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THE ANALYSIS OF DEFORMATIONS CAUSED BY LOADING APPLIED TO THE WALLS OF A CIRCULAR TUNNEL

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SUMMARY

An analytical solution to the problem of a load applied to a portion of the wall of a circular tunnel installed deep in a homogeneous isotropic material has been found. The solution is obtained by taking a Fourier transform, in the axial direction, of the field quantities and then representing these as a Fourier series in the polar angle. The integration which is necessary for the inversion of Fourier transforms has to be performed numerically using Gaussian quadrature.

The solution has been used to analyse the problem of uniform normal loading applied over rectangular patches of the tunnel wall. An indication of how the solution can be applied to the tunnel jacking problem is given.

1. INTRODUCTION

There are many occasions in tunnelling and mining practice when loads are applied directly to the faces of underground excavations. For example, a roof or wall support will exert a reaction on the surfaces of the tunnel because it is supporting some of the ground load; also, it may exert additional forces on the ground because of some initial pre-stress associated with its installation. The face plate of a pre-stressed rock bolt is one means of transferring this type of load to the rock face. Loading of the tunnel walls will also occur when *in situ* tests are carried out on the rock or soil surrounding an opening. An example is the tunnel jacking test.

If the size of the opening is large compared with the size of the loaded area, then it may be reasonable to assume that the response of the tunnel wall can be approximated by the response of the surface of an elastic half space. However, if the size of the loaded region is significant compared to the size of the tunnel, then the half space approximation may no longer be adequate.

The type of problem described above could perhaps be analysed using an approximate numerical technique such as the finite element method. The analysis would need to take into account the full three dimensional nature of the configuration and would thus involve a large computational cost.

Many situations may be modelled adequately by assuming that the opening is a long tunnel of circular cross-section placed at great depth in an elastic medium. It is shown in this paper that it is possible to find an analytical solution to the problem of loading on the walls of a circular tunnel. Specific cases of distributed loading applied to discrete portions of the tunnel wall are investigated.

The problem of loading the walls of a circular tunnel has been investigated previously, e.g. Duvall and Blake,[1,2] and solutions for some plane strain problems have been given by Jaeger and Cook.[3]

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2. PROBLEM DESCRIPTION

We consider the problem of an infinite, single phase, elastic body containing a cylindrical cavity of radius a and of infinite length. The elastic medium is considered to be homogeneous and isotropic with shear modulus G and Poisson's ratio ν.

It is assumed that loads are applied to the walls of the tunnel, causing deformations of the surrounding elastic medium. This loading may be of arbitrary distribution and direction. In particular, we will consider the problem illustrated in Figure 1, in which a specified traction is applied over a region on the tunnel wall. When analysing this problem it is convenient to adopt a cylindrical polar co-ordinate system (r, θ, z).

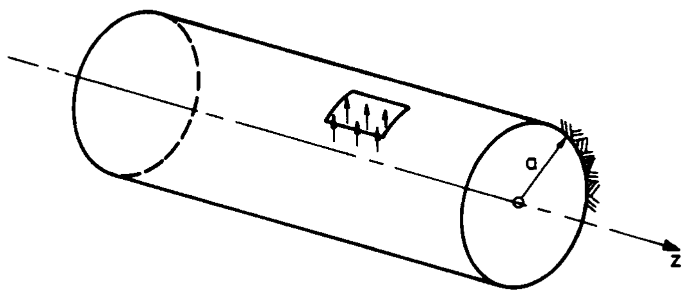

Figure 1. Problem description

In the following section, flexibility expressions are derived that relate the displacement components (u, v, w) at some point within the elastic body (at radius $r > a$), to the components of the traction $(\tau_{rr}, \tau_{r\theta}, \tau_{rz})$ applied to the tunnel face. In subsequent sections the analysis is extended to cover several specific cases of distributed loading applied to discrete portions of the tunnel wall.

3. ANALYSIS

3.1. Flexibility relations

The equations of elasticity, expressed in terms of cylindrical polar co-ordinates, are (e.g. Timoshenko and Goodier[4]),

$$
\begin{aligned}
G\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right) + (\lambda + G)\frac{\partial \varepsilon_v}{\partial r} &= 0 \\
G\left(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right) + (\lambda + G)\frac{1}{r}\frac{\partial \varepsilon_v}{\partial \theta} &= 0 \\
\nabla^2 \varepsilon_v &= 0 \\
\frac{\partial w}{\partial z} = \varepsilon_v - \frac{\partial u}{\partial r} - \frac{u}{r} - \frac{1}{r}\frac{\partial r}{\partial \theta}
\end{aligned} \tag{1}
$$

where ε_v is the volume strain, λ and G are the Lamé parameters for the isotropic elastic ground surrounding the tunnel, and u, v, w are the components of displacement.

We shall represent each of the field quantities in terms of their Fourier transforms with respect to the variable z, i.e.

$$(u, v, w, \varepsilon_v) = \int_{-\infty}^{\infty} e^{i\alpha z}(U, V, -iW, E)\,d\alpha \tag{2}$$

and similarly, the components of the traction applied at the tunnel wall will be represented as

$$(\tau_{rr}, \tau_{r\theta}, \tau_{rz}) = \int_{-\infty}^{\infty} e^{i\alpha z}(T_{rr}, T_{r\theta}, -iT_{rz})\,d\alpha \tag{3}$$

In equations (2) and (3), U, V, W, E, T_{rr}, $T_{r\theta}$ and T_{rz} are the appropriate Fourier transforms, e.g.

$$U = \int_{-\infty}^{\infty} e^{-i\alpha z} u\,dz \tag{4}$$

Each of these Fourier transforms may also be expanded as a Fourier series in the variable θ, viz.

$$\begin{aligned}(U, W, E, T_{rr}, T_{rz}) &= \sum_{n=0}^{\infty} (U^{(n)}, W^{(n)}, E^{(n)}, T_{rr}^{(n)}, T_{rz}^{(n)}) \cos(n\theta + \varepsilon_n)\\ (V, T_{r\theta}) &= \sum_{n=0}^{\infty} (V^{(n)}, T_{r\theta}^{(n)}) \sin(n\theta + \varepsilon_n)\end{aligned} \tag{5}$$

On substitution of expressions (2), (3) and (5) into the equations (1), we have

$$\begin{aligned}&G\left(\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \alpha^2 U - \frac{n^2}{r^2}U - \frac{2n}{r^2}V - \frac{1}{r^2}U\right) + (\lambda + G)\frac{dE}{dr} = 0\\ &G\left(\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \alpha^2 V - \frac{n^2}{r^2}V - \frac{2n}{r^2}U - \frac{1}{r^2}V\right) - (\lambda + G)\frac{nE}{r} = 0\\ &\frac{d^2E}{dr^2} + \frac{1}{r}\frac{dE}{dr} - \alpha^2 E - \frac{n^2}{r^2}E = 0\\ &\alpha W = E - \frac{dU}{dr} - \frac{U}{r} - \frac{nV}{r}\end{aligned} \tag{6}$$

where the superscript (n) has been omitted in order to simplify the presentation. The quantity E can be determined immediately and it is found that

$$E = A_1 K_n(\alpha r) + A_2 I_n(\alpha r) \tag{7}$$

where K_n and I_n are the modified Bessel functions of order n and the constants A_1 and A_2 are to be determined.

If we now consider E as known, a particular solution of equations (6) is

$$\begin{aligned}U_p &= -\left(\frac{\lambda + G}{2G}\right) rE\\ V_p &= 0\\ \alpha W_p &= E + \left(\frac{\lambda + G}{2G}\right)\left(\frac{r\,dE}{dr} + 2E\right)\end{aligned} \tag{8}$$

The complementary solution now must satisfy the homogeneous equations

$$
\begin{aligned}
&G\left(\frac{d^2U_c}{dr^2}+\frac{1}{r}\frac{dU_c}{dr}-\alpha^2U_c\frac{n^2}{r^2}U_c-\frac{2n}{r^2}V_c-\frac{1}{r^2}U_c\right)=0\\
&G\left(\frac{d^2V_c}{dr^2}+\frac{1}{r}\frac{dV_c}{dr}-\alpha^2V_c-\frac{n^2}{r^2}V_c-\frac{2n}{r^2}U_c-\frac{1}{r^2}V_c\right)=0\\
&W_c=-\frac{dU_c}{dr}-\frac{U_c}{r}-\frac{nV_c}{r}
\end{aligned}
\tag{9}
$$

If the fisrst two equations of this set are added we find that

$$
\frac{d^2\xi}{dr^2}+\frac{1}{r}\frac{d\xi}{dr}-\alpha^2\xi-\frac{(n+1)^2}{r^2}\xi=0 \tag{10a}
$$

where $\xi=\frac{1}{2}(U_c+V_c)$. Similarly, if these two equations are subtracted then

$$
\frac{d^2\eta}{dr^2}+\frac{1}{r}\frac{d\eta}{dr}-\alpha^2\eta-\frac{(n-1)^2}{r^2}\eta=0 \tag{10b}
$$

where $\eta=\frac{1}{2}(U_c-V_c)$. The solutions of equations (10a, b) are of course

$$
\begin{aligned}
\xi&=B_1K_{n+1}(\alpha r)+B_2I_{n+1}(\alpha r)\\
\eta&=C_1K_{n-1}(\alpha r)+C_2I_{n-1}(\alpha r)
\end{aligned}
\tag{11}
$$

and thus the complete solution of equations (6) is

$$
\begin{aligned}
U&=-\left(\frac{\lambda+G}{2G}\right)rE+\xi+\eta\\
V&=\xi-\eta\\
\alpha W&=\left(\frac{\lambda+2G}{G}\right)E+\left(\frac{\lambda+G}{2G}\right)r\frac{dE}{dr}-\left(\frac{d\xi}{dr}+\frac{(1+n)}{r}\xi\right)-\left(\frac{d\eta}{dr}+\frac{(1-n)}{r}\eta\right)
\end{aligned}
\tag{12}
$$

If we now consider only those solutions that are bounded as $r\to\infty$ we find, after some manipulation, that

$$
(\alpha U, \alpha V, \alpha W)^{T}=D(A_1, B_1, C_1)^{T} \tag{13}
$$

where the matrix D is given by

$$
D=\begin{bmatrix}
-\left(\frac{\lambda+G}{2G}\right)\alpha rK_n(\alpha r) & K_{n+1}(\alpha r) & K_{n-1}(\alpha r)\\
0 & K_{n+1}(\alpha r) & -K_{n-1}(\alpha r)\\
\left(\frac{\lambda+G}{2G}\right)(\alpha rK'_n(\alpha r)+2K_n(\alpha r))+K_n(\alpha r) & K_n(\alpha r) & K_n(\alpha r)
\end{bmatrix}
$$

The stresses can be determined from Hookes Law:

$$T_{rr} = \lambda E + 2G\frac{\mathrm{d}U}{\mathrm{d}r}$$
$$T_{r\theta} = G\left(-\frac{nU}{r} + \frac{\mathrm{d}V}{\mathrm{d}r} - \frac{V}{r}\right) \tag{14}$$
$$T_{rz} = G\left(\frac{\mathrm{d}W}{\mathrm{d}r} - nU\right)$$

so that we may write

$$\left(\frac{T_{rr}}{G}, \frac{T_{r\theta}}{G}, \frac{T_{rz}}{G}\right)^{\mathrm{T}} = S(A_1, B_1, C_1)^{\mathrm{T}} \tag{15}$$

where the matrix S is given by

$$S = \begin{bmatrix} -\left(\frac{\lambda+G}{2G}\right)2\alpha a K_n(\alpha a) - K_n(\alpha a) & 2K'_{n+1}(\alpha a) & 2K'_{n-1}(\alpha a) \\ \left(\frac{\lambda+G}{2G}\right) n K_n(\alpha a) & -K_{n+2}(\alpha a) & K_{n-2}(\alpha a) \\ \left(\frac{\lambda+G}{2G}\right)\cdot\left(2\alpha a + \frac{n^2}{\alpha a} K_n(\alpha a)\right. & K'(\alpha a) - K'_{n+1}(\alpha a) & K'_n(\alpha a) - K_{n-1}(\alpha a) \\ \qquad \left. + 2K'_n(\alpha a)\right) + K'_n(\alpha a) & & \end{bmatrix}$$

Equations (13) and (15) can be combined to obtain the symmetric flexibility relationship

$$\mathbf{\Delta} = \frac{1}{\alpha G}\Phi\mathbf{F} \tag{16}$$

where

$$\mathbf{\Delta} = (U, V, W)^{\mathrm{T}}$$
$$\mathbf{F} = (T_{rr}, T_{r\theta}, T_{rz})^{\mathrm{T}}$$

and

$$\Phi = DS^{-1}$$

3.2. Normal Loading

The problem of normal traction applied to the wall of a tunnel is of particular interest. For this case, $T_{r\theta} = T_{rz} = 0$ and thus the radial displacement can be determined from

$$U^{(n)}(r) = \frac{1}{\alpha G}\Phi_{11}^{(n)}(\alpha)T_{rr}^{(n)}(a) \tag{17}$$

Transforming back to the physical plane we have

$$u(r, \theta, z) = \sum_{n=0}^{\infty} \cos(n\theta + \varepsilon_n) \int_{-\infty}^{\infty} U^{(n)}(r)\, \mathrm{e}^{\mathrm{i}\alpha z}\, \mathrm{d}\alpha \tag{18}$$

In order to determine the value of radial displacement using this expression it is necessary to determine the Fourier coefficient $T_{rr}^{(n)}$. The form of this expression will depend on the shape of that portion of the tunnel wall which is subjected to the normal traction. Explicit expressions for two types of loading are given in the next section.

It is of course possible to analyse the effects of shear loadings on the tunnel wall, using the analysis outlined above, but it is likely that in many cases the problem of normal loading will be of greater practical interest. An important exception occurs in the case of props for tunnel boring machines. Duvall and Blake[1,2] have shown that shear loads have considerable importance in the latter application.

4. SPECIFIC EXAMPLES

Two examples are presented in order to illustrate the use of the flexibility relations derived in equations (16).

4.1. Single 'Patch' Loading

In the first example we assume that the radial traction is applied to a 'patch' of the tunnel where the shape of the patch is a curvilinear rectangle. The loaded region is defined by $|\theta| < \delta$ and $|z| < b$ and its shape is shown schematically in Figure 2(a).

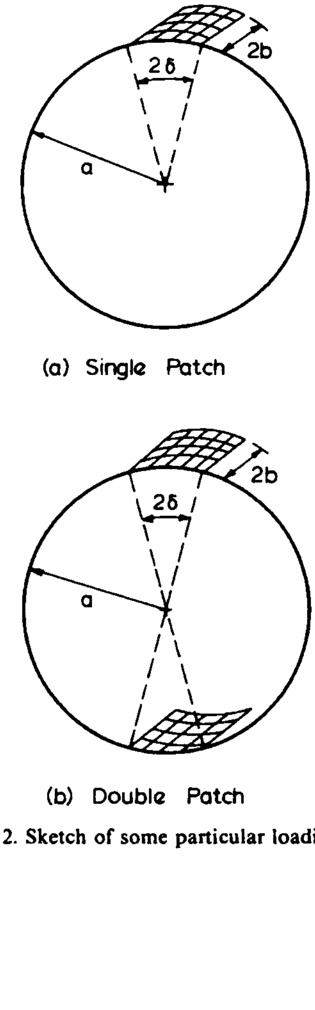

Figure 2. Sketch of some particular loading cases

The expressions defining the traction for this form of loading are

$$
\begin{aligned}
T_{rr}^{(n)} &= \frac{\delta}{\pi^2} \cdot \frac{\sin(\alpha b)}{\alpha} && n = 0 \\
T_{rr}^{(n)} &= \frac{2}{\pi^2} \cdot \frac{\sin(\alpha b)}{\alpha} \cdot \frac{\sin(n\delta)}{n} && n = 1, 2, 3, \ldots
\end{aligned}
\tag{19}
$$

The results obtained from an analysis of this case are applicable to a number of practical problems in tunnelling and mining. For example, the loading applied to a tunnel wall by a single rectangular end plate of a rock bolt or by a rectangular roof support, anchored using a number of rock bolts, may be approximated by the single patch load. Hence solutions to the present problem may provide approximate but useful solutions for the effects of such loadings on the surrounding soil or rock, particularly if the rock bolts are anchored remote from the tunnel face.

The combined effect of a number of such loads can be obtained by the appropriate superposition of results for the single patch. However, there is one special case of multiple loaded areas which warrants further attention.

4.2. Double 'Patch' Loading

In the second example we assume that a radial traction is applied to two patches of the tunnel face. The shape of each patch is a curvilinear rectangle and they are arranged so that they are 'diametrically' opposed. The loaded regions are defined by $|\theta| < \delta$, $|\theta - \pi| < \delta$ and $|z| < b$, and the arrangement is shown schematically in Figure 2(b).

The expressions defining the traction for this form of loading are

$$
\begin{aligned}
T_{rr}^{(n)} &= \frac{2\delta}{\pi^2} \cdot \frac{\sin(\alpha b)}{\alpha} && n = 0 \\
T_{rr}^{(n)} &= \frac{2}{\pi^2} \cdot \frac{\sin(\alpha b)}{\alpha} \cdot \frac{\sin(n\delta)}{n} \cdot (1 + \cos(n\pi)) && n = 1, 2, 3, \ldots
\end{aligned}
\tag{20}
$$

It can be seen that $T_{rr}^{(n)}$ vanishes when n is odd.

Probably the most important application of the solution to this problem is in the interpretation of the tunnel jacking test. In this test, usually performed in small tunnels or test adits, two sides of the tunnel are jacked apart, the force being applied to the two loaded areas by one or more hydraulic jacks aligned along a diameter of the tunnel. Observations are made of the deformation of the country rock or soil, usually beneath the loaded areas, and from these results a representative value is deduced for the elastic modulus of the surrounding ground. It has been usual in the past (e.g. Anon[5] and Goodman[6]) to interpret the results of this test by assuming that the loads are applied to two unconnected, elastic half spaces (i.e. using an integrated Boussinesq solution). It is now possible, using the more appropriate solution for the circular tunnel, to determine under what conditions the conventional interpretation becomes inaccurate and, therefore, unacceptable. This is illustrated by the results of the next section.

5. RESULTS

Equations (19) and (20) have been combined with (17) and (18) to obtain solutions for the radial displacement at the wall of the tunnel for the cases of loading over a single patch and over double patches. The integration which is necessary for the inversion of Fourier transforms has been performed numerically using Gaussian quadrature. Summation of the infinite Fourier series has also been approximated by summing over a finite but sufficiently large number of terms (typically to $n=50$ when δ is greater then 5°).

The results have been plotted in Figures 3–6 for patches of loaded area which are curvilinear squares of side $2b$, i.e. $\delta = b/a$. Results are given for elastic materials with Poisson's ratio equal to 0, 0·25 and 0·5. On each graph the ordinate is a ratio formed by dividing the solution for the circular tunnel by the corresponding solution for the half space problem. In Figures 3 and 4 the ratio is $u_{ca}/u_{c\infty}$, where u_{ca} is the radial displacement at the centre of the patch on the tunnel wall and $u_{c\infty}$ is the surface displacement at the centre of a normally loaded square on the surface of a half space. In Figures 5 and 6 the ordinate is $\bar{u}_a/\bar{u}_\infty$, where $\bar{u}_a$ and $\bar{u}_\infty$ are the average values of normal displacement for the tunnel problem and the half space problem, respectively.

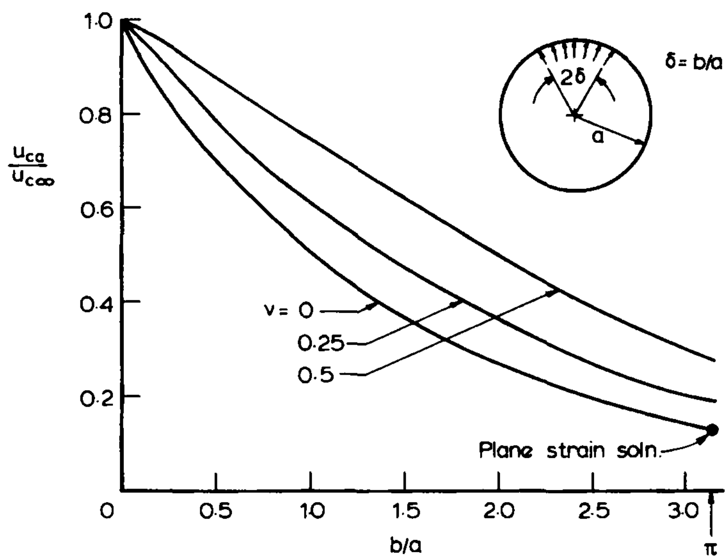

Figure 3. Central radial displacement for single patch loading

The solutions for the problem of normal loading on a half space have been reported previously, e.g. see Poulos and Davis.[7] They are given by

$$(u_{c\infty}, \bar{u}_\infty) = \frac{(1-\nu)pb}{G} . (I_{c\infty}, \bar{I}_\infty) \tag{21}$$

where

p = the magnitude of the normal traction

b = half side of the square

Values for the influence factors are

$$I_{c\infty} \approx 1 \cdot 222$$

$$\bar{I}_\infty \approx 0 \cdot 946$$

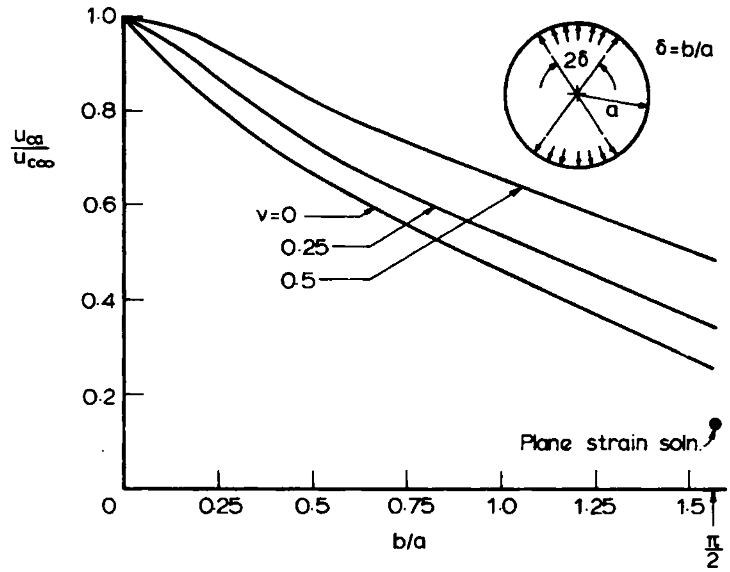

Figure 4. Central radial displacement for double patch loading

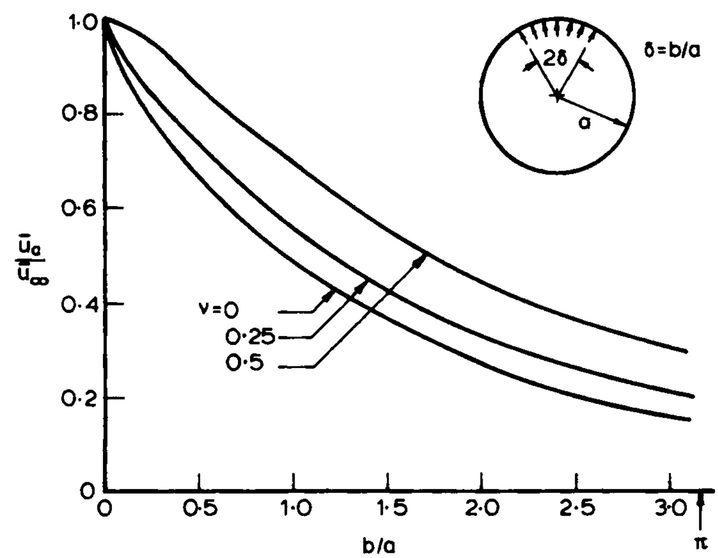

Figure 5. Mean radial displacement for single patch loading

In each figure the effect of the size of the loaded region (half side b) in relation to the size of the tunnel (radius a) is demonstrated by plotting the ratio b/a as abscissa. All curves approach an ordinate value of 1 as b/a approaches 0. This indicates that as the tunnel becomes larger compared with the size of the loaded region then the solution for the tunnel problem approaches the solution for the half space. For larger loaded areas the solution is smaller than the corresponding half space solution and the trend is for the ratio of the two displacements

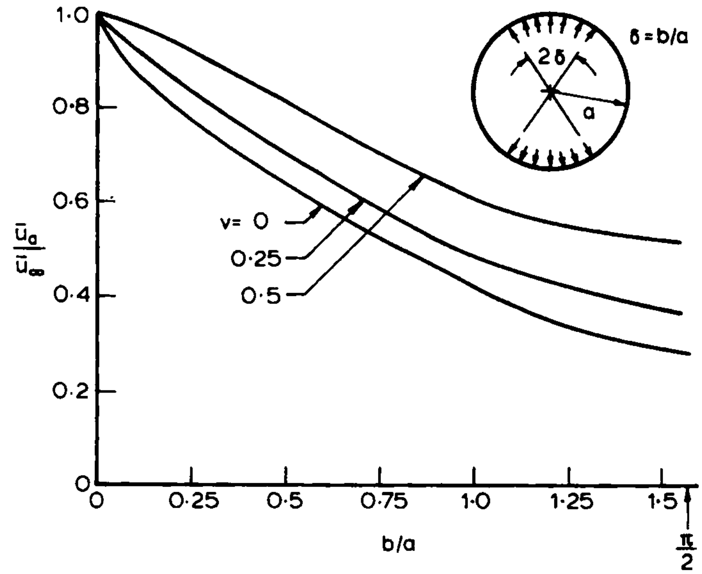

Figure 6. Mean radial displacement for double patch loading

to decrease as b/a increases. Ultimately, when the load extends around the entire circumference of the tunnel, i.e. $b=\pi a$ in the single case and $b=\frac{1}{2}\pi a$ in the double case, the tunnel displacements are between about 15 and 60 per cent of the half space displacement. Also shown on each of the Figures 3 and 4 is the plane strain solution for the case where the tunnel is loaded by uniform internal pressure. It can be seen that in some cases of patch loading over the entire circumference, the central displacement is quite close to the plane strain solution, despite the finite length along the tunnel of the loaded square patch.

It was mentioned above that the solutions for the double patch cases are relevant to the interpretation of the tunnel jacking test. Typically, in such tests, the loaded portions of the tunnel face may have a dimension of the order of 1 m (either square side or circular diameter). Tests are often carried out in tunnels of diameter 2–3 m. In such cases the ratio b/a will be in the range 0·33–0·5 and if the conventional interpretation is made, assuming an average displacement given by the half space theory, then Figure 6 indicates that the shear modulus of the ground may be overestimated by as much as 30 per cent ($\nu=0$ being the worst case). Such an overestimate can lead to non-conservative predictions of the ground deformations due to structural and foundation loads or due to subsequent excavation or tunnelling.

6. CONCLUSIONS

The analysis has been presented for the deformations due to loading applied to the walls of a circular tunnel in an infinite elastic medium. By the use of a Fourier transform with respect to the co-ordinate direction along the tunnel and a Fourier series in the circumferential direction, it has been possible to reduce the governing partial differential equations to a set of ordinary differential equations in the Fourier coefficients of the Fourier transforms of the field quantities. Solutions have been found to these transformed equations and flexibility relations have been derived for deformations in the elastic ground surrounding the tunnel. Particular solutions have been found for the case of normal traction applied to regions of the tunnel wall which have a curvilinear square shape, but the method can be extended to more complicated loading patterns. Recovery of the field quantities has required numerical inversion of the Fourier transforms and an approximation to the infinite Fourier series.

Numerical results have been plotted which may have application in the calculation of deformations due to loading on tunnel walls and in the interpretation of tunnel jacking tests. It has been demonstrated that conventional methods for analysing these tests could overestimate the elastic modulus of the ground typically by about 20 per cent but not more than 30 per cent.

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