MATHS AND STATS

Sigma Notation

Sometimes we want to add up all the numbers in a particular sequence. Say we want to add up $(2k-1)$ for every value of k from 1 to 10. When $k=1$, we add 1, then when $k=2$, we add 3, and so on up to $k=10$. This gives us the sum:

$$
\begin{aligned}
&(2\times 1-1)+(2\times 2-1)+(2\times 3-1)+\cdots+(2\times 9-1)+(2\times 10-1)\\
&=1+3+5+\cdots+17+19\\
&=100
\end{aligned}
$$

This is called summation. Sigma notation is a more convenient way to write this, especially in a formula where the upper limit on k may itself be unknown. We call k in this case the index. An index can be used to count through each time period in a problem, or each item in a list, or many other things.

Sigma notation uses a large sigma (a greek letter), with the limits on the index written above and below. What pronumeral represents the index is shown on the lower limit.

$$\sum_{k=1}^{10}(2k-1)$$

Examples

$$\sum_{t=0}^{5}t^2=0^2+1^2+2^2+3^2+4^2+5^2=55$$

$$\sum_{n=1}^{4}1000(1+i)^n=1000(1+i)^1+1000(1+i)^2+1000(1+i)^3+1000(1+i)^4$$

$$\sum_{k=0}^{n}(kp-1)=0p-1+1p-1+2p-1+\cdots+np-1$$

$$\sum_{k=0}^{n}kp-1=0p+1p+2p+\cdots+np-1$$

In each of these cases, we are adding a list of terms together, each one identical except for the value of the index. Other unknowns are not affected by this.

Notice the difference the brackets make in the last two examples. In the last example, the -1 is not part of the sum, so is only subtracted once.

To enter these into a calculator, you would still need to add each term separately. In a spreadsheet program such as excel, you could use fill down to put each term of the summation in a new cell, and then sum the values in that column.

Subscripts

We can't always use a formula to get the numbers in a sequence. For example, the interest rate or the amount invested at year n might not follow any formula.

If A is the amount invested, we can represent the amount invested in year n with A_n. So, A_4 would be the amount invested in the 4th year. A_8 would be the amount invested in the 8th year. Each amount may be unrelated, and some of them may even be 0. Using subscripts, though, we could still write a general formula involving these amounts.

$$\sum_{k=1}^{4} P_k(A_k + 1) = P_1(A_1 + 1) + P_2(A_2 + 1) + P_3(A_3 + 1) + P_4(A_4 + 1)$$

This sum gives us four terms. Each one relies on a different value of P and A to calculate, but using sigma notation, we don't need to know what those values are or even how many of them there are to write down a formula.

Common Sums

Sum of integers

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Sum of squares

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Arithmetic sequence

$$\sum_{k=0}^{n-1} (a + kd) = \frac{n}{2}(2a + (n-1)d)$$

Geometric sequence

$$\sum_{k=0}^{n-1} (ar^k) = a\left(\frac{1 - r^n}{1 - r}\right) \qquad \sum_{k=0}^{\infty} (ar^k) = a\left(\frac{1}{1 - r}\right)$$

Exercises

Evaluate the following sums:

1. $\sum_{k=1}^{4} k^3$

2. $\sum_{j=0}^{3} (1.06)^j$

3. $\sum_{p=1}^{4} A_p(1.1)^p$ where $A = \{100,300,0,100\}$

Answers

1. $1 + 8 + 27 + 64 = 100$
2. $1 + 1.06 + 1.1236 + 1.191016 = 4.375\ (3dp)$
3. $110 + 363 + 0 + 146.41 = 619.41$