Prediction of Undrained Sinkhole Collapse

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Abstract: Sinkholes are surface depressions or shafts resulting from the collapse of a submerged cavity in soil. The cavities that lead to sinkholes form as a result of underlying geology in limestone areas, or as a result of human activity such as mining or leakage from a sewer. The formation of sinkholes is often sudden and can lead to extensive damage and loss of life, especially in urban areas. Much of the literature on the subject of sinkhole formation is empirical in nature, often being associated with specific locations. This paper presents the results of a study, using numerical modeling, of the undrained stability of the submerged cavities that lead to sinkhole formation. Finite-element limit analysis techniques (using programs developed at the University of Newcastle) are used to obtain upper and lower bound values of a suitable load parameter, which bracket the exact solution. The results are compared to analytical solutions, both from literature and derived independently.

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Introduction

Sinkholes are depressions or shafts formed at a soil surface due to changes in the soil and/or rock beneath. In limestone areas, the gradual solution of rock at a depth through the passage of groundwater leads to subsidence of the overburden of residual and deposited soil and a resulting saucer-shaped region, often termed a solution depression (Sowers 1996). Limestone (or karst) areas in Florida (Ruth et al. 1985; Wilson 1995) and South Africa (Brink 1984) provide many examples of these geologically generated sinkholes. Of greater concern to engineers are the sinkholes formed from the sudden collapse of an underground cavity. These often start as small cavities at the rock–soil interface, which expand through the action of water seeping down through the overburden [Fig. 1(a)]. At some point, the cavity becomes sufficiently large that the remaining overburden is no longer able to arch across the cavity and collapses.

Many sinkholes are precipitated by other than natural causes. In limestone areas, changes in groundwater levels caused by the needs of increased urbanization or mining can lead to sinkhole formation (Bell 1988; Kannan 1999; Tharp 1999). Unstable cavities may be formed from mining, or leakage from a faulty pipe. “Chimney caving” is a term used in mining for the formation of a sinkhole due to the migration of an unsupported mined void through the overlying material (Brady and Brown 1993). Numerous instances of sinkhole formation have also been recorded in the UK due to defective Victorian brick sewers. The initial sinkhole appearance at the surface is typically only a small indicator of the final size of the cavity, as shown in Fig. 1(b). The subsequent failure of the overhanging soil around the initial hole leads to a funnel-shaped depression with a hole at the center [Fig. 1(c)]. The hole may then become filled with soil or dumped refuse material (Sowers 1996).

This paper is concerned with predicting the collapse of a submerged spherical cavity which results in the characteristic surface depression often referred to as a sinkhole or doline. The numerical approaches adopted here supplement the considerable body of empirical or semiempirical literature. The research presented here will be of use to those assessing the stability of the ground at a site where some knowledge of the likely size and location of underground cavities (perhaps obtained via geophysics methods) is available. The results of this research may also help contractors to determine the size of the plant necessary to complete dynamic compaction.

The paper begins with a review of other research in this area followed by a description of the cavity collapse problem with idealized geometry and soil properties. Novel numerical procedures, which use the bound theorems of classical plasticity to obtain the main results of the paper, are then described, although the reader is directed elsewhere for the full exposition of these methods. The results of the numerical study are shown to provide close bounds on a load parameter for various configurations of a spherical cavity, allowing stability to be assessed. The results are compared with closed-form solutions, where these are available.

Predicting Sinkhole Formation—Previous Work

As much previous research regarding sinkhole prediction is closely related to particular sites or geology, most available research is descriptive rather than analytical (e.g., Sowers 1996). The traditional approach to predicting cavity collapse in soil, as opposed to the problem of locating the cavities themselves, has been to use physical modeling and analytical techniques. The stability of soils over cavities was investigated through centrifuge modeling by Craig (1990) and Abdulla and Goodings (1996).
These centrifuge experiments used idealized cavity configurations, where collapse resulted from the overburden weight alone. Craig (1990) examined the stability of a cylindrical cavity opened up under a two-layered clay sample using two sets of tests. In the first set of tests, overburden weight was gradually increased by increasing the speed of the centrifuge, until the clay layers failed into a preformed cylindrical cavity. In the second set of tests, the centrifuge ran at constant speed while sand was extracted from a void beneath the clay layers. Craig found that the assumption of a simple cylindrical rigid-block failure in the clay was adequate for both sets of tests, providing the ratio of effective overburden depth against cavity diameter was less than unity. In a similar study, Abdulla and Goodings (1996) investigated the stability of a cemented layer of sand overlying a cylindrical cavity, with and without overburden. (The situation modeled a soil profile resulting from groundwater extraction in arid regions). The main finding of this study was that the cemented sand layer failed along steeply inclined planes forming a truncated conical section. In thicker cemented layers, however, a compression dome formed with a height of 25–30% of the cavity diameter.

Analytical approaches to this problem are surprisingly rare in literature. One example is limit equilibrium as used by Brady and Brown (1993) to determine the likelihood of chimney caving in mining. While limit equilibrium techniques are straightforward, and have been used for many years in slope stability analysis, they have some serious shortcomings in comparison to other analytical methods, notably limit analysis (the method adopted in this study, which is described in more detail later in this paper). First, limit equilibrium techniques ignore the plastic flow rule for the soil thereby often employing collapse mechanisms that are not kinematically admissible. Second, limit equilibrium techniques enforce only global equilibrium, rather than equilibrium at every point in the soil (Yu et al. 1998). The result of these (and other) shortcomings is that one cannot determine if a collapse load determined by limit equilibrium is safe or unsafe, compared to the true (and unknown) collapse load. More rigorous approaches (e.g., limit analysis) are necessary to determine true upper and lower bounds to the collapse load (Yu et al. 1998). Another limitation of standard limit equilibrium approaches is the need to specify a failure mechanism a priori.

A similar problem to sinkhole formation arising in oil production was studied by Vaziri et al. (2001). They developed an analytical model of the stability of an axisymmetric region of “caprock” above a cavity adjacent to an oil well, formed by sand production. They assumed the overburden to behave elastoplastic with a Mohr–Coulomb failure criterion and, hence, determined the load on the caprock. The caprock itself was assumed to behave as an elastic no-tension material, where support to the overburden came from a compression dome formed in the rock layer. With these models, Vaziri et al. (2001) produced an analytical model to predict the radius of the cylindrical cavity below the caprock at collapse, and a similar numerical model using FLAC.

Another analytical approach to sinkhole formation was developed by Tharp (1999) who studied the likely expansion of a cavity leading to a sinkhole, including the effects of pore–water pressure changes. Tharp’s (1999) analysis began with a linear elastic, isotropic stress field in the soil around a cavity, followed by the development of radial cracks and loss of soil into the cavity. Tharp (1999) concluded that steady-state groundwater conditions should generally promote stability and that sinkhole formation resulting from rapid drawdown of the water table could be avoided by sufficiently slow lowering.

Similar problems to submerged cavity collapse have been studied for some years by geotechnical engineers interested in the stability of lined and unlined tunnels (Atkinson and Potts 1977; Davis et al. 1980; Muhlhaus 1985; Leca and Dormieux 1990). Davis et al. (1980) studied the stability of lined tunnels in rigid-plastic soil where the (unsupported) face was pressurized. They used centrifuge testing and analytical approaches based on the Method of Characteristics and limit analysis. They showed that the techniques yielding the widest range of solutions for this problem were the bound theorems of classical plasticity (commonly termed limit analysis). These theorems state that, for certain types of elastoplastic materials, two values of a load parameter can be derived for a typical collapse problem. One solution (the lower bound) is derived from a stress field, which fulfills equilibrium boundary conditions and nowhere exceeds the yield criterion. The second solution (the upper bound) is derived from a kinematically admissible collapse mechanism. The power expended by the movement of the various loads on the system, and the soil self-weight, is equated to the power dissipated in plastic flow, to obtain a load parameter.

Using limit analysis, Davis et al. (1980) deduced rigorous bounds for undrained collapse loads for plane strain headings, plane strain tunnels, and circular headings, which are similar problems to that of a submerged cavity collapse. The bound theorems were also used to develop solutions for the stability of tunnels in drained conditions (Atkinson and Potts 1977; Muhlhaus 1985; Leca and Dormieux 1990) and have been applied to other problems in geotechnics, as described elsewhere (Chen and Liu 1990). Drumm et al. (1990) used limit analysis to investigate the stability of a vertical cylindrical shaft in soil above bedrock, modeling a sinkhole immediately as it had formed [Fig. 1(b)] and before subsequent collapse to form a funnel-shaped depression. This is, however, a study of the situation once a submerged cavity has collapsed and is therefore of little relevance to this work.

The approach to be used in this paper is limit analysis, which provides both safe and unsafe solutions to the cavity collapse problem. While limit analysis has a number of drawbacks, which will be explained next, it is particularly useful in providing rigorous upper (unsafe) and lower (safe) bounds on the true collapse load, hence, giving an indication of the error associated with the solution. A fully drained analysis is required to model the effects of changing groundwater conditions on a submerged cavity. However, since this is the first study of this problem using limit analysis, it is natural to limit this study to undrained soil conditions. This avoids the increased difficulties associated with a drained analysis (which will be discussed briefly next) and deals with the important problem of a dry cavity. It is important to state that the modeling undertaken here is not of a cavity gradually increasing in size before collapse and the formation of a sinkhole. Rather, it is the stability of the cavity at collapse that is modeled.
Simplified Model of Cavity Collapse Leading to Sinkhole Formation

Problem Layout

The layout of the undrained cavity stability problem to be used is shown in Fig. 2. The submerged cavity is assumed to be spherical, of diameter $D$ and cover $C$. A similar assumption is used in Tharp (1999). The cavity may be subject to an internal normal stress, representing an internal pressure $\sigma_T$. The ground surface is horizontal and subject to a vertical surcharge $\sigma_S$. For all analyses presented here, these stresses are taken as positive when directed into the cavity face ($\sigma_T$) or vertically downward ($\sigma_S$).

The ground around the excavation is modeled as a rigid plastic Tresca material with constant unit weight $\gamma$. This material has a single strength property, the undrained shear strength $c_u$. The failure criterion matches the Mohr–Coulomb criterion for the case of zero-friction angle. The use of a single rigid-plastic material model for the soil in this problem is necessary as the bound theorems apply only to rigid-plastic materials with associated flow rules (Chen and Liu 1990). The use of this model for undrained soils is widely accepted and well understood (Lyamin and Sloan 2000). Were the object of this research the determination of drained soils is widely accepted and well understood (Lyamin and Sloan 2000). Were the object of this research the determination of drained strength is independent of total mean normal stress also admissible for any addition of isotropic stress, since undrained strength is independent of total mean normal stress (Davis et al. 1980). Therefore, stability is affected only by the difference between $\sigma_S$ and $\sigma_T$, i.e., the top line of ($\sigma_T-\sigma_S/c_u$).

Considering the upper bound theorem, the external power expended by the loads and the self-weight of the deforming soil mass $P_{ext}$ is given by

$$P_{ext} = \sigma_S \int_{A_S} u_n^S dA - \sigma_T \int_{A_T} u_n^T dA + \gamma \int_V u dV$$

(2)

where $u_n^S$ is the downward normal velocity on the cavity face; $u_n^T$ is the outward normal velocity on the cavity face; and $A_S$ and $A_T$ are the deforming areas on the cavity face and at the surface, respectively. The last integral on the right-hand side of Eq. (2) is taken over the soil volume $V$. Since undrained behavior is assumed in this research, the soil deforms at constant volume and the volume change at the surface must equal that at the cavity. Put in mathematical terms, this is

$$\int_{A_S} u_n^S dA = \int_{A_T} u_n^T dA$$

(3)

It is then possible to rewrite Eq. (2) in terms of the dimensionless groups just given as

$$P_{ext} = \left(\frac{\sigma_S - \sigma_T}{c_u}\right) \int_{A_S} u_n^S dA + \left(\frac{\gamma D}{c_u D}\right) \int_V u dV$$

(4)

hence showing that the use of ($\sigma_T-\sigma_S/c_u$) is also acceptable for the upper bound method.

Finite-Element Formulation of Bound Theorems

The conventional use of limit analysis in stability problems in geotechnics (in Davis et al. 1980, for example) involves the determination of an admissible stress field for the lower bound case and a kinematically admissible collapse mechanism for the upper bound. An alternative approach, developed over a number of years by Sloan and others (Sloan 1988, 1989), is to discretize the problem domain into elements and then apply the bound theorems on an element-by-element basis, achieving a solution via an optimization procedure, as described in detail elsewhere (e.g., Sloan 1988; Sloan and Assadi 1991). These formulations are not variations of the displacement finite-element method but employ the same idea of discretizations of a domain to obtain solutions.

Both upper and lower bound finite-element formulations result in large optimization problems, which can be solved by linear or nonlinear programming methods. The original finite-element formulations developed by Sloan (Sloan 1988, 1989) employed linear programming techniques and, while highly successful in producing solutions for a range of two-dimensional geomechanics problems (Sloan and Assadi 1991, 1992; Yu et al. 1998) they have proved to be cumbersome to use for problems requiring three-dimensional models (as discussed in greater detail next). Recent work has led to the use of nonlinear programming methods to solve the optimization problems resulting from the three-dimensional finite-element limit analysis formulations (Lyamin 1999; Lyamin and Sloan 2002a,b). These methods have dramati-

![Fig. 2. Layout of sinkhole formation problem](image-url)
cally reduced the amount of computer time required to find a solution for a three-dimensional problem and are the methods used to produce the results presented in this paper. Clearly, the idealized sinkhole formation problem studied in this paper (Fig. 2) is axisymmetric, however, three-dimensional methods have been used as axisymmetric finite-element bound formulations are undeveloped to date. The study also demonstrates the newly developed three-dimensional techniques and shows that the extension to an unsymmetrical cavity collapse problem would be straightforward. The nonlinear programming techniques have also been used to improve the accuracy of two-dimensional solutions considerably elsewhere (Lyamin and Sloan 2000; Augarde et al., unpublished, 2002). The three-dimensional finite-element formulation for each bound theorem is now explained.

The upper bound formulation proceeds, as does a conventional upper bound solution, by seeking a kinematically admissible collapse mechanism that fits both the imposed boundary conditions and the plasticity flow rule for the material (associated in this case). The domain is covered by a mesh of four-noded tetrahedral elements. The problem variables are the three velocities at each of the element nodes in the mesh, which may vary element to element. This allows velocity discontinuities to appear in the solution, similar to those between rigid blocks in a conventional upper bound formulation. This may arise from the need to satisfy kinematic admissibility, the plastic flow rule, and to ensure zero deformation in regions inside the yield surface (rigid-plastic behavior). Additional constraints on the element stresses arise from the yield criterion. For the Tresca criterion, these constraints are nonlinear (in Cartesian stress space) and have been previously dealt with by replacing them with a set of linear constraints (Sloan 1988). The purpose of this change was to allow linear programming techniques to be used for solving the optimization problem. This is no longer necessary using the recently developed nonlinear programming solution techniques (Lyamin 1999; Lyamin and Sloan 2002a,b). In fact the yield criterion appears as a set of nonlinear constraints on the unknowns. In a similar way, the lower bound theorem is recast as an optimization problem, although for the lower bound analysis, the primary variables at the finite-element nodes are stresses, not velocities. These stresses may vary linearly over an element. The finite-element meshes are similar to those used for the upper bound analysis (i.e., comprised of four-noded tetrahedral elements) and are automatically generated from user definition of subdomains (Lyamin and Sloan, unpublished, 2002c). The objective function, to be maximized, is the integral of the normal stresses over some part of the domain. (For the case of the undrained cavity problem, this is the difference in load carried by the surcharge $\sigma_s$ and the cavity pressure $\sigma_f$.) The constraints on the solution are those imposed by equilibrium, the stress boundary conditions, and the yield criterion. Since each element node has an individual set of stresses, stress discontinuities are possible throughout the mesh, providing they do not contravene equilibrium; i.e., along element edges, shear, and normal stresses must be the same but tangential stresses can differ (Sloan 1988). Unlike the upper bound finite-element formulation, a rigorous lower bound is only obtained if the stress field satisfies equilibrium, the stress boundary conditions and the yield criterion throughout the domain, to theoretically infinite boundaries. To achieve this in the formulation, special "extension" elements are included on the semi-infinite edges of the domain. These elements extend the stress field beyond the limits of the mesh so that statical admissibility is satisfied. An alternative approach is to undertake successive analyses, progressively increasing the size of a mesh until the calculated collapse load ceases to change. The former approach, using extension elements is preferred since it guarantees a rigorous solution from a single analysis and avoids any trial-and-error solution procedure required in the latter approach. Once again, the yield criterion appears as a set of nonlinear constraints on the unknown solution variables and the problem is then solved using complex nonlinear programming techniques.

### Results

The results of 48 finite-element limit analyses of the undrained cavity collapse problem, using the methods just outlined, are presented in Table 1. All results presented here were obtained using an AMD Athlon processor running at 1200 MHz and generally required between 2000 and 4000 central processing unit seconds per analysis. These timings are impressive given the complexity of the finite-element meshes used in the analyses, and demonstrate the efficiency of the nonlinear programming approach.

The results are also presented as dimensionless stability charts in Figs. 3 and 4. (Upper bound results in Figs. 3 and 4 are referred to as “UB” and lower bound results as “LB”.) For comparison, the results of some analytical checks on the results are also included. Their derivation is discussed in the next section. The

### Table 1. Load Parameters from Analyses of Cavity Stability

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<th>$\gamma D/c_s$</th>
<th>$C/D$</th>
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<th>LB (analytical, Eq. (11) or (13))</th>
<th>LB (finite element)</th>
<th>UB (analytical, Eq. (9))</th>
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$^a$ LB results derived for a plane strain unlined tunnel from Davis et al. (1980)
charts are used to obtain upper and lower bound values of the load parameter \((\sigma_s - \sigma_T)/c_u\) given values of the weight parameter \((\gamma D)/c_u\) and geometry parameter \(C/D\). A worked example, which gives the surcharge at collapse for a cavity of diameter \(D = 3\) m and cover \(C = 3\) m, in a stiff clay and based on the finite-element limit analysis results, is given in Table 2.

In all analyses, the load parameter \((\sigma_s - \sigma_T)/c_u\) is optimized for set values of the other dimensionless groups \(C/D\) and \((\gamma D)/c_u\). Results are presented for values of the weight parameter \((\gamma D)/c_u\) of 0, 1, 2, and 3 and for values of the geometry parameter \(C/D\) from 1 to 6. These can be related to typical soil and geometry profiles by example as follows. Sowers (1996) indicates that a typical submerged dome diameter in stiff clay soils is \(2Z/3\) where \(Z\) is the depth to limestone. For weaker clays and sandy clays, the dome diameter is typically \(Z/5\). These dome sizes translate to \(C/D\) values of 0.5 and 4, respectively, assuming the spherical cavity to “sit” on the limestone. As for sinkhole formation due to the activities of man, Gregory and Walling (1987) state that sinkholes occur as a result of mining where overburden is less than 10 or 15 times the thickness of the mined seam. If the cavity formed due to mining is assumed to be twice the original seam thickness, then a \(C/D\) value of 5 is reasonable, and also lies within the range of this study. A similar exercise can be undertaken with the weight parameter \((\gamma D)/c_u\). Typical unit weights \(\gamma\) of saturated clays lie around 20 kN/m\(^3\) and undrained strengths of clays lie between 20 and 150 kN/m\(^2\). For an intermediate strength clay, therefore, these results provide solutions for cavity diameters of approximately 4, 8, and 12 m.
The results presented in Table 1, for both upper and lower bound methods, are obtained in practice by setting $\sigma_s$ to zero and optimizing for $\sigma_T$ alone. As explained herein, this is acceptable given the reliance of the undrained solution on the difference between $\sigma_s$ and $\sigma_T$. This procedure is adopted merely because it is marginally simpler to optimize for the tunnel pressure with the finite-element limit analysis software. For the upper bound, a uniform cavity pressure is simulated by imposing the loading condition on the cavity face equivalent to

$$\int_A v_T^T dA = 1$$

(5)

This does not restrict the upper bound solution to have a constant velocity profile over the moving areas of the cavity and, hence, leads to superior upper bound solutions to those produced by conventional rigid-block mechanisms.

A typical mesh used to analyze a cavity collapse, with $C/D = 1$, is shown in Fig. 5. Similar meshes are used for upper and lower bound methods although “extension” elements are also included along the soil domain boundaries for the lower bound analyses. Only a 15° sector of a cylinder of soil containing a cavity is modeled due to the axisymmetry of the problem. The mesh shown in Fig. 5 has 40,752 nodes and 10,188 tetrahedral elements. The stress boundary conditions for the lower bound analyses ensure zero shear and normal stress at the surface, zero-shear stress on the cavity face and on the vertical faces of the meshed sector. An upper bound analysis using the mesh in Fig. 5 includes velocity boundary conditions to ensure zero velocity at the outer boundary of the soil cylinder. As indicated in the description of the upper bound formulation, velocity discontinuities are present between each element in the mesh.

**Comparison of Analytical and Numerical Results**

It is important to check any numerical results against those from other analytical (i.e., non-numerical) solutions wherever possible. A simple lower bound solution for the sinkhole problem can be found for a weightless soil ($\gamma D/c_u = 0$), from the spherically symmetric annular stress field shown in Fig. 6. Once symmetry has been accounted for, equilibrium of the radial stress $\sigma_r$ and the two equivalent tangential stresses $\sigma_u$ can be written as

$$\frac{d\sigma_r}{dr} + \frac{2}{r} (\sigma_r - \sigma_u) = 0$$

(6)

For a solution that is a lower bound, the Tresca yield criterion gives

$$\sigma_u = \sigma_r + 2c_u$$

(7)

Substituting Eq. (7) and the stress boundary conditions on the annular region into Eq. (6) gives

$$\sigma_r = \sigma_T + 4c_u \ln\left(\frac{2r}{D}\right)$$

(8)

Outside the annular region of radius $C + D/2$ there is an isotropic compressive stress field $\sigma_s$. The lower bound solution, in terms of the load parameter outlined herein, therefore, is

$$\frac{\sigma_s - \sigma_T}{c_u} = 4 \ln\left(\frac{2C}{D} + 1\right)$$

(9)

Lower bound solutions, including for the effect of soil self-weight, require more complex procedures such as the Method of Characteristics, one of the methods used for the two-dimensional study of tunnel stability in Davis et al. (1980). Since the Method of Characteristics failed in many cases to provide solutions to that problem (i.e., the yield was violated) and there appears to be no other work involving Characteristics solutions using this method in three dimensions, no further solutions are derived here. However for comparison, the lower bound solutions from Davis et al.
external work is expended as a product of the block velocity and the vertical component of the force due to the internal pressure $\sigma_T$. Additional external power is expended in the movement of the weight of the rigid block if $(\gamma D)/c_u \neq 0$.

For a weightless soil, the load parameter can be determined as

$$\left. \frac{(\sigma_S - \sigma_T)}{c_u} \right|_{\gamma D/c_u = 0} = \frac{4}{\sin \alpha} \left( \frac{C}{D} + \frac{1}{2}(1 - \cos \alpha) \right)$$  \hspace{1cm} (10)

Maximizing Eq. (10) with respect to $\alpha$ gives an optimum upper bound as

$$\left. \frac{(\sigma_S - \sigma_T)}{c_u} \right|_{\gamma D/c_u = 0} = 4 \sqrt{\frac{C/D}{(1 + 2/C/D)^2}} \left( 1 + \frac{2C}{D} \right)$$  \hspace{1cm} (11)

For a soil with self-weight, Eq. (10) is replaced by

$$\left. \frac{(\sigma_S - \sigma_T)}{c_u} \right|_{\gamma D/c_u \neq 0} = \frac{4}{\sin \alpha} \left( \frac{C}{D} + \frac{1}{2}(1 - \cos \alpha) \right)$$

$$- \frac{\gamma D}{6} \left( \frac{\tan^2(\alpha/2)}{2} \left( \frac{6C}{D} + 1 + 2 \cos \alpha - \cos 2 \alpha \right) \right)$$  \hspace{1cm} (12)

For $(\gamma D)/c_u \geq 1$, the optimal upper bound solution, using the single-block mechanism of Fig. 8, is found with $\alpha = \pi/2$. This simplifies the expression for the load parameter to

$$\left. \frac{(\sigma_S - \sigma_T)}{c_u} \right|_{\gamma D/c_u = 1} = 4 \left( \frac{C}{D} + \frac{1}{2} - \frac{\gamma D}{c_u} \right) \left( \frac{C}{D} + \frac{1}{6} \right)$$  \hspace{1cm} (13)

Other rigid-block mechanisms are difficult to develop due to the three-dimensional nature of the problem.

**Discussion**

The plots shown in Figs. 3 and 4 include the analytical results developed herein. The examination of these plots and Table 1 show that in all but one case, the finite-element results give improved theoretical bounds on the load parameter, over the bounds provided by other rigorous methods. The one case in which an analytical solution provides a better solution (i.e., a lower upper bound) occurs for $C/D = 1$, $(\gamma D)/c_u = 0$. (The finite-element solution predicts a load parameter of 5.75, while the analytical solution gives 5.66.) Apart from this result, the finite-element results give much-improved bounds on load parameters for cavity collapse. The “gap” between the upper and lower bounds for finite-element results is a function of the refinement of mesh used for the analyses. Improved bounds could be found by increasing refinement, at the cost of increased computer time. The lower bound analytical solution for the weightless case [Eq. (9)] is very close to the lower bound finite-element result for all values of $C/D$. The analytical lower bounds for $(\gamma D)/c_u \neq 0$ taken from Davis et al. (1980) for a plane strain tunnel, where available, are all considerably below the lower bound solutions derived by the finite-element approach.

It is clear, however, that the analytical upper bound solution using the single-variable mechanism of Fig. 8 provides very poor results in comparison to the finite-element results particularly when the soil self-weight is high. To explain this difference, it is instructive to see the mechanisms predicted by the finite-element upper bound method. The mechanisms can be deduced from Figs. 9 and 10, where the velocity vectors at nodes in the mesh, in four optimized upper bound solutions, are shown. The mesh itself is

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Fig. 7. Unlined plane strain tunnel stability problem (after Davis et al. 1980).

Fig. 8. Upper bound rigid-block mechanism
Fig. 9. Velocity vectors from upper bound finite-element limit analysis of cavity collapse problem: (a) C/D = 1, (γD)/c_u = 0 and (b) C/D = 1, (γD)/c_u = 3

omitted for clarity, and the vectors are shown on one face of the
sector, to provide a clear two-dimensional plot. For high soil
weight, the mechanism can be seen to spread further into the
domain around the cavity. For the shallower cavity C/D = 1 (Fig.
9), the zone of influence spreads below the cavity axis as the soil
weight parameter increases from (γD)/c_u = 0 to (γD)/c_u = 3. As
the cavity is deepened, this effect becomes more pronounced (Fig.
10) with the zone of influence stretching down to the cavity in-
vert. The mechanism appears to change from a predominantly
downward movement to a downward movement plus some rota-
tion close to the cavity. The cause of this effect is likely to be a
spread in the plastic region around the cavity due to increased
vertical stress from the increased cavity depth, and due to soil
weight. This effect also appears to suggest that the failure mode
becomes localized around the cavity as it is deepened, hence,
explaining the poor results given by the single-variable block
mechanism of Fig. 8.

The same effect, of localization of failure, is seen in stability
analyses of tunneling problems (Davis et al. 1980), although the
effect is more pronounced in cohesionless soils (Lyamin and
Sloan 2000). While the initial failure as predicted by the finite-
element results may be localized, such a failure could destabilize
the ground around the cavity and lead to a failure that would
reach the surface.

The location of the exact collapse load parameter between the
upper and lower bounds shown in Figs. 3 and 4 is obviously
impossible to determine. More refined finite-element analyses
would lead to a reduction in the size of this gap between bounds,
as has happened with two-dimensional analyses of similar geo-
technical problems (Augarde et al., unpublished, 2002). As indi-
cated herein, finer meshes would isolate the collapse load with
greater accuracy. These results predict that unpressurized cavities
with zero-surface surcharge will be stable in all cases studied
here, providing the weight parameter (γD)/c_u > 1 (with the as-
sumptions of material behavior and geometry made initially).
This situation corresponds to a positive value of the load param-
eter (σ_s − σ_f)/c_u. This also holds for low C/D values for
(γD)/c_u = 2.

As is noted herein, it is difficult to obtain analytical lower
bound solutions for cases where (γD)/c_u ≠ 0. In their study of
the stability of tunnels, Davis et al. (1980) state that lower bound
solutions for (γD)/c_u > 0 can be found by adding a hydrostatic
stress field to the weightless case [i.e., Eq. (9) in this paper] to
account for the weight of soil overburden. This does not, they
admit, provide a rigorous lower bound on the original problem,
since the tunnel pressure (in their problems) must now vary “hy-
drostatically” over the tunnel depth. Since most assessments of
cavity stability will assume zero internal cavity pressure, the use
of this procedure appears unsatisfactory here.

In the absence of an analytical solution then, a rigorous lower
bound, for the ranges of the dimensionless parameters of this
study, can be derived from a fit to the finite-element results as

\[
\frac{(σ_s − σ_f)}{c_u} = 4 \ln \left( \frac{2C}{D} + 1 \right) - \frac{γD}{c_u} \frac{C^3}{D^4 + 4}
\]  

(14)

(14) is the weightless solution [Eq. (9)] less a component
based on the weight parameter and is included in the plots shown
in Figs. 3 and 4 to demonstrate its applicability. The second term
in Eq. (14) is equivalent to the vertical pressure in the soil at a
distance D/4 below the cavity center, divided by the undrained
soil strength. The significance of this point on the lower bound
solution is not apparent. Eq. (14) is a rigorous solution since the
yield criterion is convex and the flow rule is associated (Sloan and
Assadi 1992). A similar exercise can be undertaken for the upper
bound finite-element results, giving an apparent bilinear fit. Only
the lower bound is presented here, as it is likely to be of greater
use to stability assessment, providing as it does a safe value of
surface surcharge.

Since the problem is axisymmetric, it might be possible to
develop other analytical bound solutions, although this is not at-
ttempted here since the finite-element methods produce acceptable
results. As stated herein, however, truly three-dimensional ver-
sions of this problem, perhaps with nonsymmetrical surface sur-
charges, could only reasonably be attempted using the finite-
element limit analysis procedures.

Although beyond the scope of the current paper, extending the
study to a cavity in a cohesionless or cohesive–frictional soil is
straightforward using the finite-element limit analysis procedures.
Initial (two-dimensional) studies of the circular unlined tunnel
problem (Lyamin and Sloan 2000) and the plane strain heading
problem (by Augarde et al., unpublished, 2002) show that precise
results are somewhat harder to obtain, due to the increased com-
plexity of the Mohr–Coulomb failure criterion. Failure is domi-
ated by local collapse, due to the need to enforce associated flow in the limit analysis procedures. The results obtained are nonetheless closely bounded and improve on the bounds provided by the few analytical solutions already available.

Conclusion

The problem of predicting sinkhole formation resulting from the collapse of an underground cavity has, in the past, generally been approached using mainly empirical and semiempirical approaches. Limit analysis techniques, while requiring the use of yield criteria with associated flow rules, nevertheless provide robust methods for finding rigorous bounds on true collapse loads for many problems in geomechanics that are not unduly kinematically constrained. An idealization of a spherical cavity collapse that would lead to the formation of a sinkhole has been studied in this paper. Finite-element limit analysis has been employed to give rigorous bounds on a suitable load parameter, with which it is possible to assess the stability of a cavity under undrained conditions. The finite-element results improve on the few analytical (i.e., non-numerical) solutions available in the majority of cases studied here.

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Notation

The following symbols are used in this paper:

- \( A_s \) = deforming area of surface;
- \( A_r \) = deforming area of cavity;
- \( C \) = cover to cavity;
- \( D \) = cavity diameter;
- \( P_{ext} \) = externally expended power;
- \( P_{int} \) = internally dissipated power;
- \( V \) = domain volume;
- \( Z \) = depth to limestone;
- \( c_u \) = undrained shear strength;
- \( u \) = velocity;
- \( v_s \) = normal velocity of deforming surface;
- \( v_n \) = normal velocity of deforming cavity surface;
- \( \alpha \) = rigid-block mechanism characteristic angle;
- \( \gamma \) = soil unit weight;
- \( \sigma_r \) = radial direct stress;
- \( \sigma_s \) = surface surcharge;
- \( \sigma_T \) = cavity stress; and
- \( \sigma_0 \) = tangential direct stress.

References