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Finite element analysis for convective heat diffusion with phase change

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Whereas phase-change problems associated with heat conduction have been well studied during the last three decades, very little attention has been paid to phase changes taking place in convective heat diffusion. Numerical methods dealing with conventional phase change problems do not directly work in cases where a fluid is concerned. In this paper, an enthalpy method is extended to solve phase-change problems associated with fluids. By using the concept enthalpy, the governing equations are first reformulated into a single quasi-linear partial differential equation that implicitly takes into account the condition of phase change. This equation together with appropriate initial and boundary conditions are then decomposed into two sets of equations respectively representing a convection and a diffusion problem. The decomposition is accomplished in such a way that no phase contradiction occurs between the two separate problems. The convection problem is solved by the method of step by step characteristics and the diffusion problem by a Galerkin finite element method. Numerical examples demonstrate that the numerical method produces reasonable results.

1. Introduction

Numerous attempts have been made to search numerical methods solving phase change problems during the last three decades and, indeed, effective techniques have been developed for handling difficulties such as moving boundaries and non-linearity [1–5]. However, almost all of the methods are limited to phase changes taking place in stationary media and only heat conduction is considered. Recent concern for e.g. freezing of channel flow under low temperature and ice forming processes in arctic rivers, calls for attention to phase change problems associated with convective heat diffusion.

Consider a fluid flowing through a channel which risks freezing. The possibility of freezing depends not only on the rate of heat diffusion and latent heat due to phase change, but also on the velocity of the fluid. The faster the fluid is flowing, the longer distance it can flow before freezing. For a very high velocity, diffusion may even be of less importance. This can be seen more clearly by considering the heat conservation equation

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$$C(T)\frac{\partial T}{\partial t} = \nabla\cdot(\boldsymbol{K}(T)\nabla T) - \nabla\cdot(\boldsymbol{\nu}C(T)T) + q \qquad \text{in } \Omega, \text{ but } T \neq T_f, \tag{1}$$

where t denotes the time, C the volumetric heat capacity, T the temperature, ∇ the gradient operator that results in a vector, $\boldsymbol{K}$ the thermal conductivity tensor, $\boldsymbol{\nu}$ the velocity of the fluid, q the heat source, Ω the domain of interest and T_f the freezing or phase-change temperature. This non-linear equation has a dual nature because of the appearance of both a first and a second order derivative in space. When the mean value νC is very small compared with the mean thermal conductivity K, i.e. a small Peclet number, the diffusion term $(\nabla\cdot(\boldsymbol{K}\nabla T))$ dominates and (1) behaves as a second order differential equation. For large Peclet number, the convection term $(\nabla\cdot(\boldsymbol{\nu}CT))$ dominates and the equation behaves as a first order differential equation.

Without taking the phase change into account, (1) represents an ordinary convection-diffusion problem and can be solved either in an Eulerian system or in a Lagrangian system [6–8]. In a Lagrangian system, the convection term $\boldsymbol{\nu}\cdot(C\nabla T)$ can be included in the material derivative and the equation reduces to a pure parabolic type in the case of a uniform and steady state velocity field. But the need to use a deforming mesh or moving coordinates can result in mesh tangling where the geometry or the boundary conditions change in the flow direction. On the other hand, solving the equation in an Eulerian system may give rise to problems such as overshoot and numerical dispersion and, consequently, special techniques such as upstream weighting are needed to handle such problems [9].

Recently many workers have turned to Eulerian–Lagrangian methods, in which the unknown is first decomposed into a convection part and a diffusion part, whereafter the method of characteristics is used to solve the convection problem and finite element method to solve the diffusion problem [8, 10]. These methods use a fixed mesh and are also effective in eliminating numerical dispersion and overshoot. In this paper, an Eulerian–Lagrangian method is extended to solve convection-diffusion problems with phase change.

Numerical solutions to phase-change problems associated with heat conduction can be performed either in a fixed mesh or in an adaptive mesh [1–3]. The methods using an adaptive mesh refine the mesh at each time step so that the phase interface always remains on boundaries between elements. The condition of phase change is then satisfied along these internal boundaries. However, this strategy is computationally expensive and can also cause problems such as mesh tangling. An outstanding example of fixed mesh techniques is the family of enthalpy methods [4, 5]. These methods use the concept of enthalpy either as the main unknown instead of the temperature or for the definition of an apparent heat capacity that accounts for the effect of latent heat. In the family of enthalpy methods, the condition of phase change is satisfied implicitly in the enthalpy which experiences a jump at the temperature of the phase change.

In this paper, the convection-diffusion equation is first reformulated using enthalpy, so that the phase-change condition is implicitly taken into account in a single quasi-linear partial differential equation. This equation together with appropriate initial and boundary conditions is then decomposed, in a manner without creating any phase contradiction, into two sets of equations representing a convection problem and a diffusion problem, respectively. The convection problem is first solved analytically by studying the associated characteristics. The solution is used as the initial condition for the diffusion problem which is solved by a Galerkin finite element method.

2. Reformulation based on enthalpy

In a multi-phase system, the convective-diffusion equation (1) is valid within each phase but not on the phase interfaces. On a phase interface, the following partial differential equation governing the phase change must be satisfied:

$$\boldsymbol{K}\left.\frac{\partial T}{\partial \boldsymbol{n}}\right|_{T_{\mathrm{f}}^-} - \boldsymbol{K}\left.\frac{\partial T}{\partial \boldsymbol{n}}\right|_{T_{\mathrm{f}}^+} = L\,\frac{\mathrm{d}S}{\mathrm{d}t}\,, \quad T = T_{\mathrm{f}}\,, \tag{2}$$

where S denotes the phase interface, $\boldsymbol{n}$ the outward unit normal of S and L the latent heat. Equation (2) expresses an internal boundary condition and the boundary S is determined a posteriori from the temperature field. A numerical solution that explicitly satisfies (2) requires mesh refinement at each time step, in order to track the moving boundary S. This is computationally very expensive and also inconvenient where a fluid is concerned. Therefore, in this paper only an implicit satisfaction of the phase change condition is considered. To achieve this, the concept enthalpy has to be utilized.

Enthalpy is defined as the sum of sensible heat and latent heat. If a phase change takes place at a specific temperature, the enthalpy exhibits a jump at that point due to the latent heat released from the phase change. This discontinuity may cause difficulties in evaluating the enthalpy values at those nodal points having the phase change temperature. Thus, it is often assumed that the phase change takes place within a small temperature interval $[T_{\mathrm{f1}}, T_{\mathrm{f2}}]$ and the enthalpy H is defined continuously as [11]

$$H = \int_{T_{\mathrm{r}}}^{T} C(T)\,\mathrm{d}T\,, \qquad T < T_{\mathrm{f1}}\,, \tag{3}$$

$$H = \int_{T_{\mathrm{r}}}^{T_{\mathrm{f1}}} C(T)\,\mathrm{d}T + \int_{T_{\mathrm{f1}}}^{T} \left(C(T) - \frac{\mathrm{d}L}{\mathrm{d}T}\right)\mathrm{d}T\,, \quad T_{\mathrm{f1}} \leqslant T < T_{\mathrm{f2}}\,, \tag{4}$$

$$H = \int_{T_{\mathrm{r}}}^{T} C(T)\,\mathrm{d}T + L\,, \qquad T \geqslant T_{\mathrm{f2}}\,, \tag{5}$$

where T_{r} is a reference temperature. By use of the enthalpy, an apparent heat capacity $\bar{C}$ is defined as

$$\bar{C} = \mathrm{d}H/\mathrm{d}T\,. \tag{6}$$

A direct evaluation of $\bar{C}$ according to (6) is possible only when the phase change interval $[T_{\mathrm{f1}}, T_{\mathrm{f2}}]$ is fairly large. Otherwise, this direct evaluation may cause the risk of skipping the phase change interval in a single time step [12]. In order to avoid this risk, the averaging technique suggested by Lemmon [13] is used, which gives

$$\bar{C} = \left(\frac{\nabla H \cdot \nabla H}{\nabla T \cdot \nabla T}\right)^{1/2}. \tag{7}$$

In the finite element method discussed below, the enthalpy H is approximated in terms of the same shape function as used for the temperature T. When linear elements are used, the

gradients of both H and T are constants and thus (7) need only be performed once. The nodal values of H are evaluated at each time step according to integrals (3)–(5). For higher degree basis functions as in isoparametric elements, numerical integration is needed. The gradients in (7) are no longer constant in an element. In this case, the gradients as well as the apparent heat capacity can be evaluated at the integration points. As a result, inhomogeneity of elements has to be dealt with.

With the definition of the apparent heat capacity, the coupled equations (1) and (2) can be rewritten in a weak form as

$$\bar{C}\,\frac{\partial T}{\partial t} = \nabla\cdot(\boldsymbol{K}\nabla T) - \nabla\cdot(\boldsymbol{v}\bar{C}T) + q\,. \tag{8}$$

This quasi-linear equation is assumed to be valid within the entire temperature range. This together with the following initial and boundary conditions completes a well-posed problem:

$$T(\boldsymbol{x},0) = T_0(\boldsymbol{x})\,, \qquad \text{in } \Omega\,, \tag{9}$$

$$(\boldsymbol{v}CT)\cdot\boldsymbol{m}_1 + \alpha_1(T - T_{s1}) = Q_1\,, \qquad \text{on } \Gamma_1\,, \tag{10}$$

$$(-\boldsymbol{K}\nabla T)\cdot\boldsymbol{m}_2 + \alpha_2(T - T_{s2}) = Q_2\,, \quad \text{on } \Gamma_2\,, \tag{11}$$

where $\boldsymbol{x}$ is a vector defining space coordinates, Γ_1 and Γ_2 are the boundaries of the domain Ω, $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are the outward unit normals of Γ_1 and Γ_2, respectively, and T_0, T_{s1}, T_{s2}, α_1, α_2, Q_1 and Q_2 are prescribed functions. Both (10) and (11) contain three types of boundary conditions, i.e. Dirichlet when $\alpha\to\infty$, Neumann when $\alpha = 0$ and Cauchy when $0<\alpha<\infty$.

3. Decomposition and solution of convection

In a Lagrangian coordinate system, (8) takes the form

$$\bar{C}\,\frac{\mathrm{D}T}{\mathrm{D}t} = \nabla\cdot(\boldsymbol{K}\nabla T) - \bar{C}T(\nabla\cdot\boldsymbol{v}) + q\,, \tag{12}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla \tag{13}$$

denotes the material derivative. We now decompose the temperature T into the two parts, i.e. a convection part $\tilde{T}$ and a diffusion part $\hat{T}$,

$$T = \tilde{T} + \hat{T}\,, \tag{14}$$

where $\tilde{T}$ satisfies the following initial value problem:

$$\frac{\mathrm{D}\tilde{T}}{\mathrm{D}t} = \frac{\partial\tilde{T}}{\partial t} + \boldsymbol{v}\cdot\nabla\tilde{T} = 0\,, \tag{15}$$

$$\tilde{T}(\boldsymbol{x}, 0) = T(\boldsymbol{x}, 0) = T_0(\boldsymbol{x}) , \tag{16}$$

$$(\boldsymbol{v}C\tilde{T})\cdot \boldsymbol{m}_1 + \alpha_1(\tilde{T} - T_{s1}) = Q_1 \quad \text{on } \Gamma_1 . \tag{17}$$

By subtracting (15) from (12), we obtain the following diffusion problem:

$$\bar{C}\left(\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{\mathrm{D}\tilde{T}}{\mathrm{D}t}\right) = \nabla\cdot(\boldsymbol{K}\nabla T) - \bar{C}T\nabla\cdot\boldsymbol{v} + q \tag{18}$$

subject to the initial and boundary conditions

$$T(\boldsymbol{x}, 0) - \tilde{T}(\boldsymbol{x}, 0) = 0 , \tag{19}$$

$$(-\boldsymbol{K}\nabla T)\cdot \boldsymbol{m}_2 + \alpha_2(T - T_{s2}) = Q_2 , \quad \text{on } \Gamma_2 . \tag{20}$$

The set of equations (15)–(17) represents a convection problem and the set of equations (18)–(20) a diffusion problem. The convection problem can be solved analytically by the method of characteristics and the diffusion problem numerically by, for example, the finite element method.

One remaining question on the decomposition of the temperature is that phase contradiction may appear if any phase change in the convection problem is inconsistent with that in the diffusion problem. In fact, the temperature of a material particle should never pass the phase-change temperature during the convection, since the only parameter that accounts for the effect of latent heat, i.e. the apparent heat capacity $\bar{C}$, does not appear in the convection problem. To make sure that the convection problem involves no fake phase change, we have to inspect the analytical solution of the convection problem. The differential equation (15) has constant solutions along the family of characteristic curves defined by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{x}, t, \tilde{T}) . \tag{21}$$

In the case of a uniform steady state velocity field, (21) represents a family of parallel lines in the x–t space. Considering the initial condition (16), we have

$$\tilde{T}(\boldsymbol{x}, t) = T_0(\boldsymbol{x} - \boldsymbol{v}t) . \tag{22}$$

In the case where the velocity $\boldsymbol{v}$ is explicitly only a function of the temperature, the associated characteristics are [14]

$$\boldsymbol{x} = \boldsymbol{v}(T_0(\boldsymbol{h})) + \boldsymbol{h} , \tag{23}$$

where $\boldsymbol{h}$ is an arbitrary coordinate vector. Equation (23) defines $\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{x}, t)$ implicitly as a function of x and t; the solution of the convection problem is then given by

$$\tilde{T}(\boldsymbol{x}, t) = T_0(\boldsymbol{h}(\boldsymbol{x}, t)) . \tag{24}$$

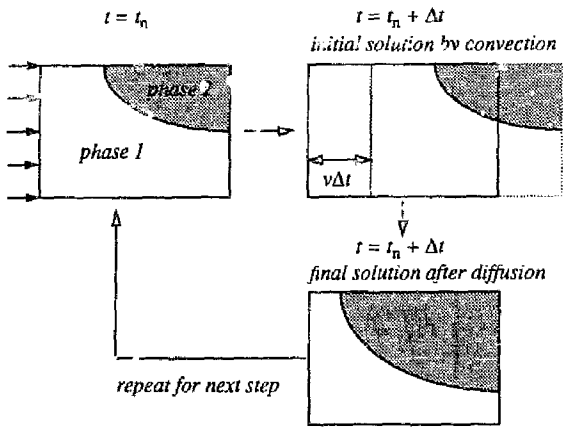

Fig. 1. Schematic diagram of the decomposition in the case of a uniform and steady state velocity field.

Both the solutions (22) and (24) represent a translation of the temperature field in the flow direction. The translation by (22) is uniform over the entire domain but that by (24) varies according to particular temperatures.

On the boundary Γ_1, $\tilde{T}$ is forced to be

$$\tilde{T} = \frac{Q_1 + \alpha_1 T_{s1}}{\alpha_1 + C(\boldsymbol{v} \cdot \boldsymbol{m}_1)} . \tag{25}$$

This inflow boundary usually contains only one phase, i.e. a liquid phase.

In general, it has been shown that no phase change is involved in the convection problem. Phase changes take place only in the diffusion problem and hence no phase contradiction is created from the decomposition.

The schematic view of the decomposition in the case of a uniform and steady state velocity field is shown in Fig. 1. We assume that the temperature distribution at the time level t_n is known as T_n and we search for the temperature distribution T_{n+1} at the time level $t_{n+1} = t_n + \Delta t$. We first let T_n be the initial condition of the convection problem and find the 'initial solution' $\tilde{T}_{n+1}$ due to convection. This process is actually equivalent to translating a fictitious particle from each node backward to its position at t_n, according to the velocity field. The heat diffusion during this translation is then computed by a finite element method to give the final solution T_{n+1} at t_{n+1}.

4. Finite element method for the diffusion problem

The diffusion problem described by (18), (19) and (20) can be solved by a Galerkin finite element method. We multiply (18) by an arbitrary test function ψ and integrate over the domain Ω. By using Green's theorem, the integral equation

$$\int_\Omega \bar{C}\left(\frac{DT}{Dt} - \frac{D\tilde{T}}{Dt}\right)\psi \, d\Omega = -\int_\Omega (\boldsymbol{K}\nabla T)\cdot\nabla\psi \, d\Omega + \int_{\Gamma_2} (\boldsymbol{K}\nabla T)\cdot \boldsymbol{m}_2\psi \, d\Gamma - \int_\Omega \bar{C}T(\nabla\cdot\boldsymbol{v})\psi \, d\Omega + \int_\Omega q\psi \, d\Omega \tag{26}$$

is obtained. Let the unknown T be approximated by

$$T(\boldsymbol{x}, t) = \sum_{i=1}^{N} \varphi_i(\boldsymbol{x}) T_i(t) , \tag{27}$$

where φ_i are ordinary shape functions defined piecewise element by element, T_i temperatures at nodal points and N the number of nodal points.

In the Galerkin method, the test function ψ is set to be identical with the shape function φ_j. Substituting (27) and (20) into (26) and replacing ψ by φ_j yields a system of ordinary differential equations,

$$\sum_{i=1}^{N}\left(\frac{DT_i}{Dt} - \frac{D\tilde{T}_i}{Dt}\right)A_{i,j} = \sum_{i=1}^{N} T_i B_{i,j} + F_j , \tag{28}$$

where

$$A_{i,j} = \int_\Omega \bar{C}\varphi_i\varphi_j \, d\Omega , \tag{29}$$

$$B_{i,j} = \int_\Omega (-(\boldsymbol{K}\nabla\varphi_i)\cdot\nabla\varphi_j - \bar{C}(\nabla\cdot\boldsymbol{v})\varphi_i\varphi_j) \, d\Omega + \int_{\Gamma_2} \alpha_2\varphi_i\varphi_j \, d\Gamma , \tag{30}$$

$$F_j = \int_\Omega q\varphi_j \, d\Omega + \int_{\Gamma_2} (Q_2 - \alpha_2 T_{s2})\varphi_j \, d\Gamma . \tag{31}$$

By use of matrix notation, (28) can be written as

$$\boldsymbol{A}\left(\frac{D\boldsymbol{T}}{Dt} - \frac{D\tilde{\boldsymbol{T}}}{Dt}\right) = \boldsymbol{B}\boldsymbol{T} + \boldsymbol{F} . \tag{32}$$

The time derivatives of the temperatures are approximated by the finite differences

$$\frac{D\boldsymbol{T}}{Dt} \approx \frac{\boldsymbol{T}_{n+1} - \boldsymbol{T}_n}{\Delta t} \quad \text{and} \quad \frac{D\tilde{\boldsymbol{T}}}{Dt} = \frac{\tilde{\boldsymbol{T}}_{n+1} - \boldsymbol{T}_n}{\Delta t} , \tag{33}$$

where $\tilde{\boldsymbol{T}}_{n+1}$ can be computed by (22) or (24) from $\boldsymbol{T}_n$, or by (25) on the inflow boundary Γ_1. Now using an Euler-backward scheme, the matrix equation (32) is solved by iteration,

$$\boldsymbol{A}_{n+1}\left(\frac{\boldsymbol{T}_{n+1} - \tilde{\boldsymbol{T}}_{n+1}}{\Delta t}\right) = \boldsymbol{B}_{n+1}\boldsymbol{T}_{n+1} + \boldsymbol{F}_{n+1} . \tag{34}$$

So far, we have discussed the problem in general terms. Let us now apply the suggested solution technique to two space dimensions, use triangular elements with linear shape functions φ_i, and assume a piecewise uniform velocity field, isotropic heat diffusion and

constant thermal properties within each phase. With these assumptions, an explicit and three level time stepping scheme can be used instead of the implicit scheme (34),

$$A_n\left(\frac{T_{n+1}-\tilde{T}_{n+1}}{\Delta t}\right)=B_n\left(\tfrac{3}{4}T_{n+1}+\tfrac{1}{4}\tilde{T}_n\right)+F_n\,, \tag{35}$$

where

$$\tilde{T}_{n+1}(x)=T_n(x-v\,\Delta t)\,, \qquad (x-v\,\Delta t) \text{ in } \Omega\,, \tag{36}$$

$$\tilde{T}_{n+1}(x)=\left.\frac{Q_1+\alpha_1 T_{s1}}{\alpha_1+(vC)\cdot m_1}\right|_{t_n}, \qquad (x-v\,\Delta t) \text{ out of } \Omega\,, \tag{37}$$

and

$$\tilde{T}_n(x)=T_{n-1}(x-2v\,\Delta t)\,, \qquad (x-2v\,\Delta t) \text{ in } \Omega\,, \tag{38}$$

$$\tilde{T}_{n-1}(x)=\left.\frac{Q_1+\alpha_1 T_{s1}}{\alpha_1+(vC)\cdot m_1}\right|_{t_{n-1}}, \qquad (x-2v\,\Delta t) \text{ out of } \Omega\,. \tag{39}$$

Equations (36) and (38) are the analytical solutions of the convection problem at the time level t_{n+1}, by particle tracking for one step and two steps backwards, respectively. Equations (37) and (39) result from the fact that the temperatures of those new particles flowing into Ω during Δt and $2\Delta t$, respectively, are controlled by the inflow boundary condition (17).

In (35), the matrices A and B as well as the vector F are evaluated based on the temperature vector $\tilde{T}_{n+1}$. If we know the temperature distributions at two time levels t_{n-1} and t_n, we can solve (35) explicitly for T_{n+1} at t_{n+1}.

5. Numerical example

The example deals with the freezing of a short channel flow. Suppose that a length of 20 m of a channel is exposed to an air temperature of −20 °C. The channel is 5 m wide and the water in it is 0.5 m deep. The water flows with an initial temperature 10 °C. The material properties are

- water: $K_w = 0.6\,\mathrm{W/mK}$, $C_w = 4.18\,\mathrm{MJ/m^3K}$;
- ice: $K_i = 2.2\,\mathrm{W/mK}$, $C_i = 1.93\,\mathrm{MJ/m^3K}$;
- latent heat of water–ice phase change: $L = 333\,\mathrm{MJ/m^3}$;
- freezing temperature interval: $[-0.001, 0.001]$ °C;
- heat transfer coefficient of the air–water interface: $\alpha = 20\,\mathrm{W/m^2K}$.

If it is a laminar flow and both phases flow with exactly the same velocity, we need only consider the heat diffusion in a cross-section and follow this section in the flow direction. The phase distribution is stabilized at the time the cross-section reaches the end of the channel. However, this Lagrangian solution does not suit the case where ice stays stationary since formation. In this case, we have to consider the problem in the plane parallel to the flow direction, as shown by Fig. 2.

Assuming that

- the velocity of the liquid phase is uniform and stable both in direction and in magnitude,
- the solid phase stays stationary since formation,
- phase change occurs within the temperature interval $[0.001, -0.001]$ °C,
- the heating effect from the base is negligible,

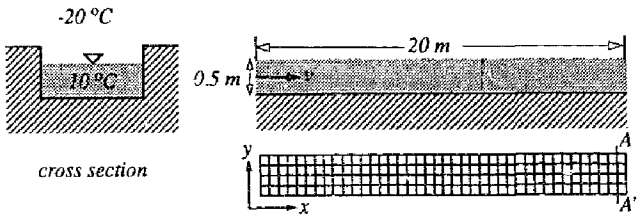

Fig. 2. The geometry and the finite element mesh of the channel.

we can use the simplified version of the finite element method described in the previous section to model the growth of the ice cover. The finite element mesh is shown in Fig. 2. The position of the phase interface is determined at each time step by interpolation with respect to the temperature gradient in the solid phase.

The computed propagation of the phase interface is shown in Fig. 3. Ice starts to form downstream and grows upstream. The rate of the growth decreases as the water velocity increases. When the velocity reaches 10 cm/s, no ice can form during 240 h.

Figure 4 plots the temperature history at the cross-section $A-A'$. It can be noted that once a temperature curve passes the freezing point, i.e. 0 °C, a corresponding flattening happens to the neighbouring curves. This characteristic behaviour of enthalpy methods is attributed to the fact that the latent heat released at the phase interface has been averaged to a neighbouring finite area [15].

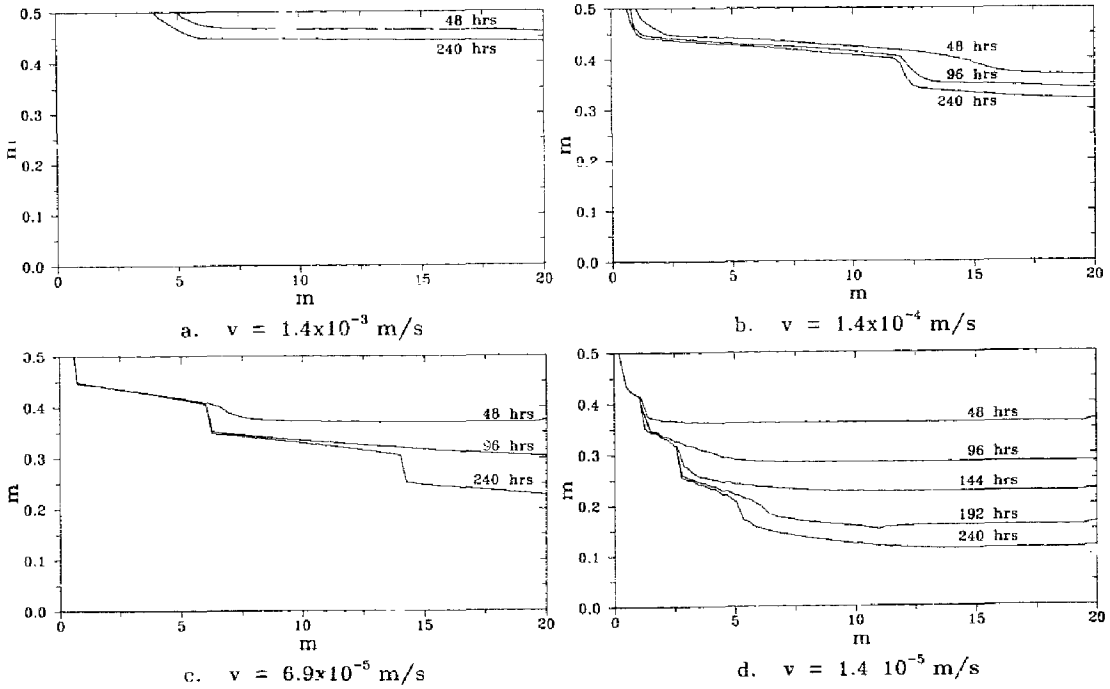

Fig. 3. Propagation of phase interface.

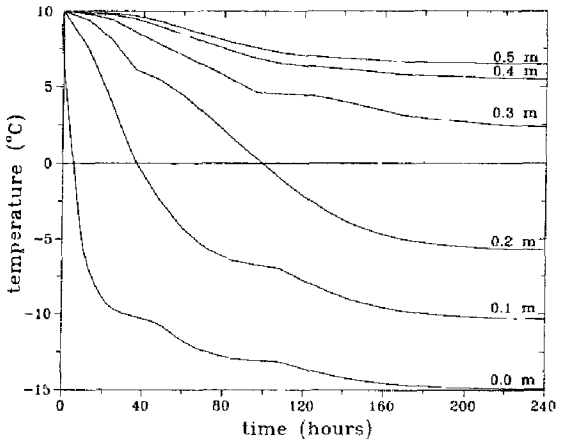

Fig. 4. Temperature history at the cross-section A–A'.

It should be noted that the assumption of a uniform and stable velocity field for the liquid phase together with a stationary solid phase may violate the law of mass conservation. In a real situation, the water velocity increases due to the decrease of the flow area as ice grows. Nevertheless, it is not impossible to apply a more complex and more realistic velocity field to this example problem.

Although Neumann claimed that the Eulerian–Lagrangian finite element method works well for Courant numbers in excess of 1, it is found during computation that using a Courant number larger than 0.5 may result in oscillation in solution. Thus the following restriction on time steps is imposed:

$$\Delta t = \min\left(\frac{CA_e}{K}, \alpha_c \frac{\Delta x}{v_x}\right), \tag{40}$$

where A_e denotes the area of an element, Δx the element length in the x direction and α_c the Courant number which is assigned to 0.1 in this example. The results shown in Figs. 3 and 4 are generally reasonable. Since no exact solution for this problem exists, validation of the numerical method can only be carried out by comparison with experiments, which, unfortunately, are lacking at the moment.

6. Conclusion

The finite element method presented in this paper provides a prior numerical solution to convective heat diffusion with phase change. The incorporation of an enthalpy method with an Eulerian–Lagrangian method has exhibited the effectiveness of handling the problem. A simplified version of the finite element method, i.e. for two space dimensions, linear shape functions, laminar flow and isotropic heat diffusion, is demonstrated giving reasonable results.

Although a general implementation of the finite element method requires tremendous

programming work, simplified versions can be employed in many cases. The critical parameter that controls the feature of the solution is the velocity $\boldsymbol{v}$. More realistic velocity fields can be obtained by taking into account mass conservation, which can consequently lead to physical elaboration of numerical modelling of for example ice-cover formation.

Appendix A. Notation

$\boldsymbol{A}$	Heat capacity matrix	
$\boldsymbol{B}$	Conductivity matrix	
C	Volumetric heat capacity	J/m^3K
$\bar{C}$	Apparent heat capacity	J/m^3K
$\boldsymbol{F}$	Heat supply vector	
H	Enthalpy	J/m^3
$\boldsymbol{h}$	Arbitrary coordinate vector	m
$\boldsymbol{K}$	Thermal conductivity tensor	W/mK
L	Latent heat	J/m^3
$\boldsymbol{m}_1$	Outward unit normal of boundary Γ_1	
$\boldsymbol{m}_2$	Outward unit normal of boundary Γ_2	
$\boldsymbol{n}$	Unit normal of phase interface	
N	Number of elements	
Q_1	Heat flux on boundary Γ_1	W/m^2
Q_2	Heat flux on boundary Γ_2	W/m^2
q	Internal heat source	W/m^3
S	Phase interface	
T	Temperature	°C
$\tilde{T}$	Convection part of T	°C
$\hat{T}$	Diffusion part of T	°C
T_f	Phase change temperature	°C
$[T_{f1}, T_{f2}]$	Temperature interval for phase change	°C
T_0	Initial temperature	°C
T_r	Reference temperature	°C
T_{s1}	Temperature on boundary Γ_1	°C
T_{s2}	Temperature on boundary Γ_2	°C
t	Time	s
Δt	Time step	s
$\boldsymbol{v}$	Velocity vector	m/s
$\boldsymbol{x}$	Coordinate vector	m
∇	Gradient operator	
$\cdot$	Scalar products of two vector	
α_1	heat transfer coefficient on boundary Γ_1	W/m^2K
α_2	heat transfer coefficient on boundary Γ_2	W/m^2K
Ω	Domain of interest	
Γ_1	Inflow boundary of Ω	
Γ_2	Boundary of Ω	
φ	Shape function	

References

[1] J. Crank, Free and Moving Boundary Problems (Clarendon Press, Oxford, 1984).
[2] J. Crank, How to deal with moving boundaries in thermal problems, in: R.W. Lewis, K. Morgan and O.C. Zienkiewicz, eds., Numerical Methods in Heat Transfer (Wiley, Chichester, 1981) 177–200.
[3] K. Morgan, R.W. Lewis and O.C. Zienkiewicz, An improved algorithm for heat conduction problems with phase change, Internat. J. Numer. Methods Engrg., 12 (1978) 1191–1195.
[4] V.R. Voller, C.R. Swaminathan and B.G. Thomas, Fixed grid techniques for phase change problems: A review, Internat. J. Numer. Methods Engrg. 30 (1990) 875–898.
[5] K.K. Tamma and R.R. Namburu, Recent advances, trends and new perspectives via enthalpy-based finite element formulations for applications to solidification problems, Internat. J. Numer. Methods Engrg. 30 (1990) 803–920.
[6] J.C. Heinrich, P.S. Huyakorn and O.C. Zienkiewicz, An upwind finite element scheme for two dimensional convective transport equation, Internat. J. Numer. Methods Engrg. 11 (1977) 131–143.
[7] E. Varoğlu and W.D.L. Finn, Utilization of the method of characteristics to solve accurately two-dimensional transport problems by finite elements, Internat. J. Numer. Methods Fluids 2 (1982) 173–184.
[8] S.P. Neuman, Adaptive Eulerian–Lagrangian finite element method for advection-diffusion, Internat. J. Numer. Methods Engrg. 20 (1984) 321–337.
[9] G.F. Pinder and W.G. Gray, Finite Element Simulation in Surface and Subsurface Hydrology (Academic Press, New York, 1977).
[10] R. Cady and S.P. Neuman, Three-dimensional adaptive Eulerian–Lagrangian finite element method for advection-dispersion, in: M.A. Celia et al., eds., Developments in Water Science: Vol. 2 Numerical Methods for Transport and Hydrologic Processes (Elsevier, Amsterdam, 1988).
[11] D. Sheng, Numerical modelling of frost and thaw penetration, Licentiate Thesis, 1990: 3L, Luleå University of Technology, Sweden, 1990.
[12] A.J. Dalhuijsen and A. Segal, Comparison of finite element techniques for solidification problems, Internat. J. Numer. Methods Engrg. 23 (1986) 1807–1829.
[13] E.C. Lemmon, Multidimensional integral phase change approximations for finite element conduction codes, in: R.W. Lewis, K. Morgan and O.C. Zienkiewicz, eds., Numerical Methods in Heat Transfer (Wiley, Chichester, 1981) 201–213.
[14] J.D. Logan, Applied Mathematics – A Contemporary Approach (Wiley, New York, 1987).
[15] D. Sheng, S. Knutsson and K. Axelsson, Sensitivity analysis of an enthalpy method, in: R.W. Lewis, J.H. Chin and G.M. Homsy, eds., Proc. 7th Internat. Conf. on Numerical Methods in Thermal Problems (Pineridge, Stanford, 1991).