MATHS AND STATS

Limits

Given a function $f(x)$, finding $\lim_{x\to a} f(x)$ involves investigating the value of the function *f* as the value of *x* approaches *a*.

Example

Given $f(x) = 3x+1$ find $\lim_{x\to 2} f(x)$.

From the table below you can see that as *x* gets closer and closer to 2 the value of *f(x)* gets closer to 7. So $\lim_{x\to 2} f(x) = \lim_{x\to 2} 3x+1 = 7$

x	1.8	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	6.4	6.7	6.97	6.997		7.003	7.03	7.3

Approaching from below → ← Approaching from above

When our function *f(x)* is continuous then $\lim_{x\to a} f(x) = f(a)$ (i.e. we can simply evaluate the function *f* at *x* = *a*.) This method of finding the limit is called **direct substitution**.

Common challenges

Zero denominator

Sometimes direct substitution results in a denominator of zero.

Find $\lim_{x\to 5} f(x)$ when $f(x) = \dfrac{x^2-3x-10}{x-5}$ (try direct substitution)

$$\lim_{x\to 5} = \frac{x^2-3x-10}{x-5} = \frac{0}{0}$$ which is undefined

Here *x* – 5 is a factor of the numerator, so we can factorise the numerator and cancel the denominator giving

$$\lim_{x\to 5} = \frac{x^2-3x-10}{x-5} = \lim_{x\to 5}\frac{(x-5)(x+2)}{(x-5)} = \lim_{x\to 5} x+2 = 7$$

Note that even though *x = 5* is not in the domain of $f(x)$, the limit as x approaches 5 still exists.

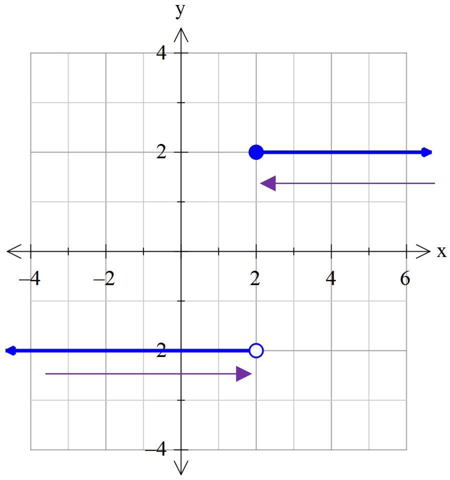
Potential limits don't match

Sometimes the limit from the left does not agree with the limit from the right.

For the piecewise function

$$f(x) = \begin{cases} 2 & \text{if } x \geq 2 \\ -2 & \text{if } x < 2 \end{cases}$$

Find $\lim_{x \to 2} f(x)$



If you let x approach 2 from below then $f(x)$ is always -2. If you let x approach 2 from above then $f(x)$ is always 2. As we get different values depending our direction of approach, we say there is no limit.

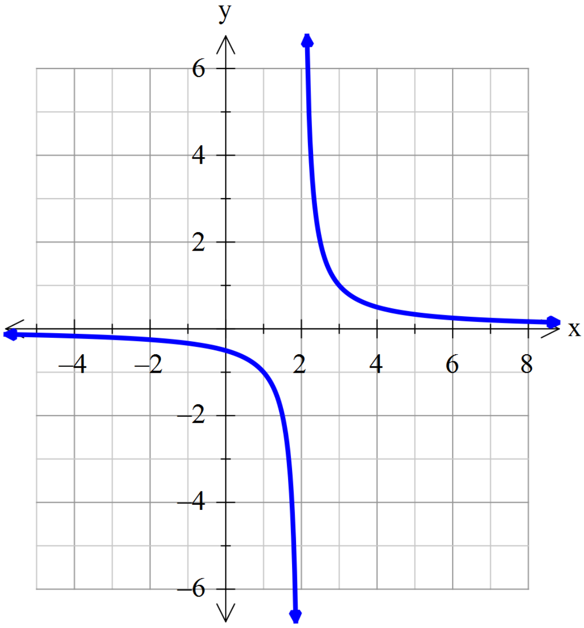
$\lim_{x \to 2} f(x)$ does not exist.

Unbounded growth

Sometimes the value of $f(x)$ grows without bound.

$$f(x) = \frac{1}{x-2}$$

Find $\lim_{x \to 2} \frac{1}{x-2}$



As x approaches 2 from above $f(x)$ grows without limit. Also as x approaches 2 from below $f(x)$ grows without limit but in the negative direction.

Again we say the limit does not exist.

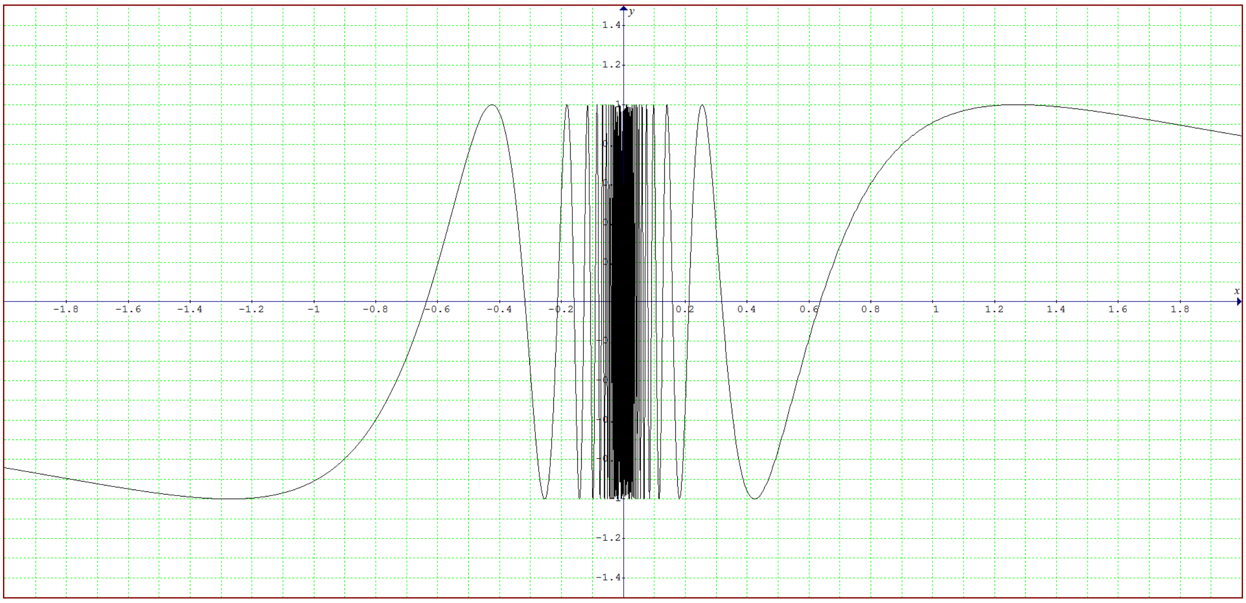
$\lim_{x \to 2} f(x)$ does not exist.

Oscillating values

Sometimes the limit does not settle down to a particular value

$$f(x) = \sin\left(\frac{2}{x}\right)$$

Consider $\lim_{x \to 0} \sin\left(\frac{2}{x}\right)$



$y = \sin\left(\frac{2}{x}\right)$

You can see from the graph that $f(x)$ keeps oscillating between -1 and 1 and will not settle to a single value.
Again we say the limit does not exist.

Exercises

Find the limit for the following questions. If the limit is not defined then write No limit.

1. $\lim_{x \to 2} x^2 - 3x$
2. $\lim_{x \to 0} 3x^2 - 7x + 4$
3. $\lim_{x \to 2} \frac{x^2 + 2x - 8}{x - 2}$
4. $\lim_{x \to 3} \frac{4}{x - 3}$
5. $\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$
6. $\lim_{x \to 5} \frac{x^2 - 1}{x - 5}$
7. $\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$
8. $\lim_{x \to 3} \frac{x^2 - 3x + 6}{x - 2}$

Answers

1. -2
2. 4
3. 6
4. No limit
5. -4
6. No limit
7. ½
8. 6