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ELASTIC CONSOLIDATION AROUND A LINED CIRCULAR TUNNEL

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Abstract—The analysis of the interaction between a thin, circular, elastic tunnel lining and the surrounding saturated, elastic soil has been presented. The loading on the lining is due to the removal of material from within the tunnel and the response of the lining is dependent on time because of consolidation within the surrounding elastic soil.

A parametric study has been carried out to investigate the effects of the elastic properties of both the lining and the soil, the thickness of the lining, and the magnitude of the initial stress state in the soil upon the behaviour of the lining. It has been found that typically the maximum hoop thrust in the lining is increased only by about 10% due to consolidation, but the maximum bending moment can be increased by as much as 100%.

NOTATION

a	radius of tunnel
c	coefficient of consolidation
d	thickness of tunnel lining
E_l	Young's modulus of lining material
E_l^*	$= E_l/(1-\nu_l^2)$
F	Fourier coefficient of hoop force in lining
G_s	shear modulus of soil skeleton
K_0	coefficient of earth pressure at rest $= \sigma'_H/\sigma'_v$
M	Fourier coefficient of bending moment in lining
n	Fourier integer
p	pore water pressure
P	Fourier coefficient of pore water pressure
p, q	radial and tangential components of loading on the tunnel lining
P, Q	Fourier coefficients of p, q
r	radial coordinate
R	Fourier coefficient of radial stress on tunnel lining
s	Laplace transform variable
S	volume of pore water which flows across unit area of soil–tunnel interface
$S_{rr}, S_{\theta\theta}, S_{r\theta}$	Fourier coefficients of polar stress components
t	time variable
T	Fourier coefficient of shear stress at tunnel-lining interface
T_v	non-dimensional time $= ct/a^2$
u_s, v_s	polar components of soil displacement
U, V	Fourier coefficients of u_s, v_s
u_l, v_l	polar components of lining displacement
U_l, V_l	Fourier coefficients of u_l, v_l
X, Y	Fourier coefficients of the components of displacement of the soil at the tunnel interface
Z	Fourier coefficient of pore water pressure at the soil-lining interface
γ_w	unit weight of pore water
λ_s	Lamé constant for soil skeleton
$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}$	polar components of total stress
σ_m	mean total stress before tunnel cutting $= \frac{1}{2}(\sigma_v + \sigma_H)$
σ_d	deviator stress before tunnel cutting $= \frac{1}{2}(\sigma_v - \sigma_H)$
σ_v	total vertical stress before tunnel cutting
σ_H	total horizontal stress before tunnel cutting

1. INTRODUCTION

When a tunnel is driven through saturated clay, it is usual to place some form of lining around the walls of the cavity. This lining not only provides support for the surrounding soil, but also helps to keep the ground water out of the tunnel opening. In order to design these linings as support systems, the geotechnical engineer is called upon to make estimates of the loads on the lining and perhaps some predictions of the deflections that can be expected. As a first approximation, the soil surrounding the tunnel may be idealised as an isotropic, linear, elastic continuum and the lining may be assumed to be another (usually

stiffer) elastic material. Opening the tunnel in general will cause the development of excess pore pressures throughout the soil. Initially these excess pore pressures will not have time to dissipate and the soil will respond as a single phase (undrained) elastic material. Subsequently the excess pore pressures will begin to dissipate and the soil will consolidate until ultimately all excess pore pressures will have dissipated completely and again the soil will respond as a single phase (drained) elastic material.

In earlier investigations, e.g. Hoeg[1], Carter *et al.*[2], the soil was considered to behave as a single phase elastic material and thus these investigations are only appropriate for the short term (undrained) behaviour or the long term (drained) condition. In this paper an examination, of the complete time dependent behaviour of a long, deep circular tunnel in an elastic saturated soil, is made. It is assumed that the tunnel cavity is lined with a thin elastic tube which is in intimate radial contact with the surrounding soil. Solutions are presented for the displacements, the hoop force and moment in the lining as well as the stresses and pore water pressure at the tunnel-soil interface. The investigation is restricted to a saturated soil with a purely elastic skeleton; no attempt is made to include the effect of creep which will be considered in a subsequent investigation.

2. PROBLEM DESCRIPTION

It will be assumed that the tunnel will be at a depth which is large when compared to its nominal radius and thus to sufficient accuracy the *in situ* stress state can be considered to be a uniform vertical stress σ_v and a uniform horizontal stress σ_H.

For a circular opening, it proves convenient to express the *in situ* stress state in terms of polar coordinates, see Fig. 1. It follows that the total stresses acting on a circular boundary are given by,

$$\sigma_{rr} = \sigma_m + \sigma_d \cos 2\theta \tag{1a}$$

$$\sigma_{\theta\theta} = \sigma_m - \sigma_d \cos 2\theta \tag{1b}$$

$$\sigma_{r\theta} = -\sigma_d \sin 2\theta \tag{1c}$$

where

$$\sigma_m = \frac{1}{2}(\sigma_v + \sigma_H) = \text{the mean total stress;}$$

$$\sigma_d = \frac{1}{2}(\sigma_v - \sigma_H) = \text{the deviator stress.}$$

It is assumed that initially the interface between the tunnel lining and the surrounding soil is circular in cross-section with radius a, and that the lining has an annular

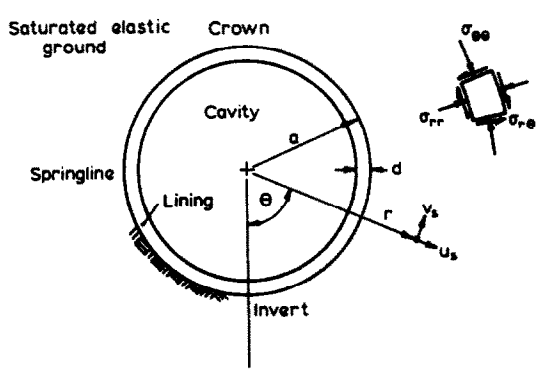

Fig. 1. Coordinate description.

cross-section with outer radius a and thickness d. It will be assumed that $d \ll a$ so that it is permissible to assume that the lining is governed by cylindrical shell theory and so there is no need to distinguish between whether a load is applied at $r = a$, or at $r = a - d$. The axis of the tunnel is assumed to be horizontal and the tunnel itself is assumed to be long so that the problem is one of plane strain. A typical vertical cross-section is shown in Fig. 1. Furthermore, it is assumed that the circular cavity is created and the lining installed in such a way that the *in situ* stress state initially is undisturbed. It is only after the lining is in place that the normal and shear stresses at the inside face of the lining are removed. This may be a reasonable model for the case in practice where the lining is jacked horizontally into the clay ahead of the excavation face. In any case, this model of the installation process is likely to over-predict the stress acting on the lining and thus, for design purposes, can be considered conservative. It will be assumed, as is usually the case, that the tunnel lining is relatively impermeable when compared with the soil and thus that the porewater pressure boundary condition is

$$\frac{\partial p}{\partial r} = 0 \text{ at } r = a. \tag{2}$$

The problem under consideration is the time dependent, structural interaction between the elastic tube lining and the surrounding, consolidating soil. In order to simplify the analysis, the *in situ* stress state can be considered as the sum of two parts; that part due to the presence of the mean total stress σ_m and the other due to the presence of a deviator stress σ_d. Thus the removal of the *in situ* stress state at the tunnel can be considered in two parts and the overall effects will be found by the superposition of the individual results. The boundary conditions for the component problems are:

Case 1. Removal of the mean stress

$$\left.\begin{aligned} \Delta\sigma_{rr} &= -\sigma_m \\ \Delta\sigma_{r\theta} &= 0 \\ \frac{\partial p}{\partial r} &= 0. \end{aligned}\right\} \text{ at } r = a \tag{3}$$

Case 2. Removal of the deviator stress

$$\left.\begin{aligned} \Delta\sigma_{rr} &= -\sigma_d \cos 2\theta \\ \Delta\sigma_{r\theta} &= +\sigma_d \sin 2\theta \\ \frac{\partial p}{\partial r} &= 0. \end{aligned}\right\} \text{ at } r = a \tag{4}$$

The symbol Δ is used to represent a change in the appropriate quantity.

The form of the *in situ* stress state suggests that the analysis can be made in terms of the Fourier components of the field quantities, e.g. at any time the field quantities for each case may be written as,

$$\begin{aligned} u_s(r, \theta, t) &= U(r, t) \cos n\theta \\ v_s(r, \theta, t) &= V(r, t) \sin n\theta \\ p(r, \theta, t) &= P(r, t) \cos n\theta \\ \sigma_{rr}(r, \theta, t) &= S_{rr}(r, t) \cos n\theta \\ \sigma_{\theta\theta}(r, \theta, t) &= S_{\theta\theta}(r, t) \cos n\theta \\ \sigma_{r\theta}(r, \theta, t) &= S_{r\theta}(r, t) \sin n\theta \end{aligned} \tag{5}$$

where u_s, v_s are the radial and circumferential components of displacement of the soil, p is the actual pore pressure and σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ are the total normal stress and shear stress components, and where U, V, P, S_{rr}, $S_{\theta\theta}$, $S_{r\theta}$ denote the corresponding Fourier coefficients. It is clear that $n = 0$ for Case 1 and $n = 2$ for Case 2.

3. ANALYSIS

It is convenient to begin an analysis of this time dependent, interaction problem by considering separately the behaviour of the consolidating soil and the behaviour of the elastic lining. Once the governing equations have been developed for each material, then the interaction effects may be investigated.

3.1 *Consolidation of elastic soil*

The problem of a cylindrical cavity in a saturated elastic medium is defined in Fig. 2. Suppose that the field quantities may be expressed in the form of eqns (5). Carter and Booker[3] have found the general solution of the equations of consolidation for this case. It follows from their analysis that the solution which remains bounded as $r \to \infty$ may be expressed in the form:

$$\begin{bmatrix} \bar{U} \\ \bar{V} \\ \dfrac{r\Delta\bar{P}}{2G_s} \\ \dfrac{r\Delta\bar{S}_{rr}}{2G_s} \\ \dfrac{r\Delta\bar{S}_{\theta\theta}}{2G_s} \\ \dfrac{r\Delta\bar{S}_{r\theta}}{2G_s} \end{bmatrix} = \begin{bmatrix} \dfrac{\beta K_n'(\beta r/a)}{\beta^2 K_n(\beta)} & \dfrac{n}{4(n-1)}\left(\dfrac{a}{r}\right)^{n-1} & \left(\dfrac{a}{r}\right)^{n+1} \\ -n\left(\dfrac{a}{r}\right)\dfrac{K_n'(\beta r/a)}{\beta^2 K_n(\beta)} & \dfrac{(n-2)}{4(n-1)}\left(\dfrac{a}{r}\right)^{n-1} & \left(\dfrac{a}{r}\right)^{n+1} \\ \left(\dfrac{\lambda_s + 2G_s}{2G_s}\right)\left(\dfrac{r}{a}\right)\dfrac{K_n(\beta r/a)}{K_n(\beta)} & \dfrac{1}{2}\left(\dfrac{a}{r}\right)^{n-1} & 0 \\ \dfrac{-\left\{n^2\left(\dfrac{a}{r}\right)K_n(\beta r/a) - \beta K_n'(\beta r/a)\right\}}{\beta^2 K_n(\beta)} & \left(\dfrac{n+2}{4}\right)\left(\dfrac{a}{r}\right)^{n-1} & (n+1)\left(\dfrac{a}{r}\right)^{n+1} \\ \dfrac{\left\{\left(\dfrac{r}{a}\right)\beta^2 + \left(\dfrac{a}{r}\right)n^2\right\}K_n(\beta r/a) - \beta K_n'(\beta r/a)}{\beta^2 K_n(\beta)} & -\left(\dfrac{n-2}{4}\right)\left(\dfrac{a}{r}\right)^{n-1} & -(n+1)\left(\dfrac{a}{r}\right)^{n+1} \\ \dfrac{-n\left\{\left(\dfrac{a}{r}\right)K_n(\beta r/a) - \beta K_n'(\beta r/a)\right\}}{\beta^2 K_n(\beta)} & \left(\dfrac{n}{4}\right)\left(\dfrac{a}{r}\right)^{n-1} & (n+1)\left(\dfrac{a}{r}\right)^{n+1} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \tag{6}$$

where C_1, C_2, C_3 are constants to be determined from the boundary conditions, the bar denotes a Laplace transform with respect to time, t, of the appropriate quantity, viz:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{7}$$

and β is defined by:

$$\beta = \sqrt{\frac{s}{c}} \cdot a \tag{8}$$

where the square root denotes the branch having a positive real part, and

$$c = \frac{k}{\gamma_w}(\lambda_s + 2G_s) \tag{9}$$

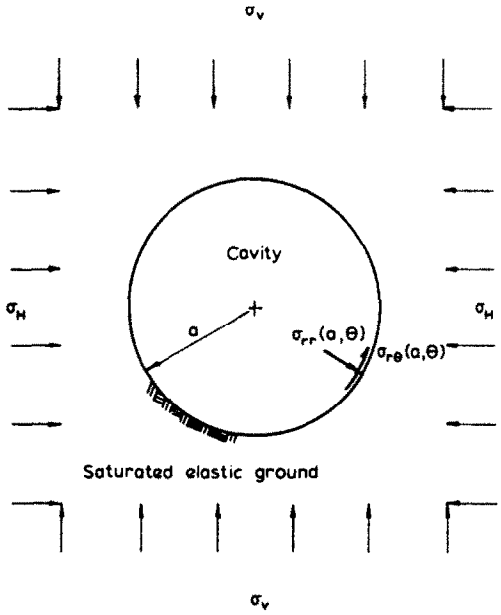

Fig. 2. The soil problem.

with k = permeability of the elastic soil; γ_w = unit weight of pore fluid; and λ_s, G_s = Lamé parameters of the elastic soil skeleton. In the case of a lined tunnel the behaviour at $r = a$ is of interest. At this inner boundary of the soil, it is convenient to define

$$\begin{aligned} X(t) &= U(a, t) \\ Y(t) &= V(a, t) \\ Z(t) &= P(a, t) \\ R(t) &= S_{rr}(a, t) \\ T(t) &= S_{r\theta}(a, t) \end{aligned} \tag{10}$$

and finally

$$S(t) = \frac{k}{\gamma_w} \int_0^t \frac{\partial P(r, t)}{\partial r} \, dt \quad (r = a)$$

where $S(t)$ denotes the total volume of pore water which has flowed out of a unit area of the soil-tunnel interface. Clearly, for an impermeable tunnel $(\partial p/\partial r) = 0$ and thus $S = 0$.

Upon Laplace transformation this last equation may be written

$$\bar{S} = \frac{a^2}{\beta^2(\lambda_s + 2G_s)} \left(\frac{\partial \bar{P}}{\partial r} \right)_{r=a}. \tag{11}$$

For $n > 1$ it can be shown that eqns (6) lead to "stiffness" relations for the soil which take the form

$$\begin{bmatrix} \bar{R} - \bar{T} \\ \bar{Z} \\ \bar{R} + \bar{T} \end{bmatrix} = \frac{1}{s} \begin{bmatrix} R_0 - T_0 \\ Z_0 \\ R_0 + T_0 \end{bmatrix} + \frac{2G_s}{a} \begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 \\ \Phi_{21} & \Phi_{22} & 0 \\ 0 & 0 & \Phi_{33} \end{bmatrix} \begin{bmatrix} \bar{X} - \bar{Y} \\ 2\bar{S} \\ \bar{X} + \bar{Y} \end{bmatrix} \tag{12}$$

where

$$\Phi_{11} = (n-1)\left\{1 - \left(\frac{2G_s}{\lambda_s + 2G_s}\right)\frac{4n(q+n)}{\beta^4 D}\right\}$$

$$\Phi_{12} = \Phi_{21} = \frac{-2(q+n)}{\beta^2 D}$$

$$\Phi_{22} = \frac{2n}{(n-1)\beta^2 D} - \left(\frac{2G_s}{\lambda_s + 2G_s}\right)\frac{2n(q+n)}{\beta^4 D}$$

$$\Phi_{33} = n + 1$$

$$D = \frac{2q}{(n-1)\beta^2} + \left(\frac{2G_s}{\lambda_s + 2G_s}\right)\frac{2n(q+n)}{\beta^4}$$

and

$$q = \frac{\beta K_n'(\beta)}{K_n(\beta)}$$

and R_0, T_0, Z_0 denote the values of R, T, Z at $t = 0^-$. For an impermeable tunnel lining eqns (12) become

$$\begin{bmatrix} \bar{R} - \bar{T} \\ \bar{R} + \bar{T} \end{bmatrix} = \frac{1}{s}\begin{bmatrix} R_0 - T_0 \\ R_0 + T_0 \end{bmatrix} + \frac{2G_s}{a}\begin{bmatrix} \Phi_{11} & 0 \\ 0 & \Phi_{33} \end{bmatrix}\begin{bmatrix} \bar{X} - \bar{Y} \\ \bar{X} + \bar{Y} \end{bmatrix} \tag{13}$$

and

$$\bar{Z} = \frac{Z_0}{s} + \frac{2G_s}{a}\Phi_{21}(\bar{X} - \bar{Y}).$$

Equations (13) are valid when $n > 1$ and thus apply to Case 2. The problem of Case 1, i.e. removal of the *in situ* mean stress, corresponds to $n = 0$ and for this special case it is a simple matter to show that the behaviour of the soil is independent of time and is governed by the equations

$$\begin{aligned} R &= \left(\frac{2G_s}{a}\right)X \\ Y &= 0 \\ Z &= Z_0 \\ T &= 0 \\ S &= 0. \end{aligned} \tag{14}$$

3.2 *Behaviour of the lining*

Consider now the deformation under conditions of plane strain of the elastic tube, shown in Fig. 3. The loading of the tube may be considered to consist of the sum of loads having the form

$$\begin{aligned} p &= P\cos n\theta \\ q &= Q\sin n\theta. \end{aligned} \tag{15}$$

These loads will lead to radial and circumferential deflections of the lining (u_l, v_l) having

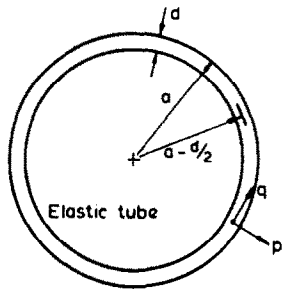

Fig. 3. The lining problem.

the form

$$u_l = U_l \cos n\theta$$
$$v_l = V_l \sin n\theta. \tag{16}$$

Carter *et al.*[2] have shown that when $n > 1$ the tunnel lining has the stiffness relations

$$\begin{bmatrix} P-Q \\ P+Q \end{bmatrix} = \left\{ \frac{S_A}{2} \begin{bmatrix} (n-1)^2 & 1-n^2 \\ 1-n^2 & (n+1)^2 \end{bmatrix} + \frac{S_B}{2} \begin{bmatrix} n^2(n-1)^2 & n^2(n^2-1) \\ n^2(n^2-1) & n^2(n+1)^2 \end{bmatrix} \right\} \begin{bmatrix} U_l - V_l \\ U_l + V_l \end{bmatrix} \tag{17}$$

where

$$S_A = E_l^* \frac{d}{a^2} = \frac{E_l d}{(1-\nu_l^2)a^2} = \frac{1}{a^2}. \quad \text{Membrane stiffness of the ring.}$$

$$S_B = \frac{E_l^* d^3}{12a^4} = \frac{E_l d^3}{12(1-\nu_l^2)a^4} = \frac{1}{a^4}. \quad \text{Bending stiffness of the ring.}$$

E_l = Young's modulus of the lining material. ν_l = Poisson's ratio of the lining material. Equations (17) are valid for Case 2 ($n = 2$) but again Case 1 ($n = 0$) must be treated as a special case and it is found that

$$P = S_A U_l$$
$$Q = 0 \tag{18}$$
$$V_l = 0.$$

3.3 *The interaction problem*

The solution to the interaction problem will depend on the boundary conditions assumed at the interface between the soil and the tunnel lining, i.e the conditions at $r = a$. It has been assumed already that the lining will act as an impermeable boundary. For the stress and displacement boundary conditions, two important extremes may be identified, viz.:

(a) a perfectly rough interface, implying compatibility of radial and circumferential displacements across the interface, and

(b) a perfectly smooth interface, implying zero shear stress at the interface together with compatibility of the radial displacement.

In practice, the interface condition will vary somewhere between these two extremes since there may be some finite limit to the shear stress which can be developed between the soil and the lining. Nevertheless, it is useful to examine the two distinct classes of problem which arise from assuming either of the conditions (a) or (b).

(a) *Perfectly rough interface*

Case 1, n = 0. For the removal of the *in situ* mean stress $n = 0$ and, if the soil and the lining are to remain intact, then $P = -R$. Combining eqns (14) and (18) leads to a solution which is independent of time, viz.

$$
\begin{aligned}
X &= -\sigma_m\left(\frac{1}{\dfrac{2G_s}{a}+S_A}\right)\\
Y &= 0\\
Z &= Z_0\\
R &= \left(\frac{2G_s}{a}\right)X\\
T &= 0.
\end{aligned}
\tag{19}
$$

In this case the force induced in the lining is given by

$$F = -S_A a X. \tag{20}$$

There are no bending effects associated with the case $n = 0$, i.e. the bending moment in the lining is everywhere zero

$$M = 0. \tag{21}$$

Case 2, n = 2. For the removal of the *in situ* deviator stress $n = 2$ and, for the case of a perfectly rough interface, the displacements of the soil and the lining must be the same, i.e. there can be no radial separation or circumferential slip between the soil and the lining. This implies that $P = -R$ and $Q = -T$. Combining eqns (13) and (17) gives

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix}\begin{bmatrix} \bar{X}-\bar{Y} \\ \bar{X}+\bar{Y} \end{bmatrix} = \frac{1}{s}\begin{bmatrix} 2\sigma_d \\ 0 \end{bmatrix} \tag{22}$$

where

$$S_{11} = \Phi_{11} + 2\frac{S_B}{a^4} + \frac{1}{2}\frac{S_A}{a^2}$$

$$S_{12} = 6\frac{S_B}{a^4} - \frac{3}{2}\frac{S_A}{a^2}$$

$$S_{22} = \frac{6G_s}{a} + 18\frac{S_B}{a^4} + \frac{9}{2}\frac{S_A}{a^2}$$

and all other quantities are defined above. The solution of equations (22) provides the Laplace transforms of the displacements of the lining, i.e. $\bar{X}$ and $\bar{Y}$. The excess pore pressure at the interface may be found from eqn (13), i.e.

$$\bar{Z} = \frac{Z_0}{s} + \left(\frac{2G_s}{a}\right)\Phi_{21}(\bar{X}-\bar{Y}). \tag{23}$$

The transforms $\bar{X}$ and $\bar{Y}$ must be inverted to provide the displacement coefficients X, Y at the appropriate time t. The displacement coefficients may then be substituted into eqns (17) in order to calculate the stresses acting at the soil-lining interface. Following the

analysis of Carter *et al.*[2], the hoop force and the bending moment induced in the lining are given by

$$F = -S_A a(X + 2Y)$$
$$M = -S_B a^2(4X + 2Y). \tag{24}$$

(b) *Perfectly smooth interface*

Case 1, n = 0. For the case where $n = 0$, the solution for a perfectly smooth interface is the same as that for a perfectly rough interface and is given by eqns (19)–(21).

Case 2, n = 2. For this case there can be no shear transfer between the lining and the surrounding soil and for $t > 0$, only normal stress may be transmitted, i.e. $T = Q = 0$ and $P = -R$. As a consequence of this, the solution for a smooth lining is given by

$$\bar{X} = \bar{U}_l = \frac{\dfrac{-2\sigma_d \Phi_{33}}{s(\Phi_{11} + \Phi_{33})}}{\dfrac{(n-1)^2 S_A S_B}{(S_A + S_B)} + \left(\dfrac{4G_s}{a}\right)\dfrac{\Phi_{11}\Phi_{33}}{\Phi_{11} + \Phi_{33}}}$$
$$\bar{Y} = \left(\frac{\Phi_{11} - \Phi_{33}}{\Phi_{11} + \Phi_{33}}\right)\bar{X} + \frac{\sigma_d a}{sG_s(\Phi_{11} + \Phi_{33})}$$
$$\bar{V}_l = -\frac{1}{n}\left(\frac{S_A + n^2 S_B}{S_A + S_B}\right)\bar{U}_l \tag{25}$$
$$\bar{Z} = \frac{Z_0}{s} + \frac{2G_s}{s}\Phi_{21}(\bar{X} - \bar{Y}).$$

Again it is necessary to invert the transforms of displacement and then to use eqns (17) to calculate the radial stress transmitted across the interface. The hoop force and bending moment can also be obtained from eqns (24).

4. INVERSION OF TRANSFORMS

In the previous analysis, solutions were found for the Laplace transforms of the quantities of interest, e.g. $\bar{X}$, $\bar{Y}$, etc. In order to recover the coefficients X, Y, etc. these transforms must be inverted. This can be achieved by an application of the Complex Inversion Theorem, i.e. for the general function $f(t)$ the inversion is given by

$$f(t) = \frac{1}{2\pi i}\int_C \bar{f}(s)\, e^{st}\, ds \tag{26}$$

where s is a complex variable and C is any contour in the complex plane such that all singularities of $\bar{f}(s)$ lie to the left of C. For the results presented in this paper, the integration was performed numerically using the contour and the efficient numerical scheme developed by Talbot[4].

5. RESULTS

A limited parametric study has been carried out in order to assess the importance of certain parameters upon the time dependence of the behaviour of the tunnel lining. The parameters considered were:

(a) (d/a) = the ratio of the lining thickness to the interface radius,

(b) (E_l^*/G_s) = the ratio of the "effective" modulus of the lining material to the shear modulus of the soil (note that $E_l^* = E_l/(1 - \nu_l^2)$),

(c) the interface condition, either perfectly rough, or perfectly smooth.

Only one value of Poisson's ratio was considered for the elastic soil skeleton, viz. $\nu_s = 0$.

It was noted previously that the solution to the overall problem can be found from

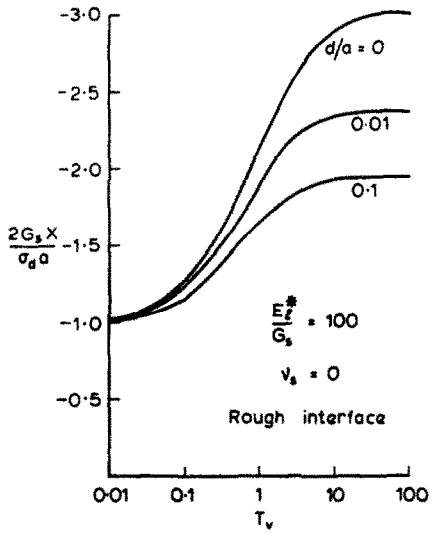

Fig. 4. Coefficient of radial displacement of tunnel lining—$n = 2$.

superposition of two sets of results; one corresponding to $n = 0$ and the other to $n = 2$ in the Fourier treatment. It was also pointed out that the solution corresponding to $n = 0$ was independent of time and explicit expression for the quantities describing the behaviour of the lining are given as eqn (19). Hence, in this section, we need consider only the components corresponding to $n = 2$. The results in this case are functions of time and are presented graphically in Figs. 4–19. On each plot the abscissa is a measure of non-dimensional time, defined by

$$T_v = c\frac{t}{a^2} \tag{27}$$

where t denotes the time elapsed from the instant of loading.

(a) *Effect of* d/a

Figures 4–10 show the results of analysis of a tunnel with a perfectly rough interface for three different values of the parameter d/a, namely 0 (corresponding to an unlined but impermeable tunnel), 0.01 and 0.1. These figures can be regarded as showing the effects

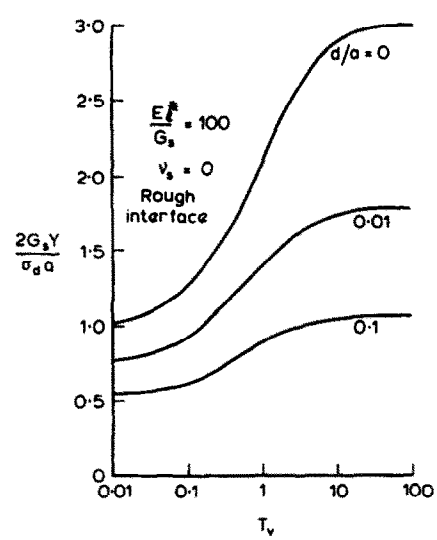

Fig. 5. Coefficient of circumferential displacement of tunnel lining—$n = 2$.

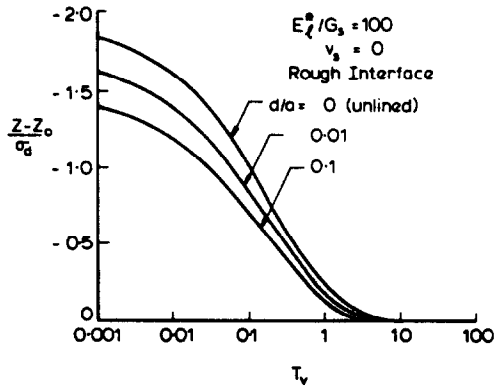

Fig. 6. Coefficient of excess pore pressure at interface—$n = 2$.

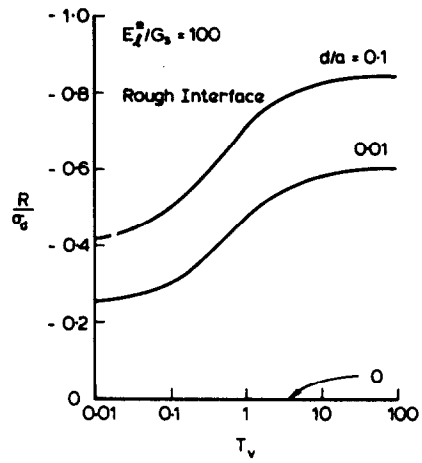

Fig. 7. Coefficient of total radial stress at interface—$n = 2$.

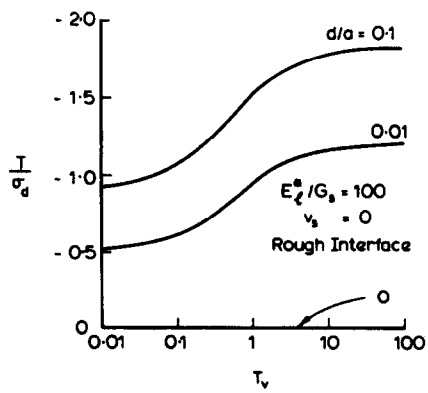

Fig. 8. Coefficient of shear stress at interface—$n = 2$.

of no lining and of two finite thicknesses of lining in tunnels where the lining material and the soil have a particular ratio of material stiffness, i.e. $E_l^*/G_s = 100$.

Figures 4 and 5 show that, as would be expected, the lining stiffness increases with lining thickness. Not only does this reduce the magnitude of the lining displacements at any given time, but it also decreases the variation with time of these displacements. Figure 6 shows the decay of excess pore pressure at the tunnel interface and demonstrates that the excess pore pressure at any time depends on the lining thickness; the thicker the lining, the smaller is the excess pore pressure. Figures 7 and 8 indicate that greater radial stress

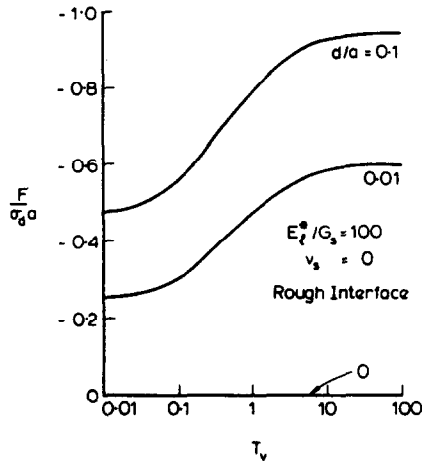

Fig. 9. Coefficient of hoop force in lining—$n = 2$.

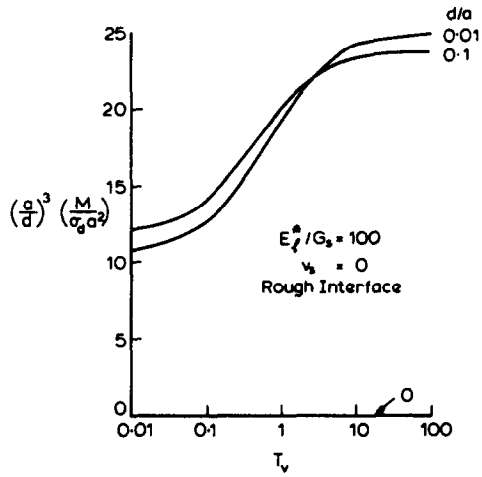

Fig. 10. Coefficient of bending moment in lining—$n = 2$.

and shear stress must be carried by the lining if its thickness is increased. In addition, the magnitude of these stresses will increase with time, asymptotically approaching finite limits which depend on the lining thickness. Figure 9 indicates a similar trend in the hoop force which will be induced in the lining as a result of this stress transfer across the interface. However, Fig. 10 indicates that within the range $0.01 < d/a < 0.1$ the bending moments induced in the lining at any time are almost directly proportional to the bending stiffness of the lining and hence proportional to the cube of the lining thickness (d^3). For the cases examined, it was found also that the bending moments increased by a factor of about 2–2.5 due to consolidation of the elastic soil.

(b) *Effect of* E_l^*/G_s

In this study, only tunnels with $d/a = 0.1$ and a rough interface were considered. Predictions of the behaviour of the lining are shown in Figs. 11–17. A number of values of the parameter E_l^*/G_s have been investigated covering the range from 0 (unlined or lining with no stiffness or perfectly rigid ground) to ∞ (perfectly rigid lining or soil with no stiffness). Figures 11–13 show that, in general, as the stiffness of the lining material is increased, the magnitudes of the displacements and pore pressures at any time are reduced and, furthermore, the variation with time of these quantities is also reduced. Figures 16 and 17 indicate that stiffer lining materials attract larger hoop forces and bending moments and, as consolidation proceeds, the relative variation of these quantities with time is

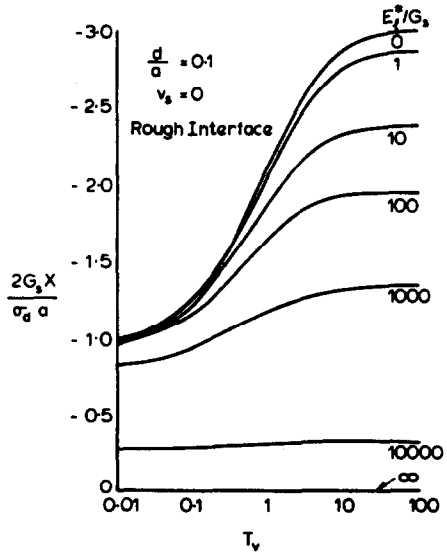

Fig. 11. Coefficient of radial displacement of tunnel lining—$n = 2$.

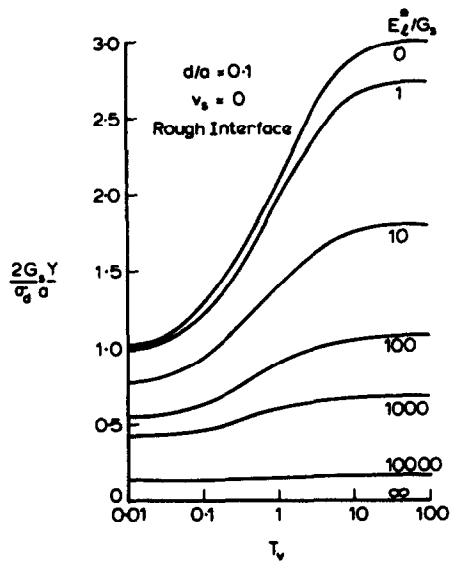

Fig. 12. Coefficient of circumferential displacement of tunnel lining—$n = 2$.

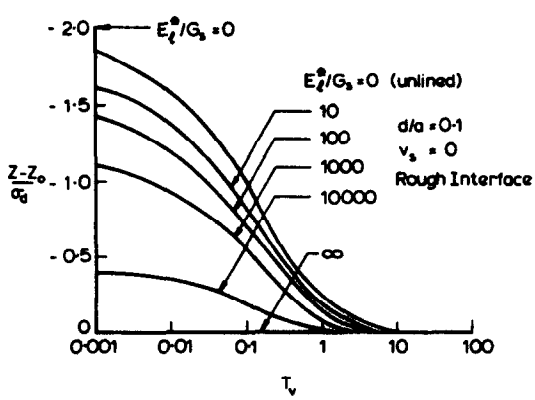

Fig. 13. Coefficient of excess pore pressure at interface—$n = 2$.

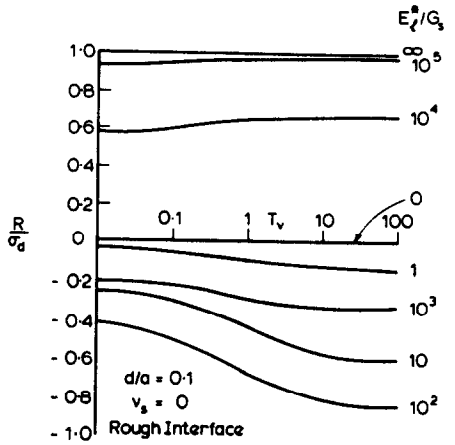

Fig. 14. Coefficient of total radial stress at interface—$n = 2$.

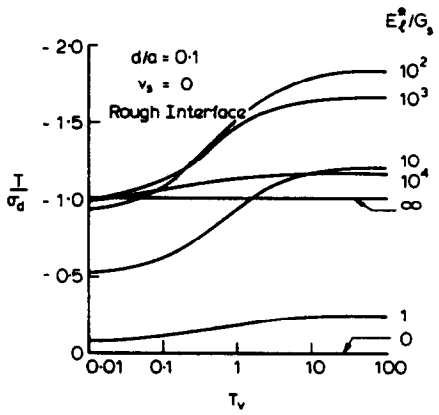

Fig. 15. Coefficient of shear stress at interface—$n = 2$.

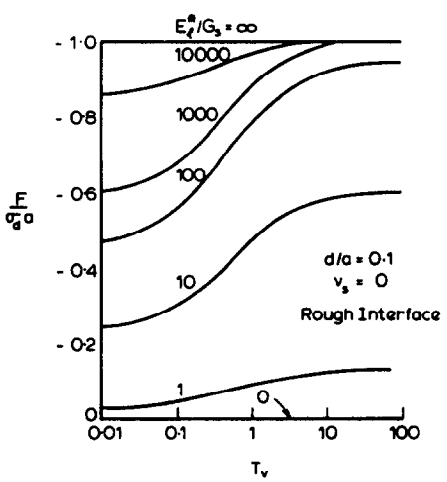

Fig. 16. Coefficient of hoop thrust in lining—$n = 2$.

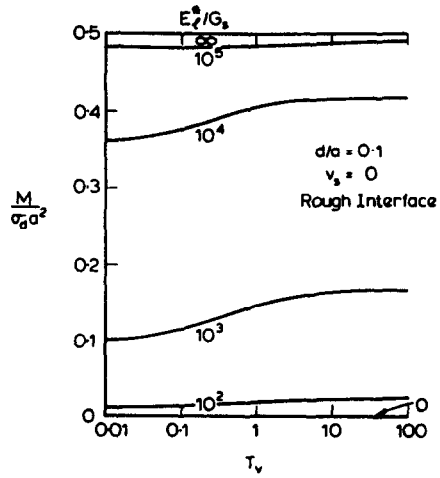

Fig. 17. Coefficient of bending moment in lining—$n = 2$.

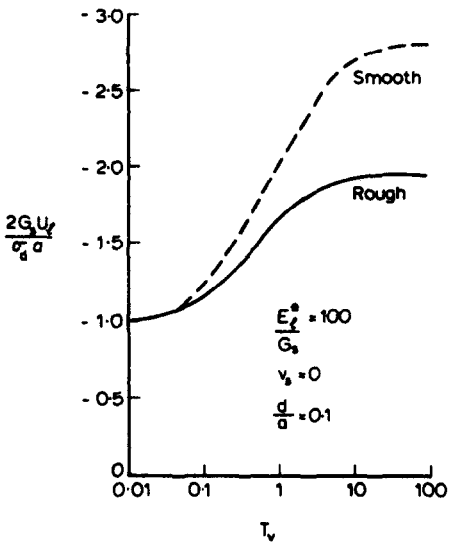

Fig. 18. Coefficient of radial displacement of tunnel lining—$n = 2$.

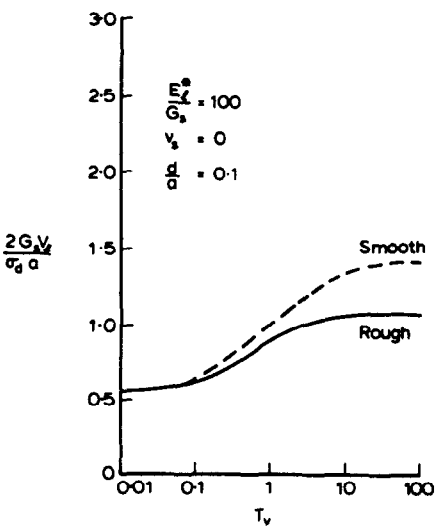

Fig. 19. Coefficient of circumferential displacement of tunnel lining—$n = 2$.

smaller for stiffer materials. Figures 14 and 15 show variations of the radial and the shear stress at the interface. For all finite values of E_l^*/G_s there is some variation in these quantities as consolidation occurs. However, the figures also show an interesting trend in behaviour with variations in the quantity E_l^*/G_s. This occurs because for thin linings, the behaviour cannot be separated independently into different functions of the two parameters d/a and E_l^*/G_s. These parameters are linked and, in fact, the response of the lining is dependent on both its relative hoop stiffness, $E_l^* t/(G_s a)$, and its relative bending stiffness $E_l^* t^3/12G_s a^3$). This matter is treated in greater detail in an earlier paper on the behaviour of lined tunnels in (non-consolidating) elastic ground[2], but the basic effects are evident from the nature of the governing equations (see eqns 22).

(c) *Effect of the interface condition*

The effect of the condition assumed for the interface between the lining and the consolidating soil can be seen in Figs. 18 and 19, where results are given for the lining displacements. Both sets of results, for a smooth and a rough interface, are for the case where $d/a = 0.1$, $E_l^*/G_s = 100$ and $\nu_s = 0$. The initial displacements are the same but the displacement components generally are larger for the smooth tunnel and the variation as consolidation occurs is also greater.

6. SOLUTION SYNTHESIS—SOME EXAMPLES

The treatment above has covered separately the two components of the overall problem and what is required now is the assembly of these components into a composite solution. In order to illustrate this procedure, and also to demonstrate the influence of the *in situ* (initial) stress state, three examples are considered. In each it is assumed that $E_l^*/G_s = 100$, $\nu_s = 0$, $d/a = 0.1$ and the interface is rough. The examples can be interpreted as three tunnels of the same size, lined with the same thickness of the same material, with each tunnel passing through the same type of soil. The only difference between each example is the magnitude of the initial stress state. Three different values have been selected for the earth pressure coefficient K_0 viz. (a) $K_0 = 0.5$; (b) $K_0 = 1$; (c) $K_0 = 2$. Values such as these cover a range from normally consolidated clay through to over-consolidated clay. If it is assumed that the water table is at the surface and that the ratio of the bulk unit weight of the soil to that of water is 2, it is possible to express all stress quantities in terms of σ_v, the initial total overburden pressure. When stresses are expressed in this way, a comparison of the results of each example can be made on the basis that each of the three tunnels is at the same depth below the surface (because σ_v is an indicator of the depth at which the tunnel is bored). A summary of the three initial stress states is given in Table 1.

The results of syntheses of Case 1 and Case 2 solutions for each of the three examples are set out in Figs. 20–26. The variation with θ of the actual displacements u_l, v_l, the excess pore pressure p, the stresses σ_{rr}, $\sigma_{r\theta}$ and the hoop force and bending moment f, m, have been shown ($\theta = 0$, $\frac{1}{2}\pi$, π correspond to the invert, spring line and crown of the tunnel,

Table 1. Initial stresses for example problems

Quantity	Example		
	(a)	(b)	(c)
$K_o = \sigma_H'/\sigma_v'$	0.5	1.0	2.0
$N = \sigma_H/\sigma_v$	0.75	1.0	1.5
σ_m/σ_v	0.875	1.0	1.25
σ_d/σ_v	0.125	0	-0.25
p_o/σ_v	0.5	0.5	0.5

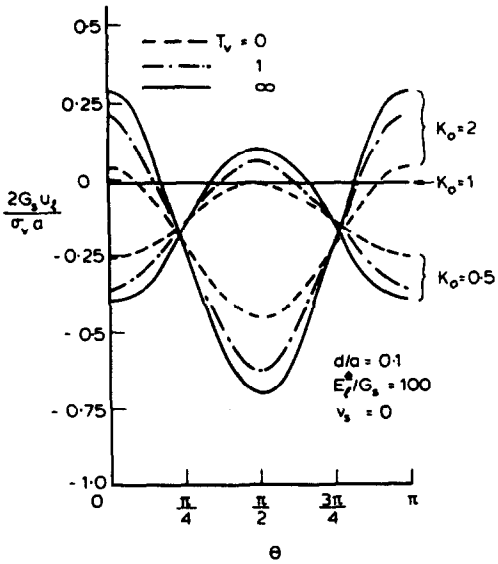

Fig. 20. Radial displacement of tunnel lining.

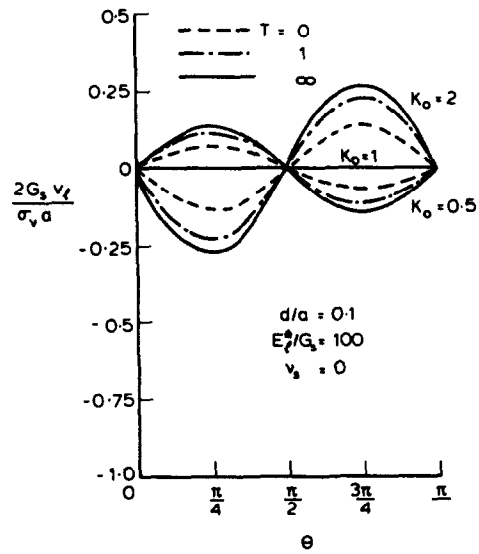

Fig. 21. Circumferential displacement of tunnel lining.

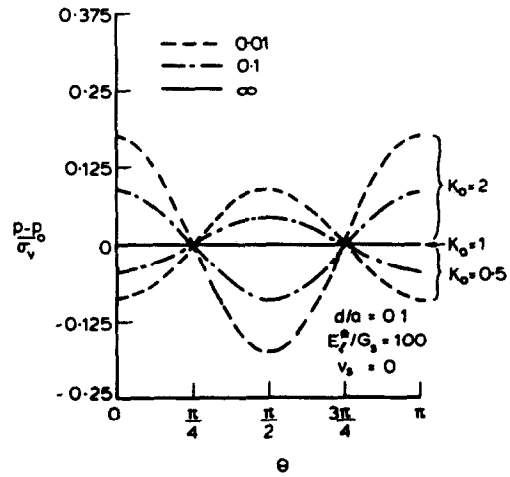

Fig. 22. Excess pore pressure at tunnel interface.

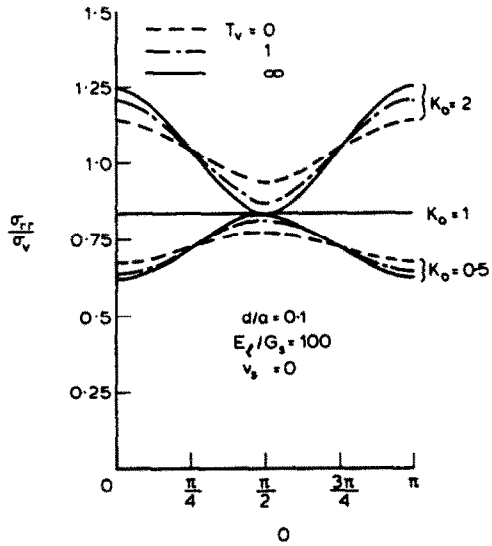

Fig. 23. Total radial stress at the interface.

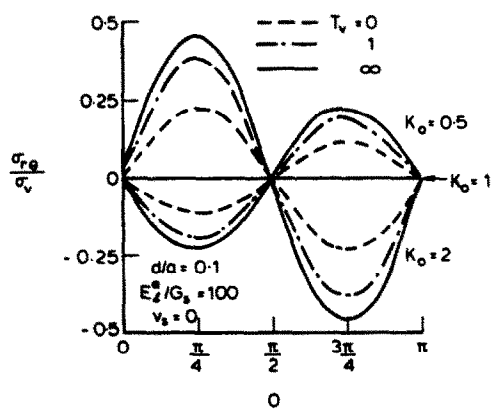

Fig. 24. Shear stress at the interface.

respectively). Curves have been plotted which correspond to various values of non-dimensional time T_v, as indicated. Some general features may be noted.

(i) The results for $K_0 = 1$ are independent of both time and position around the tunnel.

(ii) The results for $K_0 = 0.5$ ($\sigma_H < \sigma_v$) are "out of phase" with those for $K_0 = 2$ ($\sigma_H > \sigma_v$) by the angle $\pi/2$.

(iii) The maximum values of the various displacements, stresses and lining actions are greatest for the case $K_0 = 2$, since the soil in this example is initially more highly stressed than in the other examples.

(iv) The variation of each quantity with θ is of course sinusoidal and, furthermore, for $T_v > 0$ the "nodes" of each curve correspond to $\theta = (\pi/4)$, $(3\pi/4)$ or $\theta = 0$, $\pi/2$. Once the form of the initial response has been established (i.e. the curve for $T_v = 0$), then at subsequent times the curves maintain the same form but, in general, there is an increase in the magnitude of the variation from the mean response.

In particular, it can be noted that for each example the hoop force is everywhere compressive (see Fig. 25), even though the variation around the tunnel is sinusoidal. The average hoop thrust arises because of the mean *in situ* stress σ_m and variations from the mean occur when the *in situ* deviator stress σ_d has a finite value, i.e. when the *in situ* stress

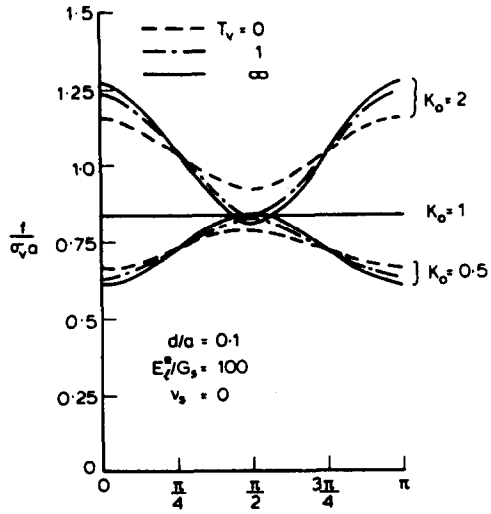

Fig. 25. Hoop force induced in the lining.

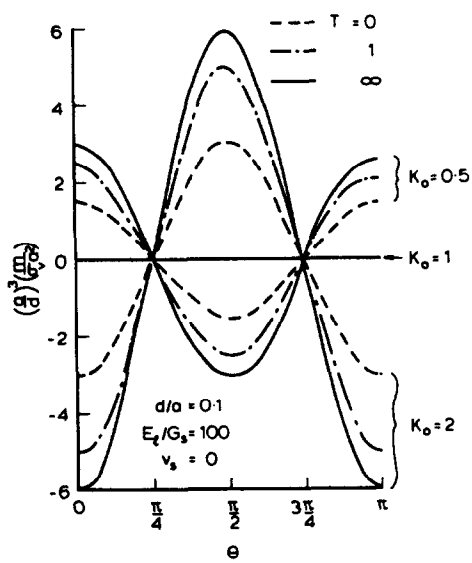

Fig. 26. Bending moment induced in the lining.

state is not hydrostatic. In contrast to this, the mean stress σ_m induces no bending moment in the lining; the bending action in the lining is due entirely to σ_d (see Fig. 26). The sinusoidal variation of the bending moment indicates that between $\theta = 0$ and π the circular beam is in contraflexure.

7. CONCLUSIONS

The analysis of the interaction between a thin, circular, elastic tunnel lining and the surrounding saturated elastic soil has been presented. The loading on the lining is due to the removal of material from within the tunnel (assumed to occur once the lining is in place) and the response of the lining is dependent on time because of consolidation within the surrounding elastic soil.

The analysis assumed that the depth of the tunnel is large when compared with its nominal radius and it was found convenient to separate the overall problem into two components. This allowed separate Fourier treatments for the removal of the mean *in situ* stress and removal of the *in situ* deviator stress.

Results of a parametric study were presented and it was noted that, if the initial state

of stress in the ground was hydrostatic, then the response of the lining was independent of time and, conversely, if the deviator stress was non-zero, then there was significant time dependence of the behaviour of the lining.

Some examples were also presented for the synthesis of the complete solution. It was found that the response of the lining will depend on the magnitude of the relative hoop stiffness and the relative bending stiffness of the lining and upon the initial state of stress in the soil. Consolidation of the elastic soil will cause the maximum, total hoop thrust in the lining to be increased typically by only about 10% of its initial value, but the maximum bending moment can be increased by as much as 100%.

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