MATHS AND STATS

# Indices and Logarithms II

**A *log* is an *index***

Logarithms are another way of expressing powers. A logarithm is an index or power.

Index /power/exponent　　　　Number

**Base** → $2^3 = 8$　　　　... called **index form**

Can be expressed as　　Number　　　　Index

$\log_2 8 = 3$　　　　... called **log form**

Base

**Note** – Base (2), index (3) and number (8) are the same but in different positions.

$5^2 = 25$ ⟷ $\log_5 25 = 2$

When the bases are the same **in index form**, it can be easy to evaluate an index $x$ .

$$2^x = 8$$
$$2^x = 2^3 \qquad \text{... same bases so the indices must be equal}$$
$$x = 3$$

But what about $2^x = 10$?

We can guess $x$ is about 3 point something as $2^3$ was 8
Using logs or putting this into log form gives $\log_2 10 = x$.
This is asking 'What is the index on 2 ($x$) that results in 10?'.

$$\log_2 10 = 3.322 \qquad \text{using a calculator, shown later}$$

(check $2^{3.322} = 10$ on your calulator)

## Exercises

1. Rewrite $x^y = a$ in log form.

2. Rewrite $\log_p r = t$ in index form.

3. Rewrite $10^y = 50$ in log form.

4. Rewrite $\log_e 694 = t$ in index form.

**5.** Determine the following

a) $\log_3 9 =$

b) $\log_4 16 =$

c) $\log_{10} 1000 =$

d) $\log_6 36 =$

e) $\log_2 16 =$

f) $\log_3 27 =$

g) $\log_5 125 =$

h) $\log_2 32 =$

i) $\log_{10}\left(\frac{1}{1000}\right) =$

j) $\log_3 \frac{1}{9} =$

## Identities

For any $x$, $\log_x x = 1$ *and* $\log_x 1 = 0$

$(or\ \ x^1 = x)$ $(or\ \ x^0 = 1)$

## Logarithms on calculators

As you can see, a log can have many different bases.

Most scientific calculators only use two bases

- base 10 indicated by the **log** button
- and base *e* indicated by the **ln** button

(The **Euler number *e*** is a mathematical constant like $\pi$ and is **approximately 2.7182**)

**WARNING**

$\mathbf{\log 12.6}$ can be confusing as the base is not given. It is **usually base *e*** but it is worth checking if it is interpreted as $\log_{10} 12$ *or* $\log_e 12$

### Exercises

**6.** Determine the following

a) $\log_{10} 16 =$

b) $\log_{10} 81 =$

c) $\log_{10} 1000 =$

d) $\ln 16 =$

e) $\ln 1000 =$

f) $\ln 81 =$

## Change of base formula

When the index is a whole number, we can calculate the log result in our head or by calculator.

- $\log_3 9 = 2$ because we know $3^2 = 9$ as mentioned above
- But what do we do with $\log_3 10$ which does not have an exact answer and is not base 10 or base *e*?

When the base is different to **10 or *e*,** we can easily change the base using:

$$\log_{old} x = \frac{\log_{new} x}{\log_{new} old}$$

### Example

**Using base 10 …**

$\log_3 10 = \frac{\log_{10} 10}{\log_{10} 3} \approx 2.096$

or

**Using base *e* …**

$\log_3 10 = \frac{\log_e 10}{\log_e 3} \approx 2.096$

We can check by changing to index form $3^{2.096} \approx 10$ You can use either base!

### Exercises

**7.** Change the base then use your calculator to evaluate each of the following

a) $\log_3 50$

b) $\log_5 10$

c) $\log_7 100$

d) $\log_6 1.362$

## Solving equations

- **If an equation has an** unknown index, change the equation into log form and solve for the unknown.

Find $x$ when $10^x = 25 \longleftrightarrow \log_{10} 25 = x$

Using the calculator $x \approx 1.398$

- If the equation is in log form, change it into index form can help to solve for the unknown.

Find $x$ when $\log_{10} x = 1.9 \longleftrightarrow 10^{1.9} = x$

Using the calculator $x \approx 79.433$

**Note** - In both cases, changing the form moved all the numbers to one side of the equation leaving just the *x* on its own (making *x* the subject of the equation).

### Exercises

**8.** Change from log to index form or vice-versa to find the value of

a) $\log_{10} x = 0.5$

b) $\log_{10} x = 1.2$

c) $\log_{10} x = 1.8$

d) $10^x = 28$

e) $10^x = 45$

f) $10^x = 136$

g) $e^x = 212$

h) $e^x = 96$

i) $e^x = 13$

j) $\ln x = 0.2$

k) $\ln x = 1.8$

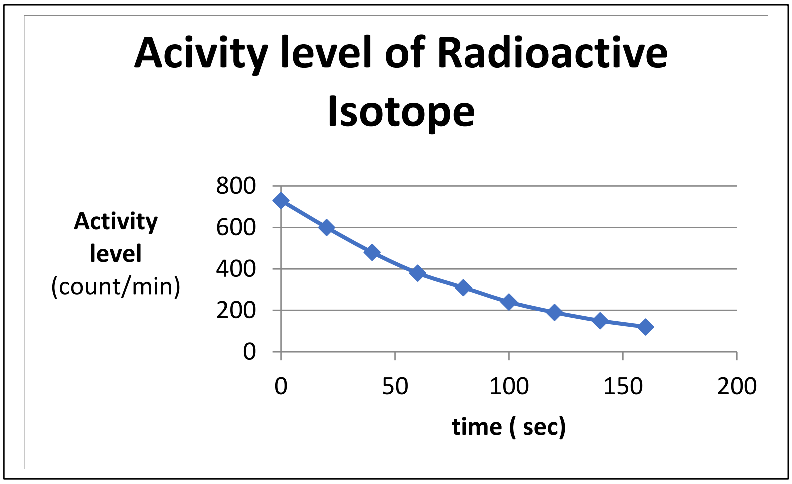
l) $\ln x = 2.3$

## Using logs

The following data shows the activity level, L, of a radioactive isotope after, t, seconds. Its graph is also shown.

Activity Level

| t (sec) | L (counts/min) |
|---|---|
| 0 | 730 |
| 20 | 600 |
| 40 | 480 |
| 60 | 380 |
| 80 | 310 |
| 100 | 240 |
| 120 | 190 |
| 140 | 150 |
| 160 | 120 |




We can see that there is a **relationship between the variables *time* and *count*** as there is a pattern formed by the points. As *time* increases, *count* decreases.

It is **useful to find equations of such relationships**. Since the graph is not a straight line it is more difficult to find its equation and this relationship.

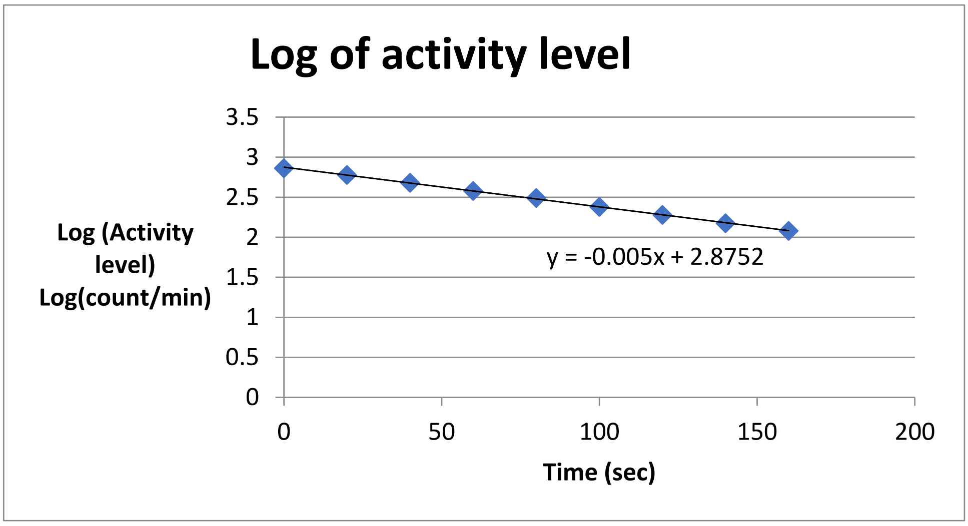
In situations like this, we can make it easier to find an equation of the relationship by changing the curve into a line. Look what happens when we **take the base 10 log** of the *count* values and draw the graph.

Activity Level

| t (sec) | Log (L) | L (counts/min) |
|---|---|---|
| 0 | $\log_{10} 730 = 2.8633228$ | 730 |
| 20 | 2.77815125 | 600 |
| 40 | 2.681241237 | 480 |
| 60 | 2.579783597 | 380 |
| 80 | 2.491361694 | 310 |
| 100 | 2.380211242 | 240 |
| 120 | 2.278753601 | 190 |
| 140 | 2.176091259 | 150 |
| 160 | 2.079181246 | 120 |

Activity Level

| t (sec) | log L (counts/min) |
|---|---|
| 0 | 2.86 |
| 20 | 2.78 |
| 40 | 2.68 |
| 60 | 2.58 |
| 80 | 2.49 |
| 100 | 2.38 |
| 120 | 2.28 |
| 140 | 2.18 |
| 160 | 2.08 |




Using logs can help to describe the relationship in a way that the equation is easy to determine.

The straight line equation is

$$y = -0.005x + 2.8752 \qquad \text{where } y = \log_{10} L$$

In terms of our variables *count* and *time*, the equation of the line becomes

$$\log_{10} L = -0.005t + 2.8752$$

It is preferable to express the dependent (y or $\log_{10} L$ ) variable without the log. The equation is expressed in log form so we can change it into index form and see how that looks.

In log form …

$$\log_{10} L = -0.005t + 2.8752$$

Identify the components …

Base is 10
Index is $-0.005t + 2.8752$ (the whole right hand side)
Number is $L$

In index form …

$$10^{-0.005t+2.8752} = L$$

Normally at this point we would use index laws to write the equation as

$$L = 10^{2.8752} \times 10^{-0.005t} \qquad \text{or} \qquad L = \frac{10^{2.8752}}{10^{0.005t}}$$

## Answers

1. $\log_x a = y$

2. $p^t = r$

3. $\log_{10} 50 = y$

4. $e^t = 694$

5.

a) $\log_3 9 = 2$
b) $\log_4 16 = 2$
c) $\log_{10} 1000 = 3$
d) $\log_6 36 = 2$
e) $\log_2 16 = 4$
f) $\log_3 27 = 3$
g) $\log_5 125 = 3$
h) $\log_2 32 = 5$
i) $\log_{10}\left(\frac{1}{1000}\right) = -3$
j) $\log_3 \frac{1}{9} = -2$

6.

a) $\log_{10} 16 = 1.204$
b) $\log_{10} 81 = 1.908$
c) $\log_{10} 1000 = 3$
d) $\ln 16 = 2.773$
e) $\ln 1000 = 6.908$
f) $\ln 81 = 4.394$

7.

a) $\log_3 50 = \frac{\log_{10} 50}{\log_{10} 3} \approx 3.56$
b) $\log_5 10 = \frac{\log_{10} 10}{\log_{10} 5} \approx 1.43$
c) $\log_7 100 = \frac{\log_{10} 100}{\log_{10} 7} \approx 2.37$
d) $\log_6 1.362 = \frac{\log_{10} 1.362}{\log_{10} 6} \approx 0.17$

8.

a) $\log_{10} x = 0.5 \to x = 10^{0.5} \to x = 3.16$
b) $\log_{10} x = 1.2 \to x = 10^{1.2} \to x = 15.85$
c) $\log_{10} x = 1.8 \to x = 10^{1.8} \to x = 63.10$
d) $10^x = 28 \to x = \log_{10} 28 \to x = 1.45$
e) $10^x = 45 \to x = \log_{10} 45 \to x = 1.65$
f) $10^x = 136 \to x = \log_{10} 136 \to x = 2.13$
g) $e^x = 212 \to x = \log_e 212 \to x = 5.36$
h) $e^x = 96 \to x = \ln 96 \text{ or } \log_e 96 \to x = 4.56$
i) $e^x = 13 \to x = \ln 13 \to x = 2.56$
j) $\ln x = 0.2 \to x = e^{0.2} \to x = 1.22$
k) $\ln x = 1.8 \to x = e^{1.8} \to x = 6.05$
l) $\ln x = 2.3 \to x = e^{2.3} \to x = 9.97$