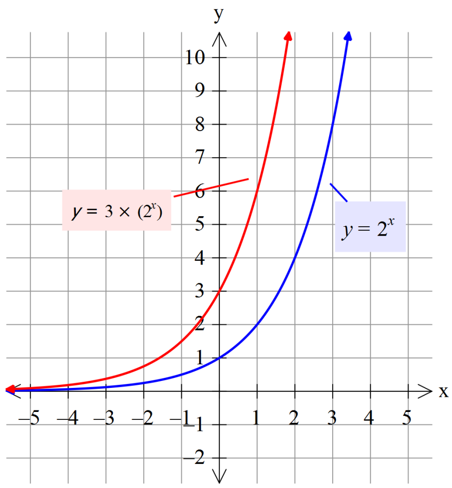
MATHS AND STATS

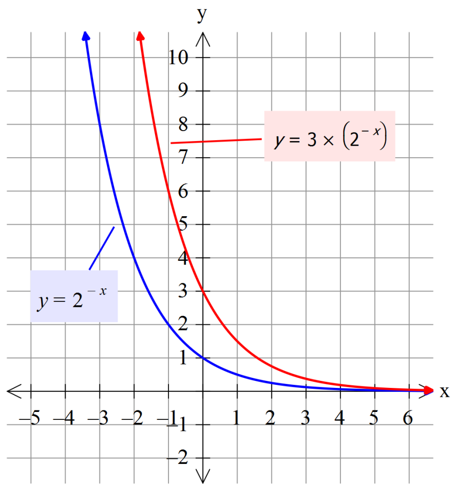
Exponential Functions

Here the independent variable (x) is the exponent of the function

$$y = ka^x$$

Examples





- The y-intercept is k
- a determines the steepness
- $y = ka^{-x}$ is a reflection of $y = ka^x$ in the y-axis
- In all these graphs the x-axis is an asymptote.

If $a>1$ (ie base a is positive) the equation $y = ka^x$ models exponential growth

If $a<1$ (ie base a is negative) the equation $y = ka^x$ models exponential decay

(N.B. this is the same as having $a>1$ and graphing $y = ka^{-x}$)

Applications: compound interest

$2000 is invested at 4% p.a. for 10 years. What is the current value (v) after the 10 years?

Let the initial amount invested (principle) be P. Look at the current value (v)

After 1 year

$$v = P + 0.04P$$
$$= P(1 + 0.04)$$
$$= P1.04$$

After 2 years

$$v = P1.04 \times 1.04$$
$$= P(1.04)^2$$

After 3 years

$$v = P(1.04)^2 \times 1.04$$
$$= P(1.04)^3$$

After 10 years

$$v = 2000(1.04)^{10}$$
$$= 2960.49$$

After *n* years

$$v = P(1.04)^n$$

Applications: radioactive decay

The decay of Carbon-14 can be modelled using the equation

$N(t) = N_0 e^{-0.000121t}$ where N = the current amount, t = time (years) and N_0= the original amount

(Equation for *N(t)* is from the web site http://physics.bu.edu/~duffy/py106/Radioactivity.html)

(a) If you start with 2 kg of C-14, how much will you have in 50 years, 1000 years and 3000 years?

(b) How long will it take you to get down to 1 kg?

Solution

(a) Mass of C-14 after …

In 50 years

$$N = 2e^{-0.000121\times 50} = 2e^{-0.00605} = 1.988$$

In 1000 years

$$N = 2e^{-0.000121\times 1000} = 2e^{-0.121} = 1.772$$

In 30000 years

$$N = 2e^{-0.00121\times 3000} = 2e^{-0..363} = 1.391$$

(b)

How long will it take you to get down to 1 kg?

We need

$$1 = 2e^{-0.000121t}$$

$$0.5 = e^{-0.000121t}$$

$$\ln 0.5 = -0.000121t$$

$$\frac{\ln 0.5}{-0.000121} = t$$

$$t = 5728 years$$

Exercises

1. \$5000 is invested at an interest rate of 5% p.a. What is the value of the investment
(a) after 4 years (b) after 10 years

2. \$1500 is invested at in interest rate of 8% p.a.
   (a) How much will the investment be worth after 5 years?
   (b) How long will it take for the investment to be worth \$5000?

3. The decay of Flourine 18 can be modelled with the equation

   $N = N_0 e^{-0.381t}$ where t is measured in hours

   If you had 5 grams of Flourine 18 :
   (a) how much would be left after 3 hours?
   (b) how much would be left after 6 hours?
   (c) How long would it take you to have only 1 gram left?

(Half-life of Flourine 18 from http://science.howstuffworks.com/nuclear2.htm)

4. Radioactive decay of Americium 241 is can be modelled by

$N = N_0 e^{-0.0015\,t}$ where t is measured in years

If you had 1 kg of Americium 241:
(a) how much would be left after 10 years?
(b) how much would be left after 20 years?
(c) How long would it take you to have only 500 grams left?

(Americium 241 half-life from http://science.howstuffworks.com/nuclear2.htm)

Answers

1. (a) $6077.53

   (b) $8144.47

2. (a) $2203.99

   (b)

$$\begin{aligned} 5000 &= 1500(1.08)^n \\ \frac{5000}{1500} &= 1.08^n \\ \ln\left(\tfrac{10}{3}\right) &= n\ln 1.08 \\ \frac{\ln\left(\frac{10}{3}\right)}{\ln 1.08} &= n \\ n &\approx 15.6 \quad years \end{aligned}$$

3. (a) 1.59g

   (b) 0.51g

3. (c)

$$\begin{aligned} 1 &= 5e^{-0.381\times t} \\ \frac{1}{5} &= e^{-0.381\times t} \\ \ln 0.2 &= -0.381\times t \\ t &= \frac{\ln 0.2}{-0.381} \\ t &\approx 4.22 \quad hours \end{aligned}$$

4. (a) 985g

   (b) 970g

   (c)

$$\begin{aligned} 500 &= 1000e^{-0.0015\times t} \\ \frac{500}{1000} &= e^{-0.0015\times t} \\ \ln 0.5 &= -0.0015\times t \\ t &= \frac{\ln 0.5}{-0.0015} \\ t &= 462 \quad years \end{aligned}$$