x^n} = x}$$

Note $8^{\frac{1}{3}} = 2$ and $(-8)^{\frac{1}{3}} = -2$ as

$2^3 = 2 \times 2 \times 2 = 8$ and $(-2)^3 = (-2) \times (-2) \times (-2) = -8$

Examples

1) Solve $x^5 = 600$ for x.

$$\begin{aligned} \sqrt[5]{x^5} &= \sqrt[5]{600} && \text{(take the fifth root of each side)} \\ x &= \sqrt[5]{600} \\ &\approx 3.5944 && \text{(4 decimal places)} \end{aligned}$$

2) Solve $10000 = 5000(1+i)^{11}$ for i.

$$\begin{aligned} \frac{10000}{5000} &= (1+i)^{11} && \text{(divide both sides by 5000)} \\ 2 &= (1+i)^{11} \\ 2^{\frac{1}{11}} &= (1+i)^{11\times\frac{1}{11}} && \text{(raise both sides to the power of an eleventh)} \\ 2^{\frac{1}{11}} &= 1+i \\ 2^{\frac{1}{11}} - 1 &= i && \text{(subtract 1 from both sides)} \end{aligned}$$

$$i \approx 0.065041 \qquad \text{(to 6 decimal places)}$$

It is important to **give an exact answer** (e.g. $2^{\frac{1}{11}} - 1$)
and
an **approximate answer** (e.g. $i \approx 0.065041$)

*** always use your exact answer in further calculations.

Even exponents

Be careful when the term you are raising to the power of n is negative (or could be negative).

Suppose we know $x^2 = 9$ and we want to know the value of x. Taking the square root of both sides gives us $x = 3$.
But there is another solution! x could also be -3, since $(-3)^2 = (-3) \times (-3) = 9$.

There are two solutions because the square of a negative number is a positive number. This is true for any even power, so whenever the power is even there could be two solutions! If you are only looking for one specific solution you might not be able to tell which one it is, or sometimes you will be able to ignore one based on the assumptions you have made.

For $\boldsymbol{n}$ even and $\boldsymbol{a}$ positive (note, an even root of a negative number is impossible)

$$\begin{aligned} \boldsymbol{x^n} &= \boldsymbol{a} \\ \sqrt[n]{\boldsymbol{x^n}} &= \pm\sqrt[n]{\boldsymbol{a}} && \text{taking the roots of both sides} \\ \boldsymbol{x} &= \pm\sqrt[n]{\boldsymbol{a}} \end{aligned}$$

Examples

1) Solve $x^2 = 100$ for x.

$$\begin{aligned} x &= \pm\sqrt[2]{100} && \text{(take the `plus or minus' square root)} \\ x &= \pm 10 \end{aligned}$$

Therefore, x can be -10 or 10.

2) Solve $10000 = 5000(1+i)^{12}$ for i, where i is an interest rate and is therefore positive.

$$\begin{aligned} \frac{10000}{5000} &= (1+i)^{12} && \text{(divide both sides by 5000)} \\ 2 &= (1+i)^{12} \\ \pm 2^{\frac{1}{12}} &= (1+i) && \text{(take the `plus or minus' twelfth root)} \\ \pm 2^{\frac{1}{12}} - 1 &= i && \text{(subtract 1 from both sides)} \end{aligned}$$

Since we know i should be positive we take the solution which is positive, that is

$$i = 2^{\frac{1}{12}} - 1 \approx 0.059463$$

Exercises

Solve the following equations for the unknown value

1) $x^2 = 256$
2) $w^6 = 3400$
3) $t^3 - 10 = 7990$
4) $30 + r^2 = 60$
5) $4z^3 = -4000$
6) $(4z)^3 = 4000$
7) $\frac{x^{12}}{2} = 848$
8) $\left(\frac{x}{2}\right)^{12} = 848$
9) $6000 = 5000(1 + x)^{24}$
10) $8000 = 6000(1 + i)^{36}$
11) $2000 = 500(1 + i)^{12}$
12) $7000 = 500(1 + i)^{360}$

Answers

1) $x = \pm\sqrt{256} = \pm 16$
2) $w = \pm\sqrt[6]{3400} \approx \pm 3.8776 \quad (4\text{ dp})$
3) $t = \sqrt[3]{8000} = 20$
4) $r = \pm\sqrt{30} \approx \pm 5.4772 \quad (4\text{ dp})$
5) $z = \sqrt[3]{-1000} = -10$
6) $z = \frac{\sqrt[3]{4000}}{4} \approx 3.9685 \quad (4\text{ dp})$
7) $x = \pm\sqrt[12]{1696} \approx \pm 1.8583 \quad (4\text{ dp})$
8) $x = \pm\sqrt[12]{848} \times 2 \approx \pm 3.5080 \quad (4\text{ dp})$
9) $x = \pm\sqrt[24]{\frac{6}{5}} - 1 \approx -2.0076 \text{ and } 0.0076 \quad (4\text{ dp})$
10) $i = \pm\sqrt[36]{\frac{4}{3}} - 1 \approx -2.0080 \text{ and } 0.0080 \quad (4\text{ dp})$
11) $i = \pm\sqrt[12]{4} - 1 \approx -2.1225 \text{ and } 0.1225 \quad (4\text{ dp})$
12) $i = \pm\sqrt[360]{14} - 1 \approx -2.0074 \text{ and } 0.0074 \quad (4\text{ dp})$