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Cavity expansion in cohesive frictional soils

J. P. CARTER,* J. R. BOOKER* and S. K. YEUNG*

Closed form solutions are presented for the expansion of cylindrical and spherical cavities in an ideal, cohesive frictional soil. An explicit solution for the pressure–expansion relationship can be obtained for infinitesimal (small strain) deformations. For finite deformations it is necessary to adopt a numerical approach to obtain the complete pressure–expansion relationship and it is found that the cavity pressure approaches a limiting value for infinite deformation. It is, perhaps surprisingly, possible to determine the precise value of this limiting pressure analytically. It is suggested that the small strain solution for a cylindrical cavity is applicable to the interpretation of pressuremeter tests in sand, and that the solutions for limit pressures have application to the problem of pile installation and the end bearing pressure of deep foundations.

L'article présente des solutions de forme fermée pour l'expansion de cavités cylindriques et sphériques dans un sol idéal cohérent à frottement. On peut obtenir une solution explicite pour le rapport entre la pression et l'expansion dans le cas des déformations infinitésimales (contraintes faibles). Pour les déformations finies il faut adopter une méthode numérique pour obtenir le rapport complet entre la pression et l'expansion et on trouve que la pression de cavité atteint une valeur limite à peu près pour la déformation infinie. Il est peut-être surprenant que la valeur précise de cette pression limite puisse être déterminée de façon analytique. On en tire la conclusion que la solution à faible contrainte pour une cavité cylindrique peut s'appliquer à l'interprétation des essais pressiométriques dans le sable et que les solutions pour les pressions limites trouvent une application au problème de l'installation des pieux et à celui de la résistance limite des fondations profondes.

KEYWORDS: elasticity; friction; plasticity; stress analysis.

INTRODUCTION

In geotechnical engineering an analytical problem of great interest concerns the expansion of a cavity in a soil or rock mass.

The analysis of a cylindrical cavity has been applied to practical problems such as the interpretation of pressuremeter tests (e.g. Gibson & Anderson, 1961; Palmer, 1972; Hughes, Wroth & Windle, 1977) and the installation of driven piles (e.g. Randolph, Carter & Wroth, 1979). In some cases solutions have been found in closed form (e.g. Gibson & Anderson, 1961) while others have required the use of numerical techniques (e.g. Randolph *et al.*, 1979).

For the spherical cavity problem some solutions have been found for cavities in various types of media. In an important contribution Chadwick (1959) presented a derivation of the pressure–expansion relationship for an elastic–perfectly plastic material. His analysis assumed that yielding would occur according to the Mohr–Coulomb criterion and that the soil would flow plastically with an associated flow rule. His solution required the adoption of a natural (or logarithmic) strain definition to describe the large deformations which occur during the expansion. His paper also includes a derivation and discussion of limit pressures for the special case of a purely cohesive (zero plastic volume change) material. Vesic (1972) also presented approximate solutions for limit pressures for a spherical cavity expansion in a cohesive frictional material and applied these solutions to the determination of bearing capacity factors for deep foundations. Ladanyi (1967) investigated the problem of cavity expansion in brittle rocks and also applied his approximate solutions for limit pressures in the spherical case to the bearing capacity problem.

In this Paper closed form solutions are presented for the expansion of both cylindrical and spherical cavities in ideal, cohesive frictional soils. An explicit expression for the pressure–expansion relation can be found for small strain deformation.

The determination of the entire pressure–expansion curve including the large strain region requires the use of numerical techniques and this is not pursued here; however, such numerical studies show that for very large deformations the cavity pressure approaches a limiting value. It is shown that this limiting value need not be determined numerically and can, perhaps surprisingly, be found explicitly.

BASIC ASSUMPTIONS

The analyses of spherical and long cylindrical cavities are quite similar and can be treated in a single analysis by the introduction of a parameter k which takes the following values

$$k = 1 \text{ (cylindrical cavity)}$$

$$k = 2 \text{ (spherical cavity)}$$

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It will be assumed that the cavity is expanded in an infinite medium which is initially in a hydrostatic stress state, i.e. $\sigma_1 = \sigma_2 = \sigma_3 = p_0$, where σ_1, σ_2 and σ_3 are the principal stress components. (For a long vertical cylindrical cavity this restriction may be relaxed to the case where there is no variation in the normal stresses in a horizontal plane.) The analysis will thus be applicable to deep cavities where any variation in stress due to body force can be neglected. Hence conditions of axial symmetry and plane strain prevail for the expansion of the cylindrical cavity while conditions of spherical symmetry hold for the expansion of a spherical cavity. This greatly simplifies the analysis and allows a one-dimensional description of the problem because the displacements in the medium are everywhere radial. Since large deformations may occur the radial co-ordinate of a typical particle may change significantly during the course of the cavity expansion. The problem will involve both geometric and material non-linearities and so it is convenient to adopt a rate formulation.

Initially, at time $t = 0$, the cavity has a radius a_0 and an internal pressure p_0. At time t later the cavity has enlarged and has a current radius a, while the internal pressure has increased to p. A typical material point of the continuum now has a radial co-ordinate r, having moved to this position from its original location r_0. The total stress at this position must be in equilibrium with the current boundary tractions. In the absence of body forces this requirement can be expressed as

$$\frac{\partial \sigma_r}{\partial r} + k\,\frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1a}$$

$$\sigma_r = p \text{ at } r = a \tag{1b}$$

$$\sigma_r = p_0 \text{ at } r = \infty \tag{1c}$$

The behaviour of the cylindrical cavity is described in terms of cylindrical polar co-ordinates (r, θ, z) and the behaviour of the spherical cavity is described in terms of spherical polar co-ordinates (r, θ, ω). The expansion of the cylindrical cavity occurs under conditions of plane strain and, provided that σ_z remains the intermediate principal stress and there is no component of plastic strain in the z direction, the increase in σ_z can be calculated from

$$\Delta\sigma_z = \nu(\Delta\sigma_r + \Delta\sigma_\theta) \tag{2}$$

where ν is Poisson's ratio.

It also follows from symmetry that for the spherical cavity

$$\sigma_\omega = \sigma_\theta \tag{3}$$

and all shear components of stress in the chosen co-ordinate system are zero. Adopting the convention of compression positive means that during cavity expansion σ_r is the major and σ_θ the minor principal stress.

The constitutive equation for the material of the continuum may be written as a relationship between the rates of change of stress and strain

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}\dot{\boldsymbol{\varepsilon}} \tag{4}$$

where

$$\dot{\boldsymbol{\sigma}} = (\dot{\sigma}_r, \dot{\sigma}_\theta)^{\mathrm{T}}$$

$$\dot{\boldsymbol{\varepsilon}} = (\dot{\varepsilon}_r, k\dot{\varepsilon}_\theta)^{\mathrm{T}}$$

$$\dot{\varepsilon}_r = -\frac{\partial \dot{u}}{\partial r}$$

$$\dot{\varepsilon}_\theta = -\dot{u}/r$$

$$u = r - r_0$$

The symbol u has been used to denote the total radial displacement of a material point in the interval from 0 to t and the dot indicates differentiation with respect to time. Even though the displacements may be large, equation (4) is sufficiently general for the purposes here since the kinematic constraints do not permit a rotation of principal stress and strain directions. If a cavity is created in a saturated porous medium then the total stress rate in equation (4) should be replaced by the effective stress rate. For simplicity, attention here will be restricted to a single-phase (dry) soil medium. The coefficients of the matrix $\mathbf{D}$ in equation (4) depend on the type of material analysed and details are given in the next section.

CONSTITUTIVE RELATIONS FOR AN IDEAL FRICTIONAL MATERIAL

It is assumed that the continuum is an isotropic elastic, perfectly plastic solid. It behaves elastically and obeys Hooke's law until the onset of yield, which is determined by the Mohr–Coulomb criterion. In the first instance it is assumed for simplicity that the material is purely frictional and thus the yield condition takes the form

$$\sigma_1 = N\sigma_3 \tag{5}$$

where

$$N = \frac{1 + \sin\phi}{1 - \sin\phi}$$

and ϕ is the friction angle. As noted previously σ_r and σ_θ are the major and minor principal stresses during cavity expansion, so that equation (5) may also be written as

$$\sigma_r = N\sigma_\theta \tag{6}$$

It is also assumed that while yield is occurring the total strain is made up of an elastic and a plastic component, i.e.

$$\dot{\varepsilon} = \dot{\varepsilon}^{\mathrm{E}} + \dot{\varepsilon}^{\mathrm{P}} \tag{7}$$

and that the material dilates plastically at a constant rate. For the general case Davis (1969) postulated the flow rule

$$\frac{\dot{\varepsilon}_1{}^{\mathrm{P}}}{\dot{\varepsilon}_3{}^{\mathrm{P}}} = -\frac{1}{M} \quad \dot{\varepsilon}_2{}^{\mathrm{P}} = 0 \tag{8a}$$

where

$$M = \frac{1 + \sin\psi}{1 - \sin\psi}$$

ψ is the dilatancy angle and $\dot{\varepsilon}_1{}^{\mathrm{P}}$, $\dot{\varepsilon}_2{}^{\mathrm{P}}$ and $\dot{\varepsilon}_3{}^{\mathrm{P}}$ denote the major, intermediate and minor principal plastic strain rates respectively. For triaxial compression where the intermediate and minor principal plastic strain rates are equal equation (8a) becomes, because of symmetry

$$\frac{\dot{\varepsilon}_1{}^{\mathrm{P}}}{2\dot{\varepsilon}_3{}^{\mathrm{P}}} = -\frac{1}{M} \quad \dot{\varepsilon}_2{}^{\mathrm{P}} = \dot{\varepsilon}_3{}^{\mathrm{P}} \tag{8b}$$

Thus the expansion of both the cylindrical and the spherical cavities is covered by a single equation for the flow rule

$$\frac{\dot{\varepsilon}_r{}^{\mathrm{P}}}{\dot{\varepsilon}_\theta{}^{\mathrm{P}}} = -\frac{k}{M} \tag{8c}$$

Equation (8a) is the conventional definition of the angle of dilation for plane strain conditions and the difference between it and equation (8b) should be carefully noted. When these definitions of dilation angle are used to interpret the dilational behaviour of real materials, it is possible that different values of ψ could be assigned to the plane strain and triaxial cases.

It is shown in the appendix that the constitutive matrix **D** defined in equation (4) is given by

$$\mathbf{D} = \mathbf{D}_{\mathrm{E}} = \begin{bmatrix} \lambda + 2G & \lambda \\ \lambda & \lambda + 2G/k \end{bmatrix} \tag{9}$$

for purely elastic deformations and by

$$\mathbf{D} = \frac{2G}{\chi}\begin{bmatrix} 1 & 1/M \\ 1/N & 1/MN \end{bmatrix} \tag{10}$$

for deformations which involve plastic yielding. The quantities in equations (9) and (10) are defined by

$$M = \frac{1 + \sin\psi}{1 - \sin\psi}$$

$$N = \frac{1 + \sin\phi}{1 - \sin\phi}$$

$$\chi = \frac{k(1 - \nu) - k\nu(M + N) + [(k - 2)\nu + 1]MN}{[(k - 1)\nu + 1]MN}$$

where G and ν are the elastic shear modulus and Poisson's ratio for the ideal material and λ is the Lamé modulus, i.e.

$$\lambda = \frac{\nu}{1 - 2\nu}\, 2G$$

ANALYSIS

Consider the situation shown schematically in Fig. 1 where the cavity has a radius a, an internal pressure p and plastic yield is occurring throughout the region $a \leqslant r \leqslant R$. Beyond the elasto-plastic interface, i.e. for $r > R$, the material remains elastic.

If it is assumed that deformations in the elastic region are infinitesimal, then it is not difficult to show that there is no volume strain in this region and that

$$u = \varepsilon_R \left(\frac{R}{r}\right)^k R \tag{11}$$

where, following Hughes *et al.* (1977), $\varepsilon_R = (\sigma_R - p_0)/2Gk$. The symbols ε_R and σ_R have been used to represent the circumferential strain and the radial stress at the elastic–plastic boundary ($r = R$).

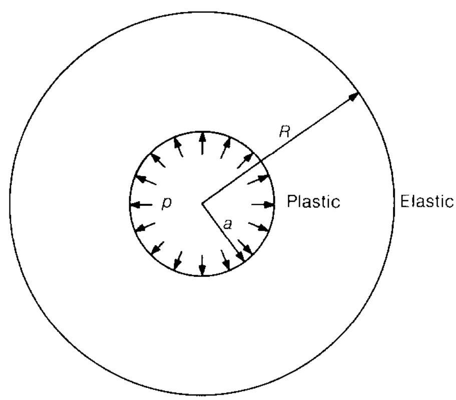

Fig. 1. Cavity expansion problem

At the elastic–plastic interface it is found that both $\sigma_R + k\sigma_\theta = (1 + k)p_0$ and the Mohr–Coulomb condition, equation (6), hold and so

$$\sigma_R = \frac{1+k}{N+k} N p_0 \tag{12}$$

$$\varepsilon_R = \frac{N-1}{N+k} \frac{p_0}{2G} \tag{13}$$

It will often be more convenient to consider the quantity σ_R, rather than the in situ stress p_0, as fundamental.

Within the plastic zone the equilibrium condition (equation (1a)) and the yield criterion (equation (6)) must be satisfied, which means that

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + k\,\frac{N-1}{N}\,\frac{\sigma_r}{r} = 0 \tag{14}$$

and hence that the distribution of radial stress within the plastic zone is given by

$$\frac{\sigma_r}{\sigma_R} = \left(\frac{r}{R}\right)^{\beta - 1} \tag{15}$$

where

$$\beta = 1 - k\left(\frac{N-1}{N}\right)$$

The constitutive relation in the plastic zone (see the appendix) reduces to the single differential equation

$$\frac{\partial \dot{u}}{\partial r} + \alpha\frac{\dot{u}}{r} = -\chi\,\frac{\dot{\sigma}_r}{2G} \tag{16}$$

where $\alpha = k/M$. When this equation is combined with the rate form of equation (15) it is found that

$$\frac{\partial \dot{u}}{\partial r} + \alpha\frac{\dot{u}}{r} = -k(k+1)\varepsilon_R\,\chi\left(\frac{r}{R}\right)^{\beta-1}\frac{\dot{R}}{R} \tag{17}$$

and so recalling from equation (11) that at the elastic–plastic interface ($r = R$)

$$\dot{u} = (k+1)\varepsilon_R \dot{R} \tag{18}$$

then

$$\dot{u} = (k+1)\varepsilon_R\left\{\left(\frac{R}{r}\right)^{\alpha} - \left(\frac{k\chi}{\alpha+\beta}\right)\left[\left(\frac{r}{R}\right)^{\beta} - \left(\frac{R}{r}\right)^{\alpha}\right]\right\}\dot{R}$$

or

$$\dot{u} = \varepsilon_R\left[T\left(\frac{R}{r}\right)^{\alpha} - Z\left(\frac{r}{R}\right)^{\beta}\right]\dot{R} \tag{19}$$

where

$$T = (k+1)\left(1 + \frac{k\chi}{\alpha+\beta}\right)$$

$$Z = (k+1)\,\frac{k\chi}{\alpha+\beta}$$

Equation (19) is valid throughout the plastic region for all magnitudes of deformation but with the restriction that deformations in the elastic zone are infinitesimal. In most real soils first yield will occur at small values of strain so that equation (19) is likely to have a very wide practical application.

SMALL STRAIN SOLUTION

It is not possible to integrate equation (19) in closed form unless the additional restriction of small strains throughout the plastic region is also imposed. Despite such a restriction it is useful to pursue this solution because it has application to the interpretation of pressuremeter tests (e.g. Hughes *et al.*, 1977). It is unusual in this type of in situ testing to encounter strains in excess of 10%.

Thus assuming small deformations equation (19) can be integrated with respect to time and, after applying the appropriate boundary condition at the elastic–plastic interface, the variation in radial displacement throughout the plastic zone is obtained, i.e.

$$\frac{u}{r} = \varepsilon_R\left[A\left(\frac{R}{r}\right)^{1+\alpha} + B\left(\frac{R}{r}\right)^{1-\beta} + C\right] \tag{20}$$

where

$$A = \frac{T}{1+\alpha}$$

$$B = \frac{-Z}{1-\beta}$$

$$C = 1 - A - B = 1 - \frac{T}{1+\alpha} + \frac{Z}{1-\beta}$$

Hence the relationship between cavity displacement δ and internal cavity pressure p is given by

$$\frac{\delta}{a} = \varepsilon_R\left[A\left(\frac{p}{\sigma_R}\right)^{\gamma} + B\,\frac{p}{\sigma_R} + C\right] \tag{21}$$

where

$$\sigma_R = \frac{1+k}{N+k} N p_0$$

$$\gamma = \frac{1+\alpha}{1-\beta} = \frac{1}{k}\,\frac{N}{M}\,\frac{M+1}{N-1}$$

It is interesting to compare equation (21) with the expression derived by Hughes *et al.* (1977) for the expansion of a cylindrical cavity in a frictional soil. In terms of the notation used in this Paper their expression is

$$\frac{\delta}{a} \approx \varepsilon_R \left(\frac{p}{\sigma_R}\right)^{\gamma} \tag{22}$$

It can be seen that the major difference between equations (21) and (22) is the constant term and a linear term in p/σ_R in equation (21), which are absent from equation (22). The difference arises because in the present analysis elastic strains in the plastic region are taken into account, whereas in the analysis of Hughes *et al.* they are ignored. In effect the analysis of Hughes *et al.* has assumed a flow rule which relates the relative magnitudes of the components of total strain rather than plastic strain. A simple evaluation of expressions (21) and (22) indicates that the differences in predicted pressure–expansion curves are small whenever both large values of relative elastic stiffness (G/p_0) are specified and strain levels are small (typically less than 10%). In contrast with this, a numerical solution of equation (19) (Carter & Yeung, 1985) indicates that at larger deformations the elastic strains in the plastic zone are significant.

LIMIT PRESSURES

It has been stated previously that exact integration of equation (19) to obtain the entire pressure–expansion curve will require a numerical technique. However, it is possible to use equation (19) to investigate the limit of an infinitely large deformation. It is likely, and numerical solution confirms, that at very large deformations a pseudo steady state configuration will be approached for which the ratio R/a of the plastic radius to the current cavity size will approach a constant value. For a pre-existing cavity this limiting value of R/a will only be reached when the cavity is made infinitely large, i.e. when both R and a approach infinity. For a cavity which is created, i.e. a cavity for which $a = 0$ initially, this limiting condition will be reached before any radial expansion takes place, since infinite strain must occur at $r = 0$ ($\dot{\varepsilon}_\theta = \dot{u}/r$).

The governing differential equation (19) can also be written as

$$\frac{\dot{u}}{\dot{R}} = \varepsilon_R \left[T\left(\frac{R}{r}\right)^{\alpha} - Z\left(\frac{r}{R}\right)^{\beta} \right] \tag{23}$$

For steady state deformation $\dot{u}/\dot{R} \rightarrow r/R$ and so

$$1 = \varepsilon_R \left[T\left(\frac{R}{r}\right)^{1+\alpha} - Z\left(\frac{R}{r}\right)^{1-\beta} \right] \tag{24}$$

and also from equation (15)

$$\frac{\sigma_r}{\sigma_R} = \left(\frac{R}{r}\right)^{1-\beta} \tag{25}$$

In particular when $r = a$ equations (24) and (25) establish the relationship between the limiting internal pressure p_L and the initial in situ stress p_0. It is convenient to express this relation parametrically as

$$\frac{2G}{p_0} = \frac{N-1}{N+k} [T\rho_L{}^{1+\alpha} - Z\rho_L{}^{1-\beta}] \tag{26a}$$

$$\frac{p_L}{p_0} = \frac{1+k}{N+k} N\rho_L{}^{1-\beta} \tag{26b}$$

where

$$\rho_L = \left(\frac{R}{a}\right)_L = \underset{a\to\infty}{\text{limit}} \left(\frac{R}{a}\right)$$

Alternatively, p_L may be expressed inversely as

$$\frac{2G}{p_0} = \frac{N-1}{N+k} \left[T\left(\frac{p_L}{\sigma_R}\right)^{\gamma} - Z\frac{p_L}{\sigma_R} \right] \tag{27}$$

Either of these expressions can be used to determine the value of p_L/p_0 for specific values of $2G/p_0$, ν, ϕ and ψ. For convenience the definitions of all the terms in equations (26) and (27) are repeated here.

$$T = (k+1)\left(1 + \frac{k\chi}{\alpha+\beta}\right)$$

$$Z = (k+1)\frac{k\chi}{\alpha+\beta}$$

$$\sigma_R = \frac{1+k}{N+k} N p_0$$

$$\alpha = k/M$$

$$\beta = 1 - k\left(\frac{N-1}{N}\right)$$

$$\gamma = \frac{1+\alpha}{1-\beta}$$

$$M = \frac{1+\sin\psi}{1-\sin\psi}$$

$$N = \frac{1+\sin\phi}{1-\sin\phi}$$

$$\chi = \frac{k(1-\nu) - k\nu(M+N) + [(k-2)\nu+1]MN}{[(k-1)\nu+1]MN}$$

COHESIVE FRICTIONAL MATERIAL

The analysis for an ideal cohesive frictional material, with cohesion c, for which the yield criterion can be expressed as

$$\sigma_1 = N\sigma_3 + 2cN^{1/2} \tag{28a}$$

or

$$\sigma_1 + c \cot \phi = N(\sigma_3 + c \cot \phi) \tag{28b}$$

is identical with that for a purely frictional material provided that all stresses are augmented by the quantity $c \cot \phi$.

Thus for small deformations equation (21) becomes

$$\frac{\delta}{a} = \varepsilon_{R'}\left[A\left(\frac{p + c \cot \phi}{\sigma_R + c \cot \phi}\right)^{\gamma} + B\left(\frac{p + c \cot \phi}{\sigma_R + c \cot \phi}\right) + C\right] \tag{29}$$

where for this case

$$\varepsilon_{R'} = \frac{N-1}{N+k}\frac{p_0 + c \cot \phi}{2G}$$

$$\sigma_R + c \cot \phi = \frac{1+k}{N+k} N(p_0 + c \cot \phi)$$

Similarly, equation (27) is suitably modified to give an expression for the limit pressure in a cohesive frictional material

$$\frac{2G}{p_0 + c \cot \phi} = \frac{N-1}{N+k}\left[T\left(\frac{p_L + c \cot \phi}{\sigma_R + c \cot \phi}\right)^{\gamma} - Z\left(\frac{p_L + c \cot \phi}{\sigma_R + c \cot \phi}\right)\right] \tag{30}$$

PURELY COHESIVE MATERIAL

Care is needed in evaluating the expressions (29) and (30) when $\phi \rightarrow 0$ and for a purely cohesive ($\phi = 0$) material

$$\frac{\delta}{a} = \frac{c}{G}\frac{1}{1+k}\left(\frac{T}{1+\alpha} \times \left\{\exp\left[\frac{1+\alpha}{k}\left(\frac{p-p_0}{2c} - \frac{k}{1+k}\right)\right] - 1\right\} - \frac{Z}{k}\left(\frac{p-p_0}{2c} - \frac{k}{k+1}\right) + 1\right) \tag{31}$$

for the small strain behaviour and

$$\frac{G}{c} = \frac{1}{k+1}\left\{T \exp\left[\frac{1+\alpha}{k}\left(\frac{p_L - p_0}{2c} - \frac{k}{k+1}\right)\right] - Z\right\} \tag{32}$$

for the limit pressure. Of particular interest in soil mechanics is the undrained behaviour of saturated clays for which $\phi = 0$, $\psi = 0$ and $\nu = 0{\cdot}5$. Equations (31) and (32) simplify greatly in this case and become

$$\frac{\delta}{a} = \frac{c}{G}\frac{1}{k+1}\exp\left(\frac{k+1}{k}\frac{p-p_0}{2c} - 1\right) \tag{33}$$

$$\frac{G}{c} = \exp\left(\frac{k+1}{k}\frac{p_L - p_0}{2c} - 1\right) \tag{34}$$

For the cylindrical cavity ($k = 1$) equation (34) may be written alternatively as

$$p_L = p_0 + c\left[1 + \ln\left(\frac{G}{c}\right)\right] \tag{35}$$

and this is precisely the same as the formula for the limit pressure in a purely cohesive, constant volume material derived independently by Bishop, Hill & Mott (1945), Hill (1950) and Gibson & Anderson (1961).

For a spherical cavity ($k = 2$) equation (34) may be written as

$$p_L = p_0 + \frac{4c}{3}\left[1 + \ln\left(\frac{G}{c}\right)\right] \tag{36}$$

and this is precisely the solution derived by Hill (1950) for the spherical cavity.

RESULTS

Solutions for the limit pressures p_L and the ratios $(R/a)_L$ have been evaluated for materials with Poisson's ratio $\nu = 0{\cdot}3$, friction angles ϕ of 20°, 30° and 40° and various values of dilation angle ψ. These results are presented graphically in Figs 2–7 for both the cylindrical and the spherical cases. The solutions are applicable to both purely frictional ($c = 0$, $\phi \neq 0$) and more general cohesive frictional ($c \neq 0$, $\phi \neq 0$) materials and have been plotted in non-dimensional form. In each case the quantity $p_L + c \cot \phi$ has been normalized by the quantity $\sigma_R + c \cot \phi$. For purely frictional materials $c = 0$ and σ_R is given by

$$\sigma_R = \frac{1+k}{N+k} N p_0$$

For cohesive frictional materials the normalizing stress is given by

$$\sigma_R + c \cot \phi = \frac{1+k}{N+k} N(p_0 + c \cot \phi)$$

Perhaps the most interesting feature of these results is the strong dependence of the normalized limit pressure $(p_L + c \cot \phi)/(\sigma_R + c \cot \phi)$ on the ratio $G/(p_0 + c \cot \phi)$. Indeed the value of $G/(p_0 + c \cot \phi)$ has the greatest single influence

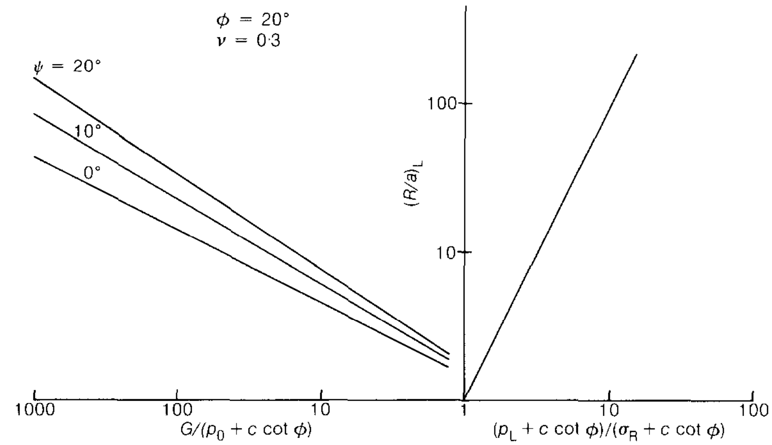

Fig. 2. Limit solution for the $\phi = 20°$ cylindrical cavity

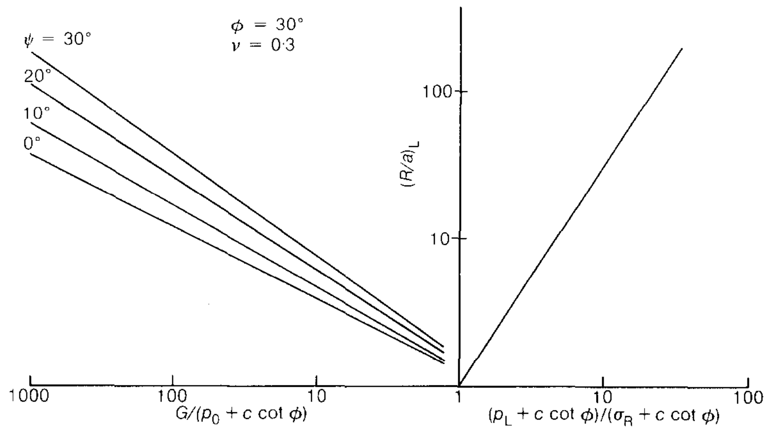

Fig. 3. Limit solution for the $\phi = 30°$ cylindrical cavity

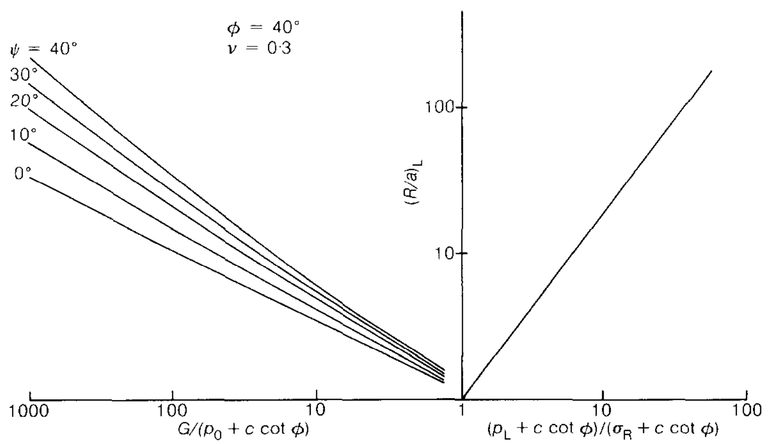

Fig. 4. Limit solution for the $\phi = 40°$ cylindrical cavity

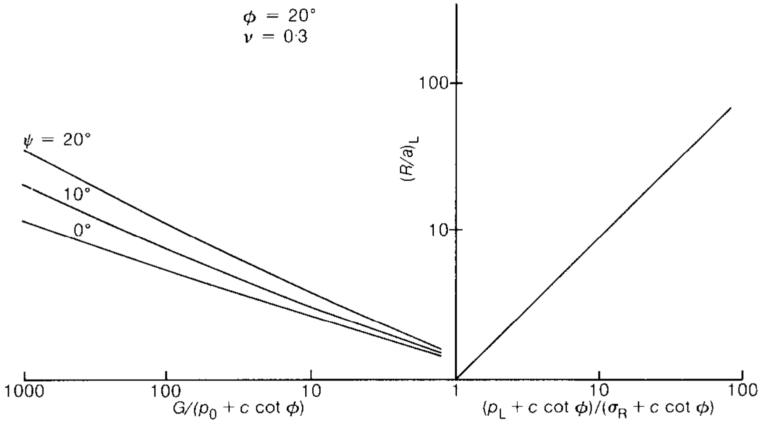

Fig. 5. Limit solution for the ϕ = 20° spherical cavity

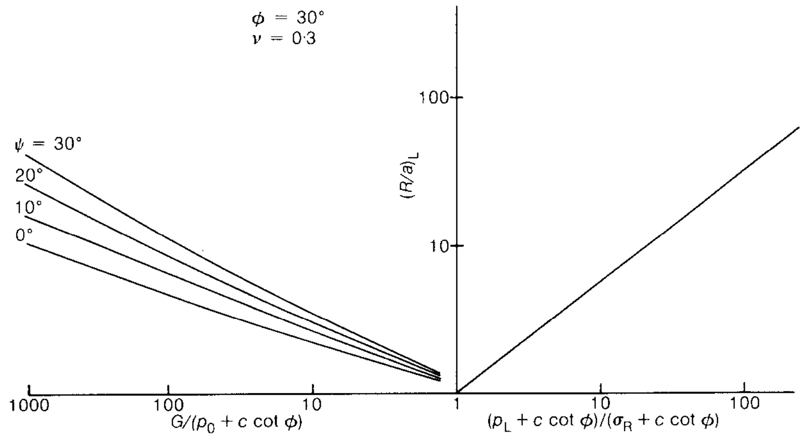

Fig. 6. Limit solution for the ϕ = 30° spherical cavity

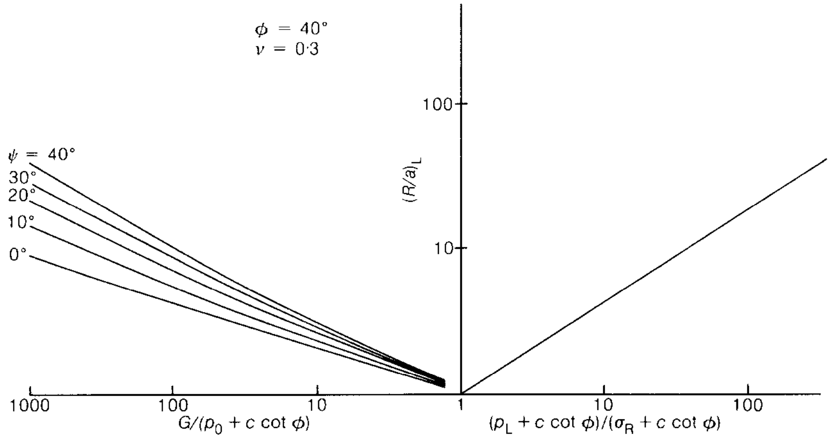

Fig. 7. Limit solution for the ϕ = 40° spherical cavity

on the limit condition and this will be significant when the results of the present analysis are applied to the installation of driven piles in sands.

CONCLUSIONS

Closed form solutions have been presented for the expansion of cylindrical and spherical cavities in an ideal, cohesive frictional soil. The solution for the pressure–expansion curve is obtainable whenever the restriction of small strains is imposed. When this restriction is relaxed only the solution for the limit condition at infinitely large deformation is obtainable in closed form. Numerical techniques must be used to obtain the entire pressure–expansion curve including the large strain portion. The use of the numerical procedure has not been pursued here but is treated in detail in another paper (Carter & Yeung, 1985), where it is shown that strain softening and limited dilation can also be analysed.

The closed form solutions will have application to the following practical problems. The small strain solutions for the cylindrical cavity are applicable to the interpretation of pressuremeter tests and it has been demonstrated that an earlier solution by Hughes *et al.* (1977) for the pressuremeter problem in sand approximates that presented here when the soil is relatively stiff, i.e. when G/p_0 is large. The closed form solutions for the limiting case of large deformations around a cylindrical cavity are applicable to the installation of driven piles in sand. The limit pressure should provide a reasonable estimate for the normal stress acting on the pile shaft after installation and the normal stress will be important in determining the capacity of the pile shaft. Limit solutions for the spherical cavity may also have application to the determination of the end bearing capacity of deep foundations in cohesive frictional soil. The limit solutions presented here for the general case of a cohesive frictional soil include as particular cases solutions presented earlier for a purely cohesive, incompressible soil (Bishop *et al.*, 1945; Hill, 1950; Gibson & Anderson, 1961).

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APPENDIX 1

The purpose of this appendix is to derive the elasto-plastic stress–strain matrix for an ideal, purely frictional soil (equation (4)).

In rate form the constitutive law for an elasto-plastic material may be written as

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}\dot{\boldsymbol{\varepsilon}} \tag{37}$$

where $\dot{\boldsymbol{\varepsilon}} = (\dot{\sigma}_r, \dot{\sigma}_\theta)^{\mathrm{T}}$ and $\dot{\boldsymbol{\varepsilon}} = (\dot{\varepsilon}_r, k\dot{\varepsilon}_\theta)$ with $k = 1$ or $k = 2$ corresponding to a long cylindrical cavity and a spherical cavity respectively. For elasto-plastic deformations the strain is composed of an elastic and a plastic part, i.e.

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{E}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{P}} \tag{38}$$

The elastic component $\dot{\boldsymbol{\varepsilon}}^{\mathrm{E}}$ can be determined from Hooke's law

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{E}} = \mathbf{D}_{\mathrm{E}}^{-1}\dot{\boldsymbol{\sigma}} \tag{39}$$

where

$$\mathbf{D}_{\mathrm{E}} = \begin{bmatrix} \lambda + 2G & \lambda \\ \lambda & \lambda + 2G/k \end{bmatrix}$$

and λ and G are the Lamé and shear moduli for elastic behaviour. The flow rule for the ideal material gives the plastic strains as

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{P}} = \dot{m}\boldsymbol{a} \tag{40}$$

In equation (40) the quantity $\dot{m}$ is a multiplier and not a material constant. The vector $\boldsymbol{a}$ contains the gradients with respect to stress of the plastic potential. It is assumed here that the material dilates plastically and that the relationship between the rates of plastic volumetric and shear strains is given by

$$\dot{\varepsilon}_{\mathrm{v}}^{\mathrm{P}} = \dot{\varepsilon}_r^{\mathrm{P}} + k\dot{\varepsilon}_\theta^{\mathrm{P}} = -(\dot{\varepsilon}_r^{\mathrm{P}} - k\dot{\varepsilon}_\theta^{\mathrm{P}}) \sin \psi \tag{41}$$

where ψ is the angle of dilatancy. For such a material the vector $\boldsymbol{a}$ is given by

$$\boldsymbol{a} = (1, -M)^{\mathrm{T}} \tag{42}$$

It is also assumed that the material is perfectly plastic with yield determined by the Mohr–Coulomb criterion

$$\sigma_1 + c \cot \phi = N(\sigma_3 + c \cot \phi) \tag{43}$$

where

$$N = \frac{1 + \sin \phi}{1 - \sin \phi}$$

with ϕ defined as the friction angle and c the cohesion. In the cavity expansion problem the principal stresses are simply $\sigma_1 = \sigma_r$ and $\sigma_3 = \sigma_\theta$. The gradients of the yield function with respect to the stress components are given by the vector $\boldsymbol{b}$, where

$$\boldsymbol{b} = (1, \ -N)^{\mathrm{T}} \tag{44}$$

It is well known that the constitutive matrix $\mathbf{D}$ for an ideal elastic, perfectly plastic solid is given as

$$\mathbf{D} = \left[\mathbf{I} - \frac{\mathbf{D}_{\mathrm{E}}\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}}{\boldsymbol{b}^{\mathrm{T}}\mathbf{D}_{\mathrm{E}}\boldsymbol{a}}\right]\mathbf{D}_{\mathrm{E}} \tag{45}$$

where $\mathbf{I}$ represents a unit matrix. Recognition that $\boldsymbol{b}^{\mathrm{T}}\mathbf{D} = \mathbf{0}^{\mathrm{T}}$ and $\mathbf{D}\boldsymbol{a} = \mathbf{0}$ where $\mathbf{0}$ is a null vector leads to the simplified expression for $\mathbf{D}$, i.e.

$$\mathbf{D} = \frac{2G}{\chi}\begin{bmatrix} 1 & 1/M \\ 1/N & 1/MN \end{bmatrix} \tag{46}$$

Direct substitution for $\mathbf{D}_{\mathrm{E}}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ into equation (45) reveals that

$$\chi = \frac{k(1 - \nu) - k\nu(M + N) + [(k - 2)\nu + 1]MN}{[(k - 1)\nu + 1]MN}$$