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Computers and Geotechnics 35 (2008) 500–503
COMPUTERS AND GEOTECHNICS
www.elsevier.com/locate/compgeo

Technical Communication

Flow rule effects in the Tresca model

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Received 17 April 2007; received in revised form 15 June 2007; accepted 18 June 2007
Available online 17 August 2007

Abstract

The Tresca failure criterion is used regularly in geotechnical engineering to compute the failure loads of clay soils deforming under undrained conditions. When this criterion is used together with the finite element method a plastic flow rule must also be incorporated in the elasto-plastic soil model.

The effects of the flow rule on the performance of a non-linear analysis using an elastic perfectly plastic soil model obeying the Tresca failure criteria are discussed in this note. Application of this model in a three-dimensional analysis causes computational difficulties, due to the gradient discontinuities that exist at the corner of the Tresca yield surface. Such discontinuities can be removed from the yield (or failure) surface using different methods. Two of the most widely used methods in removing singularities from the yield surface and their overall performances in a three-dimensional finite element analysis are discussed.

The motivation for this study comes from a concern raised by Randolph and Puzrin [Randolph MF and Puzrin AM Upper bound limit analysis of circular foundations on clay under general loading. Geotechnique, (2003);53(9):785–796, [5]] about reported instances of under predictions of the collapse loads by finite element analysis [Taiebat HA and Carter JP Numerical studies of the bearing capacity of shallow foundations on cohesive soil subjected to combined loading. Geotechnique, (2000);50(4):409–418, [7]] and [Taiebat HA and Carter JP Bearing capacity of strip and circular foundations on undrained clay subjected to eccentric loads. Geotechnique, (2002);52(1):61–64, [8]], [Gourvenec S and Randolph M Effect of strength non-homogeneity on the shape of failure envelopes for combined loading of strip and circular foundations on clay, Geotechnique, (2003);53(6):575–586, [4]], when it is usually expected that finite element results should overestimate the true collapse loads. The intent of this study is to demonstrate and reiterate that although the finite element method is an extremely powerful analytical tool for solution of engineering problems, it is nevertheless subjected to approximation errors due to simplifications that are necessarily made to prevent other numerical difficulties.

Keywords: Tresca model; Gradient discontinuities; Flow rule; Bearing capacity

1. Tresca model

The Tresca failure criterion can be regarded as a particular case of the Mohr–Coulomb failure criterion. Both are widely used in non-linear analyses of many geotechnical problems. In particular, the Tresca yield surface is a hexagonal prism when plotted in three-dimensional principal stress-space (Fig. 1), while the Mohr–Coulomb criterion plots as a hexagonal cone. The intersection of the Tresca yield surface with a deviatoric plane results in a regular hexagon. The vertices of the hexagon, which are at triaxial compression and extension points, create singularities of the normal gradient to the yield surface.

The Tresca failure surface, F, and plastic potential function, G, can be expressed as

$$F = \sqrt{J_2}\cos\theta - s_u = 0, \qquad G = \sqrt{J_2}\cos\theta - K = 0 \qquad (1)$$

In the above equation J_2 is the second stress invariant, which is a measure of the distance between the current stress state and the hydrostatic axis in the deviatoric plane, θ is the Lode angle, which defines the orientation of the stress state with respect to the principal stresses in this plane, s_u is the undrained shear strength of the soil, and

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doi:10.1016/j.compgeo.2007.06.012

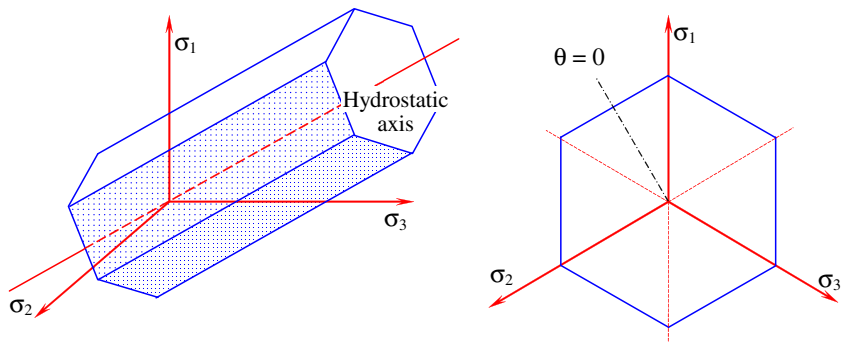

Fig. 1. Tresca failure criterion in a three-dimensional stress space.

K is a vector of state parameters whose values are not important in the present context.

Having defined the yield and the plastic potential functions, the elasto-plastic stiffness matrix, $\mathbf{D}^{ep}$, which relates increments of stress and strain, for the elastic perfectly plastic model can be written as

$$\mathbf{D}^{ep} = \mathbf{D}^{e} - \frac{\mathbf{D}^{e}\left(\frac{\partial G}{\partial \sigma}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{T}\mathbf{D}^{e}}{\left(\frac{\partial F}{\partial \sigma}\right)^{T}\mathbf{D}^{e}\left(\frac{\partial G}{\partial \sigma}\right)} \tag{2}$$

where $\mathbf{D}^{e}$ is the elastic stiffness matrix and σ is the vector of generalised stresses.

The partial derivatives of the yield and plastic potential functions required in calculation of $\mathbf{D}^{ep}$ cannot be uniquely defined at the vertices of the yield or plastic potential functions, where the gradients are singular. Attempts have been made in the past to remove this singularity from the yield and plastic potential functions. Two methods of removing the singularity are briefly explained in this note and their performance when used to predict the collapse load of an axially loaded circular foundation is demonstrated.

2. Tresca–Mises model

In this model the yield and plastic potential functions are approximated as a cylinder, which has a circular cross-section in the deviatoric plane, so that the yield and plastic potential functions are expressed as

$$F = \sqrt{J_2} - R = 0, \qquad G = \sqrt{J_2} - K = 0 \tag{3}$$

where R is a material constant representing the shear strength of the soil. In its present form Equation (3) represents the von–Mises failure criterion. The yield and plastic potential surfaces are completely smooth, providing a unique definition of the direction of flow and $\mathbf{D}^{ep}$. This failure criterion is not a function of θ, therefore its partial derivative with respect to θ is zero.

A simple modification to the von–Mises failure surface is possible by relating the value of R to the undrained shear strength of the soil, $R = s_u/\cos\theta$, as defined by the Tresca model. Therefore the new failure function, together with the von-Misses plastic potential function, is defined as:

$$F = \sqrt{J_2} - \frac{s_u}{\cos\theta} = 0, \qquad G = \sqrt{J_2} - K = 0 \tag{4}$$

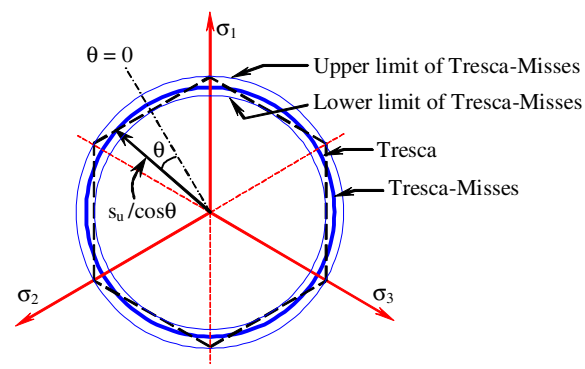

Fig. 2. Tresca–Mises failure criteria.

Although the new failure criterion is a function of θ, the partial derivative of F with respect to θ is still assumed to be zero. In this new failure criterion, the radius of the circular failure surface in the deviatoric plane evolves from s_u at $\theta = 0$ to 1.155 s_u at $\theta = 30°$ (Fig. 2). This method has been used in the commercially available finite element program ABACUS [1].

3. Rounded Tresca

This method provides an approximation to the Tresca failure surface using trigonometric rounding techniques in the deviatoric plane [6,2]. The failure surface and the plastic potential surface have a form similar to Eq. (3). The state variables R and K are selected in such a way that the deviatoric cross-section of the surfaces is similar to the Tresca cross-section, except it is rounded. Away from the singular vertices of the Tresca surface, which occur at $\theta = \pm 30°$ in the deviatoric plane, the rounded surfaces are identical to those of the Tresca surface. In the vicinity of the singularities, where $\theta > \theta_T$, and θ_T is a specific transition angle, an alternative form of R or K is defined [6]. For example, the value of R is defined as

$$R(\theta) = \begin{cases} 1/\cos\theta & |\theta| \leqslant \theta_T \\ 1/(A + B\sin 3\theta) & |\theta| > \theta_T \end{cases} \tag{5}$$

where A and B are defined as

$$\begin{aligned} A &= \cos\theta_T + (\sin\theta_T \times \tan 3\theta_T)/3 \\ B &= \sin\theta_T/(3\cos 3\theta_T) \end{aligned} \tag{6}$$

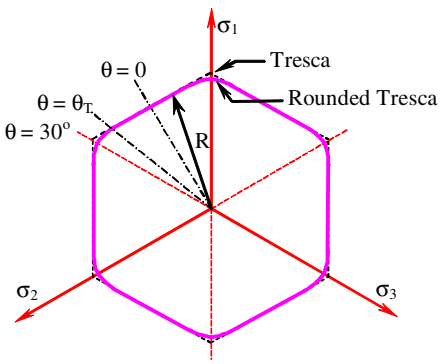

Fig. 3. Rounded Tresca failure criterion.

The value of the transition angle lies within the range $0 \leqslant \theta_T < 30°$, with larger values giving better fits to the Tresca failure criterion. Abbo and Sloan [2] recommended a typical value of $\theta_T = 25°$, indicating that θ_T should not be too close to 30° to avoid ill-conditioning of the approximation. An intersection of the rounded Tresca failure surface with the deviatoric plane is shown in Fig. 3. The failure surface is continuous and does not posses any gradient singularity.

4. Finite element idealisation

The two constitutive models that approximate the Tresca criterion have been used in finite element analyses of the vertical (axial) bearing capacity of a circular footing, with a diameter D, resting on the surface of a uniform homogeneous weightless soil that deforms under undrained conditions. The soil has a uniform undrained shear strength of s_u and an undrained Young's modulus of $E_u = 300\ s_u$. A Poisson's ratio of $\nu \approx 0.5$ (=0.49) was assumed for the soil to approximate the constant volume response of the soil under undrained conditions. The footing has a Young's modulus of $E_f = 1000E_u$ and therefore it can be considered as effectively rigid. Identical finite element meshes have been used in these analyses and they consist of 698 isoparametric (20 node) brick elements. Only a wedge-shaped segment of a cylinder of the soil, with a central angle of 15°, has been modelled, of which a section close to the footing is shown in Fig. 4. The boundaries were set 4D away from the footing in both vertical and horizontal directions.

All elements were integrated numerically by Gauss quadrature rules using a reduced integration scheme, i.e., using $2 \times 2 \times 2$ Gauss points. An incremental approach has been selected in the solution of the non-linear finite element equations where of a total vertical displacement of $\delta v = 0.125$ D was applied to the footing in 2500 equal increments.

5. Accuracy and performance

The accuracy and performance of the two constitutive models are evaluated by comparing the results of the displacement finite element analyses with the exact solution available for the foundation. Two different transition angles, $\theta_T = 25°$ and $\theta_T = 29.5°$, have been considered for the rounded Tresca model. The results of the analyses are presented in Fig. 5, where the load-displacement curves obtained from different approximations are shown.

The bearing capacity of the foundation can be expressed as

$$V = N_c \cdot s_u \cdot A \tag{7}$$

where N_c is the undrained bearing capacity factor and A is the area of the foundation. The exact solution to this problem indicates a bearing capacity factor of $N_c = 6.05$ [3]. The bearing capacity factors obtained by the finite element analyses are 5.87, 6.03 and 6.13 for the Tresca–Misses model, the rounded Tresca model with $\theta_T = 25°$ and the rounded Tresca model with $\theta_T = 29.5°$, respectively. The last two values will increase slightly if the original Tresca criterion is used to define failure but the rounded Tresca model is used for calculation of gradients to the yield surface.

It is expected that the results of the finite element analysis will improve if a model closer to the original Tresca, i.e., the rounded Tresca with $\theta_T = 29.5°$, is used in the analysis. Therefore, it is reasonable to expect that this analysis will be the most accurate analysis of the three for the particular mesh used in the finite element modelling.

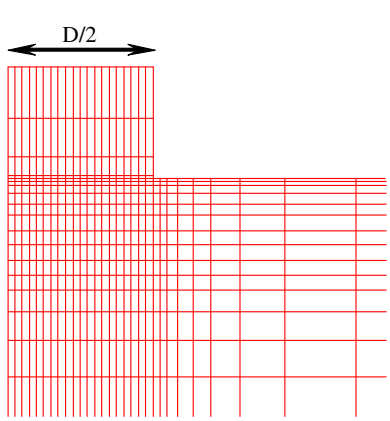

Fig. 4. Finite element mesh close to the footing.

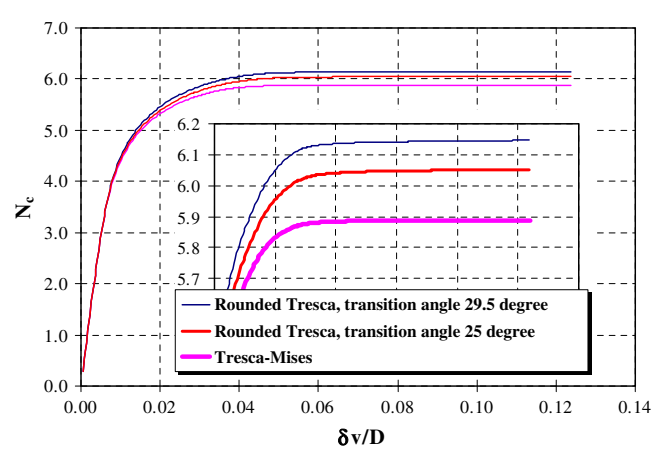

Fig. 5. Load–deflection curves obtained from different approximations.

The rounded Tresca with $\theta_T = 25°$ gives a result closer to the exact solution, even though it would normally have been expected to under-predict the collapse load. This is due to a combination of two approximations: an under-prediction due to the approximation of the original Tresca yield surface and an over-prediction that is associated with discretisation errors that are inevitable in the finite element method.

The Tresca–Misses criterion is evidently the worst of the three models considered here. This model significantly under estimates the collapse load even though it is generally expected that finite element analyses over predict the true collapse load.

6. Conclusions

The results of this study show that smooth approximations to the yield and plastic potential functions can, and in general probably do, result in underestimation of the collapse load for an axially loaded circular footing resting on undrained clay. Of the two approximate models considered here, the rounded Tresca model gives a closer answer to the exact solution. Although the model with $\theta_T = 25°$ gives the closest answer to the exact solution, a transition angle of $\theta_T = 29.5°$ in the rounded Tresca should be, theoretically, a more accurate model. This study demonstrates how a simple approximation, which is necessarily made in the formulation of the plastic potential function to avoid singularities, affects the accuracy of the finite element solution.

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