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ANALYSIS OF TUNNEL DISTORTION DUE TO AN OPEN EXCAVATION IN JOINTED ROCK

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ABSTRACT

General elastic solutions for tunnel deformations due to a nearby basement excavation are presented for jointed rock masses that contain two orthogonal joint sets. The method used to obtain these solutions involves a boundary element formulation for planar problems in anisotropic elasticity. The numerical solutions presented here allow predictions to be made of the changes in key tunnel dimensions, i.e. measures of the tunnel distortion, as the basement is excavated over a range of depths. Data that is typical for tunnels, basements and rocks in the central business district of Sydney have been used to illustrate the use of these solutions in geotechnical engineering.

INTRODUCTION

In recent times deep excavations have become a feature of urban construction and reconstruction. The excavation of a deep basement in the city environment may have a number of important influences on existing structures and services. An important effect is the deformations that may be induced in the neighbouring structures due to the removal of material from within the excavated hole.

Rock masses are usually composed of blocks of intact material separated by joints or discontinuities. The behaviour of these rock masses is complex as it is governed not only by the properties of the intact rock but also by the characteristics of the discontinuity planes. In attempting to model the mechanical behaviour at least two approaches are possible: either the joints are included explicitly in any mathematical model of the mass, in which case the discontinuous nature of the mass is addressed directly, or their effects are incorporated implicitly in the choice of constitutive relations used to represent the mass as an equivalent continuum. The latter approach has the major attraction that it is computationally more efficient than the explicit approach. Its validity will of course depend on the scale of the problem, e.g. how widely spaced the joints are compared to the size of any loaded region within the rock mass. For surface loading problems and some excavation problems, it has been shown (Refs 1 and 2) that the implicit approach is sufficiently accurate, even when the uniform spacing of the joints is of the same order as the dimension of the loaded area.

In this paper a continuum representation of the jointed rock mass is adopted for the analysis of excavation problems. The mechanical behaviour of the jointed rock mass is represented by a set of compliance relations for an anisotropic medium. This constitutive model, which allows for the presence of different planar joint sets, has been incorporated into a boundary element formulation to analyse a range of typical excavation problems, and to predict the effects that these excavations may have on neighbouring tunnels. Plane strain problems have been considered, and the Green's functions produced by Lekhnitskii (Refs 6 and 7), for the problem of line loading in a general anisotropic body have been employed in this boundary element formulation. Numerical solutions for the tunnel problem have been presented in the form of non-dimensional influence coefficients, and thus they have application to a wide range of practical problems. An example calculation has been included to illustrate the application of the theoretical solutions.

ANISOTROPIC ROCK MASS

It is common practice to assume that the joints which pervade a rock mass are planar features. The mechanical behaviour of a jointed rock mass may be described to sufficient accuracy by a set of anisotropic constitutive relations, whenever the joint planes have a regular spacing and orientation and whenever the sample size contains sufficient joints, i.e. when they are "ubiquitous". The development of relations that can be used to treat the jointed mass as a continuum is considered below. The case of a rock mass containing a single set of joints is dealt with first, followed by the general case of n joint sets.

Consider an ideal rock mass where the intact material is isotropic and elastic, with Young's modulus E and Possion's ratio ν. The blocks of intact material are separated by a single set of parallel discontinuities or joints. All joints in the set have the same orientation and mechanical properties and their spacing, S, is constant. The elastic behaviour of each joint in a set is characterised by a shear stiffness K_s and a normal stiffness K_n, i.e. the shear and normal modes of joint behaviour are assumed to be uncoupled. K_s is defined as the ratio of the shear stress applied along the joint plane to the resulting shear displacement, and K_n is

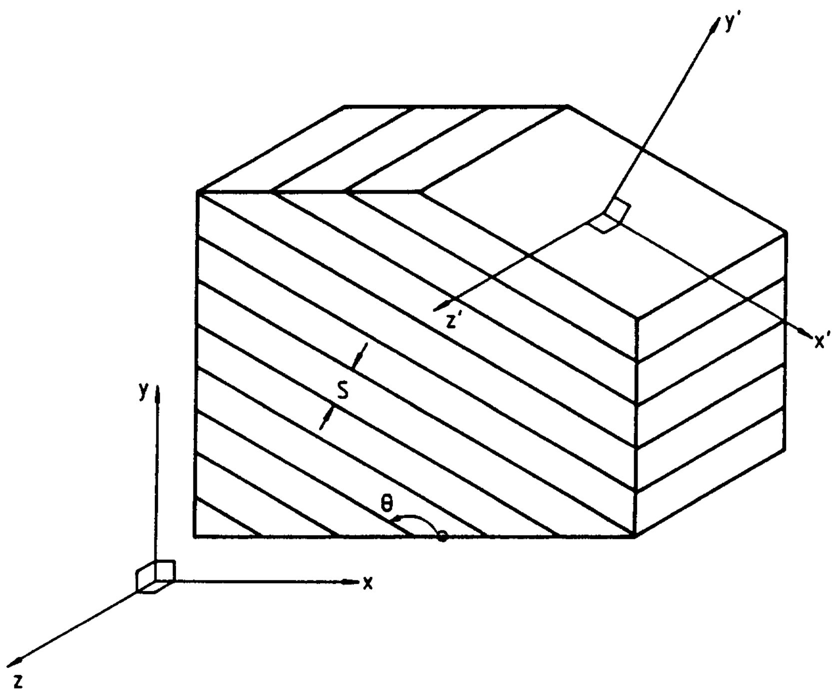

Figure 1 Set of Aligned Discontinuities

defined as the ratio of normal stress to normal displacement at the shear plane, i.e. both have units of stress divided by displacement. It can be shown (e.g. Ref. 4) that the overall behaviour of a rock mass with one set of joints is equivalent to that of a transversely anisotropic elastic continuum. The anisotropy will be more general than this if the rock mass contains more than one joint set. The equations governing this type of behaviour are set out below.

Consider the local coordinate system (x',y',z') attached to the joint plane so that the direction y' coincides with the normal to the joint, as shown in Figure 1. For plane strain conditions (no strain in the z' direction), the stress-strain relationships for the rock mass with a single joint set may be written as follows:

$$\epsilon' = C'\sigma' \tag{1}$$

in which ϵ' and σ' are vectors of the strain and stress components in the local coordinate system, i.e.

$$\epsilon' = (\epsilon_{x'}, \epsilon_{y'}, \gamma_{x'y'})^T \tag{2}$$

and

$$\sigma' = (\sigma_{x'}, \sigma_{y'}, \tau_{x'y'})^T \tag{3}$$

The compliance matrix for the jointed rock mass in the local coordinate system is given by:

$$C' = (1/E)\begin{bmatrix} 1-\nu^2 & -\nu(1+\nu) & 0 \\ 0 & 1-\nu^2+R_n & 0 \\ 0 & 0 & 2(1+\nu)+R_s \end{bmatrix} \tag{4}$$

where

$$R_s = E/(SK_s) \tag{5a}$$

$$R_n = E/(SK_n) \tag{5b}$$

R_s and R_n represent the relative shear and normal compliances of the joint set with respect to the surrounding intact rock.

It is also assumed that the global coordinate axis z is parallel to the local axis z'. Thus, a simple coordinate transformation allows the stress-strain relations to be written in terms of the global coordinate system (x,y,z), for which the compliance matrix becomes:

$$C = TC'T^T \tag{6}$$

T is a geometric transformation matrix for the joint set. It is assumed that the parallel joint planes make an angle θ with respect to the global x axis, i.e. θ is the anti-clockwise angle from the x axis to the x' axis, as indicated in Figure 1. The matrix T is given by:

$$T = \begin{bmatrix} \mathrm{Cos}^2\theta & \mathrm{Sin}^2\theta & -\tfrac{1}{2}\mathrm{Sin}2\theta \\ \mathrm{Sin}^2\theta & \mathrm{Cos}^2\theta & \tfrac{1}{2}\mathrm{Sin}2\theta \\ \mathrm{Sin}2\theta & -\mathrm{Sin}2\theta & \mathrm{Cos}2\theta \end{bmatrix} \tag{7}$$

The compliance relations may also be inverted to give:

$$\sigma = D\,\epsilon \tag{8}$$

where

$$D = C^{-1} \tag{9}$$

C and D are the global compliance and rigidity matrices describing the plane strain behaviour of an elastic rock mass with a single set of elastic joints.
Because the behaviour of the intact rock and the joints is assumed to be linear and elastic, the principle of superposition may be applied in cases where more than one joint set pervades the rock mass. For such cases it is easy to demonstrate that the constitutive behaviour can be described by the compliance matrix C_e, defined as:

$$C_e = C_r + \sum_{i=1}^{n} T_i C_i' T_i^T \tag{10}$$

where C_r and C_i' are the compliance matrices for the intact rock material and the joint set i respectively, i.e.

$$C_r = (1/E)\begin{bmatrix} 1-\nu^2 & -\nu(1+\nu) & 0 \\ 0 & 1-\nu^2 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{11}$$

and

$$C_i' = (1/E)\begin{bmatrix} 0 & 0 & 0 \\ 0 & R_{ni} & 0 \\ 0 & 0 & R_{si} \end{bmatrix} \tag{12}$$

in which

$$R_{si} = E/(S_i K_{si}) \tag{13a}$$

$$R_{ni} = E/(S_i K_{ni}) \tag{13b}$$

R_{si} and R_{ni} now represent the relative shear and normal compliances of joint set i with respect to the surrounding intact rock. Clearly, the introduction of these terms into the overall constitutive relations for the rock mass implies that its behaviour is anisotropic and more compliant than the intact rock alone.

If all parallel joint planes in set i make an anticlockwise angle θ_i with respect to the horizontal x axis, then the transformation matrix T_i is given by equation (7) with $\theta = \theta_i$.

BOUNDARY ELEMENT SOLUTION

The numerical solution of problems in elasticity by the direct boundary element method is based on an application of the reciprocal theorem, or alternatively, the use of a weighted residual approach. In the absence of bodyforces, the problem is governed by a well known integral equation, i.e.

$$\beta \mathrm{I} \mathbf{u}^i = \int_\Gamma \mathbf{U}^* \mathbf{t} \, d\Gamma - \int_\Gamma \mathbf{T}^* \mathbf{u} \, d\Gamma \tag{14}$$

in which Γ is the surface bounding the body of interest and I is the unit matrix. The vector $\mathbf{u}^i$ is used here to represent the displacements at the source point (the point of application of concentrated force), and $\mathbf{u}$ and $\mathbf{t}$ are vectors used to represent the displacements and tractions at the field points, or points of interest. In general, along the boundary Γ, some of the displacements and tractions will have specified values, and the remainder will be unknown and must be solved for during the analysis.

The matrices $\mathbf{U}^*$ and $\mathbf{T}^*$ appearing in equation (14) contain the fundamental solutions or Green's functions for the problem of interest. $\mathbf{U}^*$ contains the functions for displacement and $\mathbf{T}^*$ contains functions for the tractions. For the problem discussed in this paper, the appropriate Green's functions are the solutions to the Kelvin problem of a line load buried in an infinite, general anisotropic elastic body. These fundamental solutions for the displacement and stress components have been derived by Lekhnitskii (Refs 6 and 7) and for convenience they are provided in the Appendix. The components of $\mathbf{U}^*$ and $\mathbf{T}^*$ may easily be determined from the appropriate expressions in the Appendix.

The scalar β appearing in equation (14) takes a value which depends on the location of the source point. In particular for two-dimensional, plane strain problems it can be shown, even for the case of general anisotropy, that:

$\beta = 1$, when the source point is inside the domain bounded by Γ,
$\beta = 1/2$, when the source point is on the smooth boundary Γ, and
$\beta = 0$, when the source point is outside the domain bounded by Γ.

In the conventional boundary element method, problems in elasticity are solved approximately by discretising the bounding surface Γ and applying equation (14) at each element of the boundary. The influence of each element on itself and on all others on the boundary is considered. In this way a set of simultaneous equations is developed in the known and unknown boundary displacements and tractions. This numerical technique is well established and so further detail is not given here. Constant displacement and stress boundary elements were used to determine the solutions presented later in this paper.

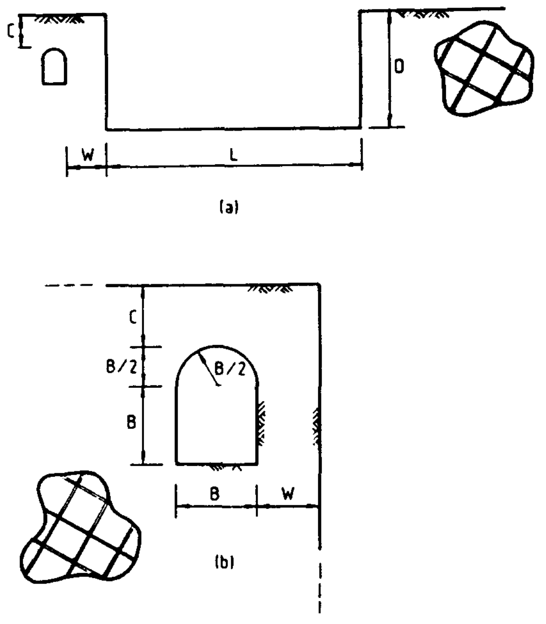

Figure 2

DEFINITION OF THE EXCAVATION PROBLEM

The problem considered here is an excavation which runs parallel to a straight section of a tunnel. The excavation is assumed to be long in the direction along the line of the tunnel, so that over most of its length conditions of plane strain may be asssumed. The sides of the excavation are vertical and of depth D. The final base level is horizontal and of width L. The tunnel is assumed to have a semi-circular arched roof with a diameter B. A section perpendicular to the line of the excavation is depicted in Figure 2a, while Figure 2b shows more detail of the tunnel and its key dimensions. Note that the symbol C represents the thickness of the rock mass above the tunnel (i.e. the cover thickness) and W is the width of the rock mass between the tunnel and the basement (i.e. the pillar width).

The rock mass is assumed to contain two orthogonal joint sets which strike parallel to the longitudinal axis, z. As described above, it is possible to define non-dimensional parameters to characterize the jointed rock mass, i.e. R_s and R_n. The special case of an orthotropic material may be defined by assuming that each joint set in the rock mass is characterized by:

$$R_s = R_n = R \tag{15}$$

where the value of R is the same for each joint set. For this case it can be shown (Ref. 2) that the response of the rock mass is not dependent on the orientation of the joint planes. In a more general class of problems the individual values of R_s and R_n may not be the same for both joint sets and for such cases the orientation of the joint planes is important.

GENERAL SOLUTIONS FOR THE TUNNEL PROBLEM

The authors have investigated the influence of geometrical parameters, such as width of excavation L, depth of cover C, and width of pillar W for the case of an excavation in an isotropic rock mass (Ref. 3). In particular, values of the ratios $L/B = 8$, $C/B = 1$ and $W/B = 0.5$ were investigated in detail as these are typical in Sydney. These values are also adopted in this study. A series of analyses, for a range of excavation depths, D (i.e. $D = 0$ to $5B$) and a range of values of the compliance ratio R (i.e. $R = 0, 1$ and 10) were performed. Also, a series of analyses for jointed rock masses containing horizontal and vertical joints with non-equal values of R_s and R_n was carried out.

Deformations in the rock mass caused by excavation depend on the original in situ stresses. In this problem the total vertical stress (the original overburden) is assumed to vary with depth according to:

$$\sigma_v = \gamma d \tag{16}$$

in which γ = the unit weight of the rock, and
d = the depth beneath the surface.

The horizontal total stress is assumed to vary as:

$$\sigma_h = K\sigma_v + q \tag{17}$$

in which K = the horizontal stress coefficient, and
q = the surface value of the horizontal stress.

The parametric study of the tunnel problem has been carried out using the boundary element technique with the constitutive model described above. Computations have been performed using fixed lateral boundaries and a fixed boundary underlying the rock layer. These boundaries were selected to be sufficiently remote from the tunnel and the excavated basement so as not to affect significantly the computed values of the tunnel distortions. Fixed remote boundaries were required because Green's functions for an infinite medium were used in the analysis of the problem with a free surface. The results of the parametric study are presented in non-dimensional form, so that they may be applied to a wide range of practical problems. In adopting this type of presentation, the principle of superposition has been employed and each displacement of interest can be expressed conveniently in the form:

$$\delta = (\gamma DB/E^*)I_\gamma + (K\gamma DB/E^*)I_K + (qB/E^*)I_q \tag{18}$$

where I_γ, I_K and I_q are non-dimensional influence coefficients for displacement, that represent the effects of the unit weight, γ, the lateral stress ratio, K, and the surface value of the horizontal stress, q, on the quantity of interest. The parameter E^* is defined as a representative modulus of the rock mass, i.e.

$$E^* = E/(1+R) \tag{19}$$

where E is the Young's modulus of the intact rock. For the particular case of the orthotropic rock mass, the value of R is given by equation (15). For more general cases in which $R_s \neq R_n$, the value of R used in equation (19) is the maximum of R_s and R_n. In computing these numerical solutions a value of 0.25 has been assumed for Poisson's ratio of the intact rock. As can be easily deduced from equation (19), $R = 0$ corresponds to an isotropic rock mass without joints.

Equal Stiffness Ratios

The results for special orthtropic materials, i.e. when $R_s = R_n = R$ in both joint sets, are presented in Figures 3 to 5. Figure 3 shows a plot of the influence factors I_γ, I_K and I_q for the change in tunnel width at the invert level, Figure 4 indicates the influence factors for the change in width at the springline, while Figure 5 shows influence factors for the change in tunnel height. As can be seen in the figures, the non-dimensional solutions are almost independent of the parameter R.

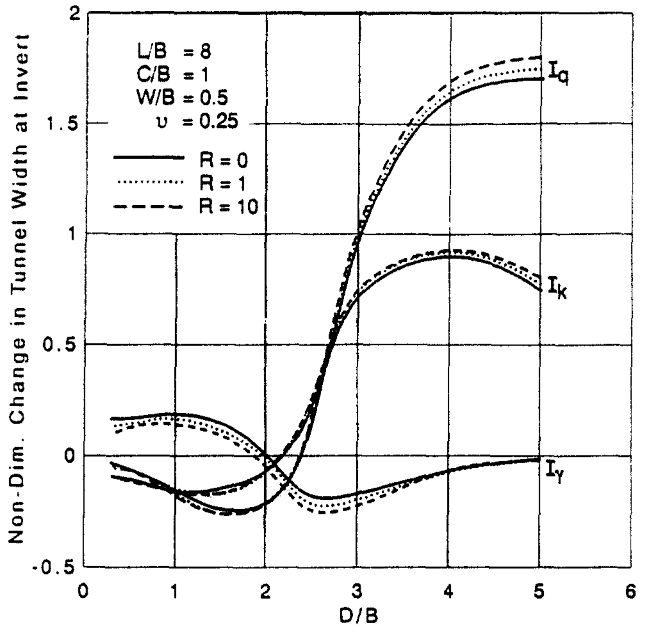

Figure 3

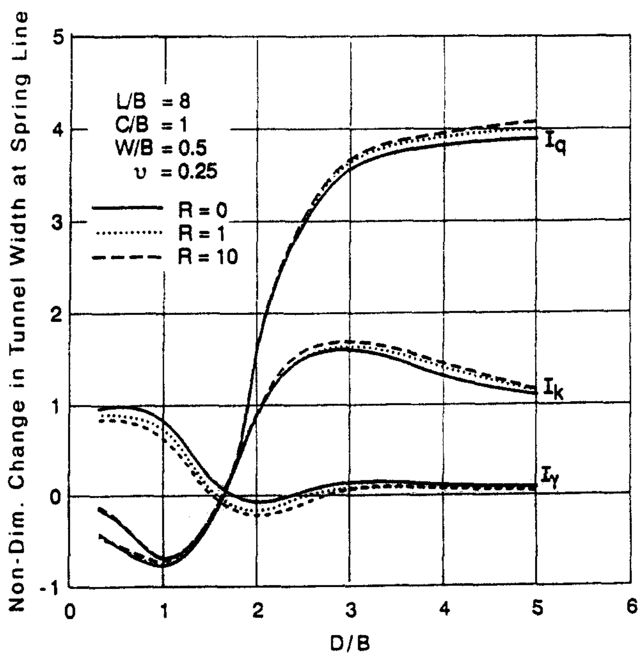

Figure 4

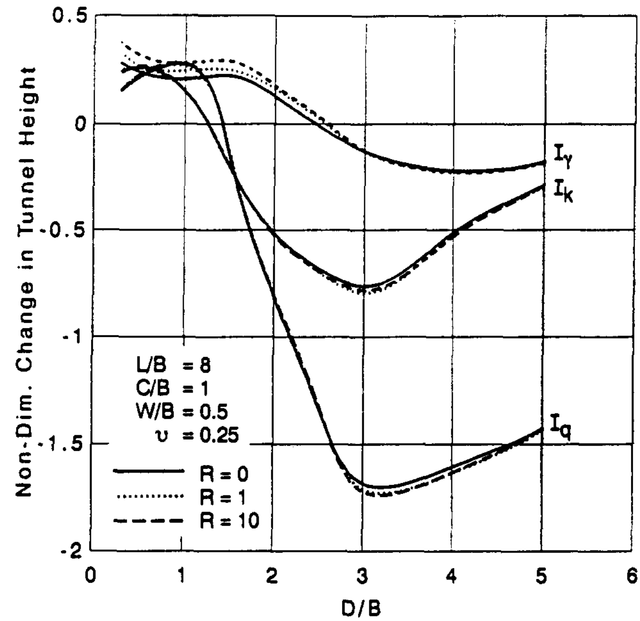

Figure 5

Unequal Stiffness Ratios

In this section the case of a rock mass with vertical and horizontal joints is considered. The mechanical behaviour of the mass is characterized by the parameters R_{sh} and R_{nh} for the horizontal joints and R_{sv} and R_{nv} for the vertical joints. Selected combinations of values for these four parameters, in the range 1 to 10, have been considered.

As before, the solutions are presented in the form of non-dimensional coefficients in order to be applicable to a wider range of boundary value problems of this kind. These solutions are presented in Figures 6 to 8. In Figure 6 the influence coefficients are presented for the change in tunnel width at the invert level. Similarly, influence coefficients for the change in width at the springline and the change in tunnel height at the centre line are presented in Figures 7 and 8. The curves in these figures are labelled 1 to 4, and the values of the compliance ratios corresponding to each curve are indicated in the figure captions.

As can be seen from the figures, and in contrast to the previous case, the

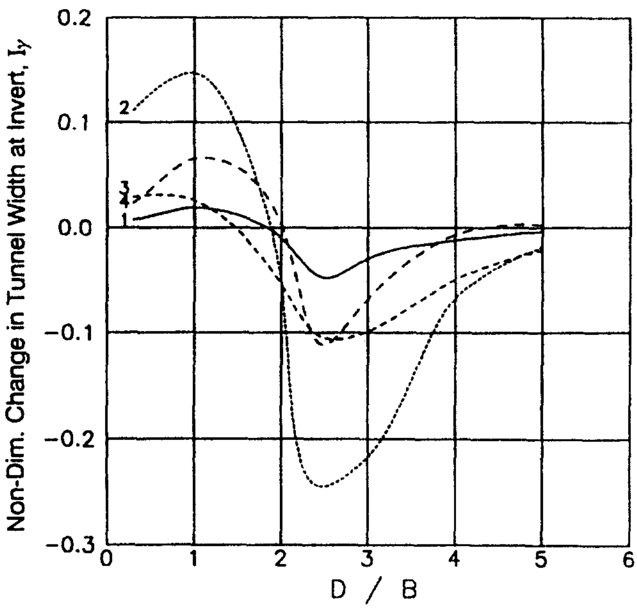

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Figure 6a

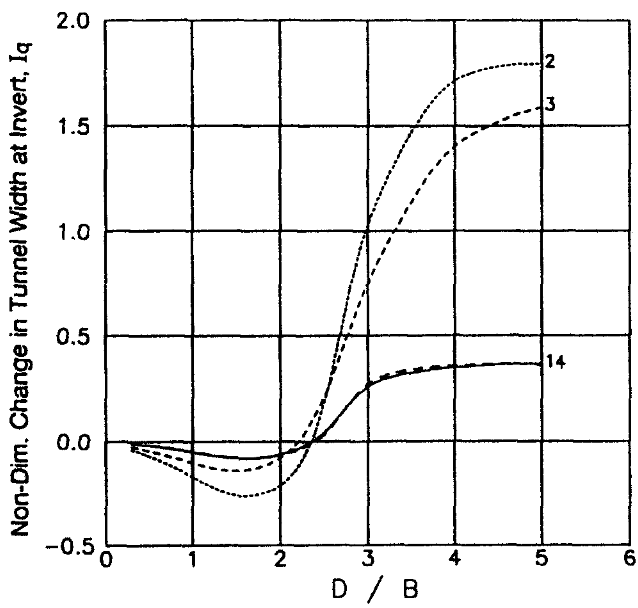

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Figure 6b

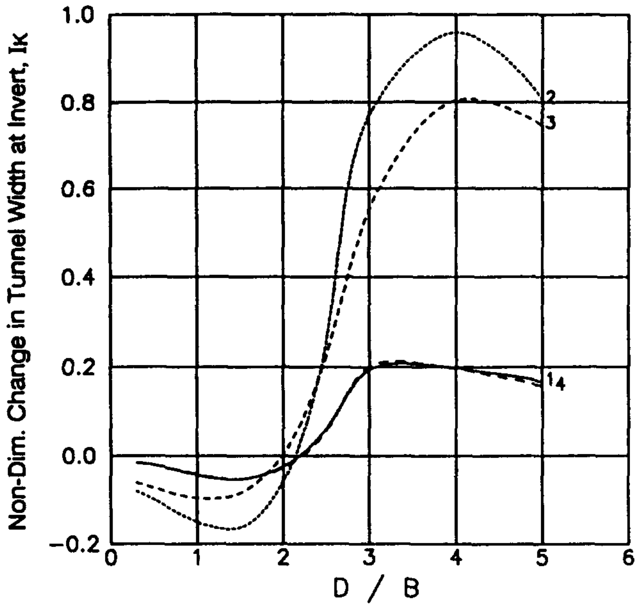

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Figure 6c

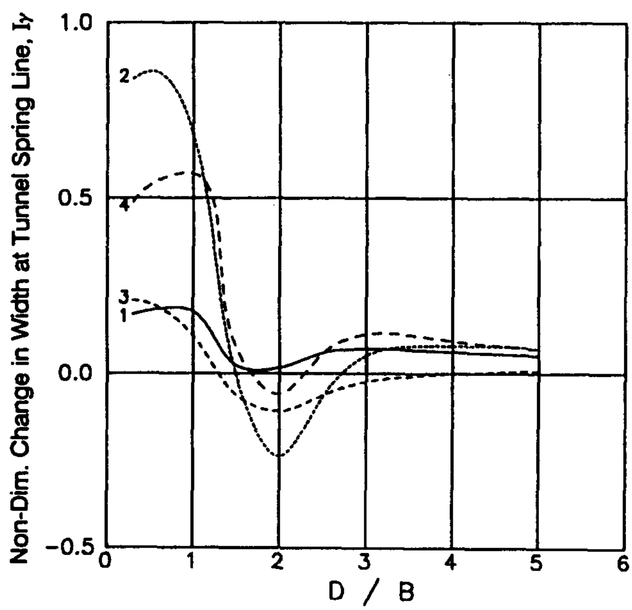

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Figure 7a

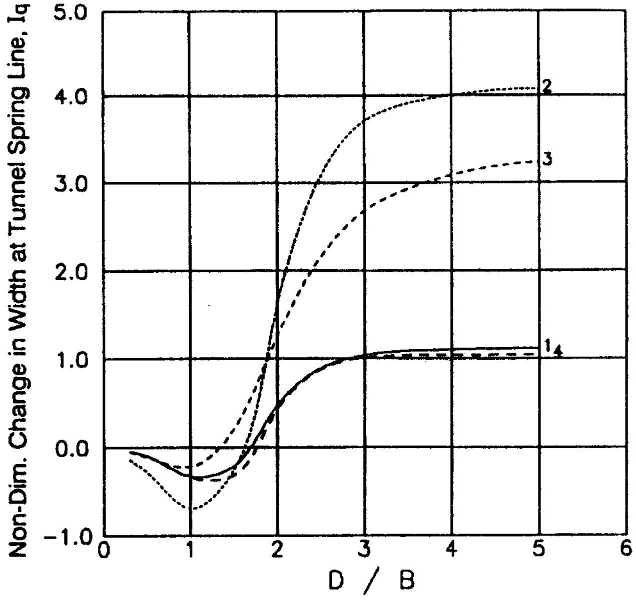

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Figure 7b

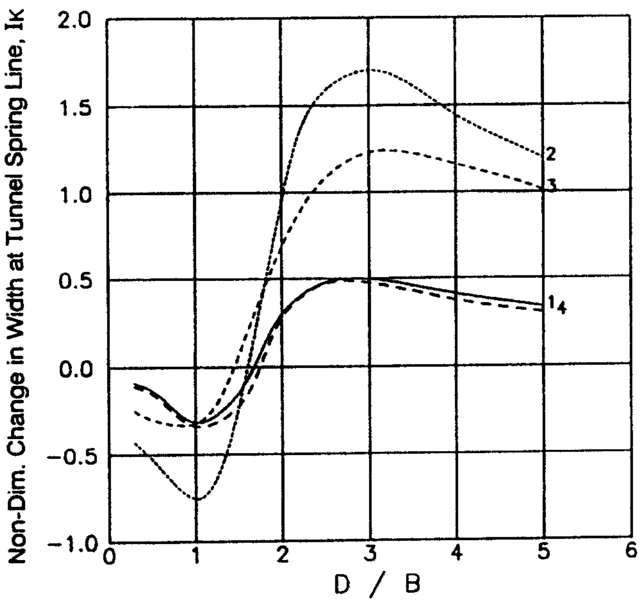

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Figure 7c

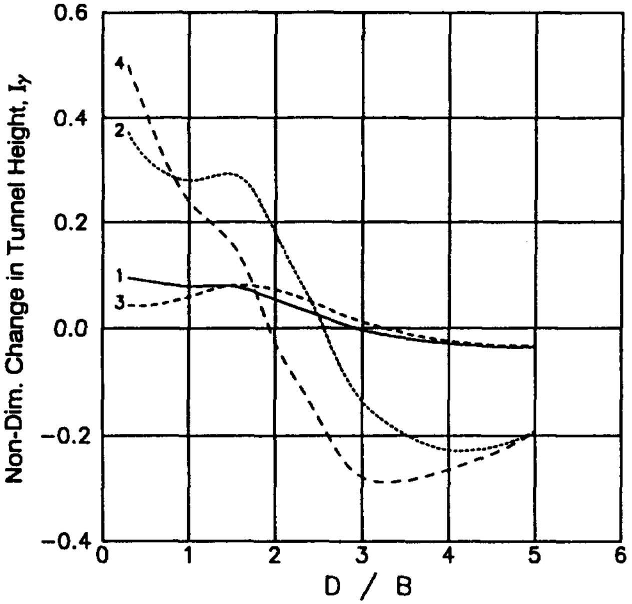

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Figure 8a

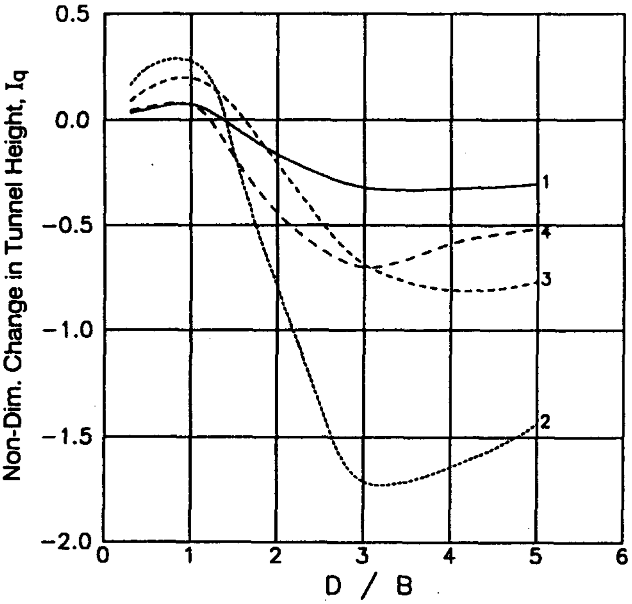

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Figure 8b

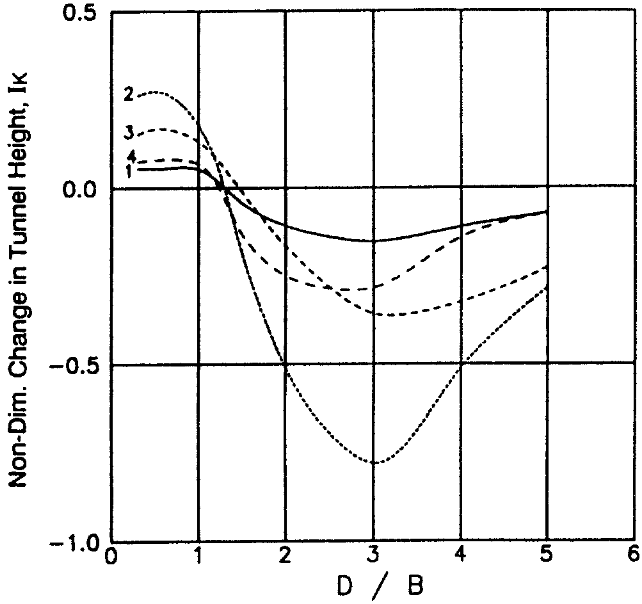

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Figure 8c

solutions now depend significantly on each of the joint compliance ratios. When using Figures 7 and 8 to determine tunnel deformations, it should be noted that the value assigned to R in equation (19) must be the maximum of R_{sh}, R_{sv}, R_{sh} and R_{nv} (i.e. R = 10 in the present case).

ILLUSTRATIVE EXAMPLE

As an application of the solutions, consider a tunnel of width B = 5 m and height H = 7.5 m, in an orthotropic jointed rock mass with equal values of R_s and R_n. An excavation of depth D = 25 m is constructed near this tunnel (W = 2.5 m) in the rock mass which has 2 orthogonal joint sets, both characterized by the following relative stiffnesses: R = R_s = R_n = 10. It is assumed that γ = 25 kN/m³, K = 2, q = 500 kPa, and for the intact rock blocks E = 2.5 GPa and ν = 0.25. The changes in the key dimensions of the tunnel due to this excavation can be estimated using the procedure outlined below.

For this case the representative modulus E^* is computed as:

$$E^* = E/(1+R) = 2.5/(1+10) = 0.227 \text{ GPa}.$$

and other dimensional quantities are computed as:

$$\gamma DB/E^* = 13.75 \text{ mm}^{-1}$$

$$K\gamma DB/E^* = 27.5 \text{ mm}^{-1}$$

$$qB/E^* = 11.0 \text{ mm}^{-1}$$

The influence coefficients corresponding to D/B = 5 are obtained from Figures 3 to 5. These values are listed in Table 1. Together with the dimensional

TABLE 1 INFLUENCE COEFFICIENTS

Location	I_γ	I_K	I_q
Invert level	-0.018	0.802	1.788
Springline	0.054	1.207	4.078
Tunnel height	-0.190	-0.288	-1.439

quantities calculated above, the values of these influence factors may be substituted in equation (18) to determine the deformations as follows:

(i) Change in tunnel width at the invert level -

$$\begin{aligned}\delta_1 &= 13.75 \times (-0.018) + 27.5 \times (0.802) + 11 \times (1.788) \\ &= 41.5 \text{ mm}\end{aligned}$$

(ii) Change in tunnel width at springline -

$$\begin{aligned}\delta_2 &= 13.75 \times (0.054) + 27.5 \times (1.207) + 11 \times (4.078) \\ &= 78.8 \text{ mm}\end{aligned}$$

(iii) Change in tunnel height at the centreline -

$$\begin{aligned}\delta_3 &= 13.75 \times (-0.190) + 27.5 \times (-0.288) + 11\times(-1.439) \\ &= -26.4 \text{ mm}\end{aligned}$$

In cases (i) and (ii) the tunnel width is enlarged. In case (iii) the tunnel height is reduced.

Calculations have also been carried out to obtain solutions for a similar problem in a rock mass without joints. The geometry of the problem is the same as before, however, in the absence of jointing the rock mass is now considered to be isotropic (i.e. $R = 0$). In Table 2 changes in the key tunnel dimensions for both the anisotropic and isotropic rock masses are compared. Note the significant influence of the jointing on the tunnel distortions.

TABLE 2 CHANGES IN KEY TUNNEL DIMENSIONS (mm)

Location	Rock Mass: Orthotropic $R_s = R_n = 10$	Rock Mass: Isotropic $R_s = R_n = 0$
Invert Width	41.5	3.6
Springline Width	78.8	6.8
Centreline Height	- 26.4	- 2.4

CONCLUSIONS

Predicting tunnel deformations due to a nearby excavation in a jointed rock mass is a complicated problem. At working stress levels, reasonable predictions may be obtained by using an elastic analysis, provided appropriate values of the elastic parameters are used to represent the behaviour of the jointed rock mass. A method of analysis capable of dealing with such problems has been proposed in this paper. The rock mass has been modelled as an anisotropic elastic continuum and the governing equations have been solved numerically using the direct boundary element method. Typical solutions for the excavation problem have been evaluated and these solutions have been presented conveniently in non-dimensional form as a series of charts. By considering a typical example problem it has been shown that including the effects of jointing in the calculations has a significant influence on the deformations of the tunnel located close to the excavation.

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APPENDIX - KELVIN SOLUTION FOR AN ANISOTROPIC BODY

Consider the problem of a line load embedded in a general anisotropic elastic continuum of infinite extent. The line load runs parallel to the global z coordinate direction and has, for the general case, non-zero components in the x and y coordinate directions - see Figure A1.

For plane strain conditions (zero strains in the z direction), the constitutive behaviour of the anisotropic continuum is described by a set of compliance relations of the form:

$$\epsilon = C \sigma \tag{A1}$$

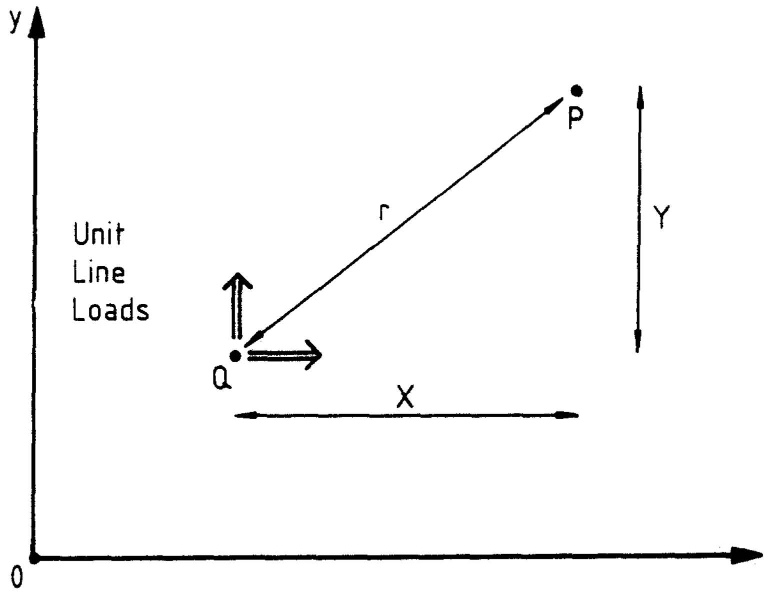

Figure A1

where $\epsilon^T = (\epsilon_x, \epsilon_y, \gamma_{xy})$

$\sigma^T = (\sigma_x, \sigma_y, \tau_{xy})$

The vectors ϵ and σ are the planar strain and stress components, respectively, in the global coordinate system.

The compliance matrix C is defined by:

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \qquad \text{(A2)}$$

in which $c_{12} = c_{21}$, $c_{13} = c_{31}$, $c_{23} = c_{32}$.

This Kelvin problem for the anisotropic material was investigated by both Lekhnitskii (Refs 6 and 7) and Green (Ref. 5) and involves the prediction of the displacements and stresses at a distance r from the line load, where

$$r = \sqrt{(X^2 + Y^2)} \tag{A3}$$

i.e. X and Y are the Cartesian components of the distance between the source point at Q and the field point at P, as depicted in Figure A1. Following Lekhnitskii, the solutions for the displacements (u_x^*, u_y^*) and stress components (σ_x^*, σ_y^*, τ_{xy}^*) at P due to a unit line load acting in the x or in the y coordinate direction at Q may be written in the form:

$$\begin{bmatrix} u_x^* \\ u_y^* \end{bmatrix} = 2\,\mathrm{Real}\left\{ \begin{bmatrix} U_{11} & U_{12} \\ U_{11} & U_{12} \end{bmatrix} \begin{bmatrix} \ln z_1 \\ \ln z_2 \end{bmatrix} \right\} \tag{A4}$$

and

$$\begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \tau_{xy}^* \end{bmatrix} = 2\,\mathrm{Real}\left\{ \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \\ S_{31} & S_{32} \end{bmatrix} \begin{bmatrix} 1/z_1 \\ 1/z_2 \end{bmatrix} \right\} \tag{A5}$$

where "Real" indicates the real parts of the components of the matrix product in brackets. The parameters z_1 and z_2 that appear in the above equations are defined by:

$$z_1 = X + \rho_1 Y \tag{A6}$$

$$z_2 = X + \rho_2 Y \tag{A7}$$

where the complex numbers ρ_1 and ρ_2 and their complex conjugates are the roots of the equation:

$$c_{11}\rho^4 - 2c_{13}\rho^3 + (2c_{12}+c_{33})\rho^2 - 2c_{23}\rho + c_{22} = 0 \tag{A8}$$

It should be noted that for the general problem of anisotropy, it is assumed that no pairs of the roots of equation (A8) are equal. The case of equal roots corresponds to a special class of problems in elasticity which is not dealt with here.

The complex coefficients U_{11}, U_{12}, etc., and S_{11}, S_{12}, etc., that appear in equations (A4) and (A5), are defined as follows.

For a unit line load acting in the x direction:

$$U_{11} = a_1 \; A_1 \tag{A9a}$$
$$U_{12} = b_1 \; B_1 \tag{A9b}$$
$$U_{21} = a_2 \; A_1 \tag{A9c}$$
$$U_{22} = b_2 \; B_1 \tag{A9d}$$

and

$$S_{11} = \rho_1{}^2 A_1 \tag{A10a}$$
$$S_{12} = \rho_2{}^2 B_1 \tag{A10b}$$
$$S_{21} = A_1 \tag{A10c}$$
$$S_{22} = B_1 \tag{A10d}$$
$$S_{31} = -\rho_1 \; A_1 \tag{A10e}$$
$$S_{32} = -\rho_2 \; B_1 \tag{A10f}$$

For a unit line load acting in the y direction:

$$U_{11} = a_1 \; A_2 \tag{A11a}$$
$$U_{12} = b_1 \; B_2 \tag{A11b}$$
$$U_{21} = a_2 \; A_2 \tag{A11c}$$
$$U_{22} = b_2 \; B_2 \tag{A11d}$$

and

$$S_{11} = \rho_1{}^2 A_2 \tag{A12a}$$
$$S_{12} = \rho_2{}^2 B_2 \tag{A12b}$$
$$S_{21} = A_2 \tag{A12c}$$
$$S_{22} = B_2 \tag{A12d}$$
$$S_{31} = -\rho_1 \; A_2 \tag{A12e}$$
$$S_{32} = -\rho_2 \; B_2 \tag{A12f}$$

The parameters a_1, a_2, b_1, b_2 are given by:

$$a_1 = c_{11}\rho_1{}^2 + c_{12} - c_{13}\rho_1 \tag{A13}$$
$$b_1 = c_{11}\rho_2{}^2 + c_{12} - c_{13}\rho_2 \tag{A14}$$
$$a_2 = c_{12}\rho_1 + c_{22}/\rho_1 - c_{23} \tag{A15}$$
$$b_2 = c_{12}\rho_2 + c_{22}/\rho_2 - c_{23} \tag{A16}$$

The complex coefficients A_1, A_2, B_1, B_2 and their complex conjugates are obtained from the solution of the following two sets of simultaneous equations.

For unit loading in the x direction:

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ \rho_1 & \rho_2 & -\bar{\rho}_1 & -\bar{\rho}_2 \\ \rho_1{}^2 & \rho_2{}^2 & -\bar{\rho}_1{}^2 & -\bar{\rho}_2{}^2 \\ 1/\rho_1 & 1/\rho_2 & -1/\bar{\rho}_1 & -1/\bar{\rho}_2 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ \bar{A}_1 \\ \bar{B}_1 \end{bmatrix} = \left(\frac{1}{2\pi i}\right) \begin{bmatrix} 0 \\ -1 \\ -c_{13}/c_{11} \\ c_{12}/c_{22} \end{bmatrix} \tag{A17}$$

For unit loading in the y direction:

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ \rho_1 & \rho_2 & -\bar{\rho}_1 & -\bar{\rho}_2 \\ \rho_1{}^2 & \rho_2{}^2 & -\bar{\rho}_1{}^2 & -\bar{\rho}_2{}^2 \\ 1/\rho_1 & 1/\rho_2 & -1/\bar{\rho}_1 & -1/\bar{\rho}_2 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ \bar{A}_2 \\ \bar{B}_2 \end{bmatrix} = \left(\frac{1}{2\pi i}\right) \begin{bmatrix} 1 \\ 0 \\ -c_{12}/c_{11} \\ c_{23}/c_{22} \end{bmatrix} \tag{A18}$$

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