Shakedown of a cohesive-frictional half-space subjected to rolling and sliding contact

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Received 22 June 2006; received in revised form 16 September 2006
Available online 7 November 2006

Abstract

The problem of rolling and sliding contact of a cylinder on the surface of a half-space of cohesive-frictional material is considered. Three shakedown multipliers, of which two are upper bounds and one is exact are computed using a simple numerical procedure. This latter solution differs significantly from previously published analytical solutions which, for realistic material parameters, typically overestimate the shakedown load by a factor of 1.5–2.5.

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Keywords: Shakedown; Rolling; Hertz contact; Pavement design

1. Introduction

The shakedown limit has since the 1960’s been recognized as the rational design criterion for metallic contacts such as rails, roller bearings, and traction drives (Johnson, 1987). More recently, starting with the work of Sharp and Booker (1984), it has been suggested that the shakedown limit would be an equivalently rational design criterion for road pavements subjected to traffic loads. Although the available experimental evidence is not entirely conclusive, it would seem that shakedown theory does offer important insights into the mechanics of the progressive degradation of road pavements (Sharp and Booker, 1984; Raad et al., 1988, 1989, 2005; Radovsky and Murashina, 1996; Shiau, 2001). Recently, shakedown concepts have also been used to describe the ratcheting behaviour of granular materials in general (Garcia-Rojo and Herrmann, 2005; Garcia-Rojo et al., 2005).

The application of shakedown theory to the geomaterials that usually make up the pavement subgrade requires consideration of a general cohesive-frictional yield criterion (as opposed to the purely cohesive criteria used in metal plasticity). This feature significantly complicates the determination of the shakedown limit, both in numerical and in analytical computations. In the past a number of attempts of both types of calculations have been made. Firstly, in their pioneering paper, Sharp and Booker (1984) introduced the so-called method

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of conics, which can be seen as an application of Melan’s lower bound theorem, and solved a number of two-dimensional problems where the load was approximated in terms of a trapezoidal pressure distribution. This method was later given a kinematic interpretation by Collins and Cliffe (1987) and the resulting generalized upper bound method was applied to a number of more realistic three-dimensional problems (Collins et al., 1993; Collins and Boulbibane, 2000; Boulbibane et al., 2005).

Secondly, the problem has been treated numerically using a combination of finite elements and linear and nonlinear programming techniques (Yu and Hossain, 1998; Shiau, 2001; Boulbibane and Ponter, 2005; Li and Yu, 2006). Although these procedures are often advocated as being either upper or lower bound procedures, they can in fact only be seen as being approximate since rigorous application of the shakedown theorems implies that one knows the exact elastic stresses at every point of the domain. Furthermore, due to the lack of exact shakedown solutions, these procedures are generally difficult to validate and one can observe a large scatter in the results for even the simplest problems (Krabbenhoft et al., 2006).

Recently, Yu (2005) has sought to improve this unsatisfactory state of affairs by providing an analytical solution for two classical benchmark problems, namely those of a homogeneous half-space subjected to either two or three-dimensional Hertzian contact. This constitutes an important contribution towards establishing exact solutions by which numerical procedures can be validated. The solutions of Yu (2005) were derived by using the lower bound theorem of elastic shakedown. For cases with subsurface failure, Yu’s solutions are shown to give rigorous lower bounds.

When surface failure becomes critical, however, Yu’s solutions may exceed those from a rigorous lower bound analysis due to the fact that the residual stresses are not constrained to satisfy the yield condition and equilibrium (unlike the solutions of Yu and Hossain (1998)). In this paper we employ the general approach of Yu (2005), amended to include the effects of the residual stress yield condition and residual stress equilibrium, to derive shakedown results for the problem of two-dimensional Hertzian contact. The full procedure is described in detail, and rigorous results for the elastic shakedown load are obtained for all values of the surface-roller friction coefficient.

2. Problem definition

The general problem of a pavement subjected to traffic load can, as a first approximation, be idealized in terms of a homogeneous half-space subjected to the action of an infinitely long roller (Radovsky and Murashina, 1996; Yu and Hossain, 1998). As such the problem may be modeled as one of 2D plane strain elasticity/plasticity as sketched in Fig. 1.

2.1. Contact pressures

The pressure distribution due to the roller is modeled as a Hertzian contact with vertical and horizontal components given by

![Fig. 1. Half-space of cohesive-frictional material subjected to rolling and sliding contact.](image-url)
\[ p_z = p = p_0 \sqrt{1 - (x/a)^2} \]
\[ p_x = \mu p, \]
\[ \] (1)

where \( \mu \) is the roller-surface friction coefficient and \( 2a \) is the contact width (which depends on the properties of the half-space as discussed by Johnson (1987)).

### 2.2. Equilibrium equations

Assuming a weightless material, the equilibrium equations for this problem are
\[
\begin{align*}
\frac{\partial \rho_x}{\partial x} + \frac{\partial \rho_z}{\partial z} &= 0 \quad \text{in } \Omega, \\
\frac{\partial \rho_z}{\partial x} + \frac{\partial \rho_x}{\partial z} &= 0 \quad \text{in } \Omega, 
\end{align*}
\]
\[ (2) \]

where \( \Omega \) is the half-space domain. The boundary conditions on the unloaded part of the top surface are given by
\[
\begin{align*}
\rho_z &= 0 \quad \text{on } z = 0, \ |x| > a, \\
\rho_x &= 0 \quad \text{on } z = 0, \ |x| > a.
\end{align*}
\] (3)

Furthermore, since all sections along the direction of travel experience the same loading, the residual stresses cannot vary in this direction. This gives the conditions
\[
\frac{\partial \rho_x}{\partial x} = 0; \quad \frac{\partial \rho_z}{\partial x} = 0; \quad \frac{\partial \rho_x}{\partial x} = 0.
\] (4)

Combining these with the equilibrium and boundary conditions, we have
\[
\rho_x = \rho_x(z); \quad \rho_z = 0; \quad \rho_x = 0.
\] (5)

Thus, only a residual stress \( \rho_{xx} \) exists and this varies in the \( z \) direction but is constant in the \( x \) direction.

### 2.3. Yield condition

In the following, we will use the Mohr–Coulomb criterion which under plane strain conditions reads
\[
F(\sigma_{xx}, \sigma_{zz}, \sigma_{xz}) = \sqrt{(\sigma_{xx} - \sigma_{zz})^2 + 4\sigma_{xz}^2} + (\sigma_{xx} + \sigma_{zz}) \sin \phi - 2c \cos \phi = 0,
\] (6)

with \( c \) being the cohesion, \( \phi \) the internal friction angle, and tensile stresses taken as positive.

### 2.4. Elastic stresses

The elastic stresses resulting from the Hertzian contact pressure distribution are given by Johnson (1987)
\[
\begin{align*}
\chi_{xx} &= -p_0 a [(m + mk^2 + mkn^2 - 2z) + \mu(-2n + nk^2 - nkm^2 + 2x)], \\
\chi_{zz} &= -p_0 a [(m - mk^2 + mkn^2) + \mu(nkm^2 - nk^2)], \\
\chi_{xz} &= -p_0 a [nkm^2 - nk^2] + \mu(m + mk^2 + mkn^2 - 2z)],
\end{align*}
\] (7)

where
\[
\begin{align*}
m^2 &= \frac{1}{2} \sqrt{(a^2 - x^2 + z^2)^2 + 4x^2z^2 + 1/2(a^2 - x^2 + z^2)}, \\
n^2 &= \frac{1}{2} \sqrt{(a^2 - x^2 + z^2)^2 + 4x^2z^2 - 1/2(a^2 - x^2 + z^2)}, \\
k &= (n^2 + m^2)^{-1}, \\
m &= |m| \text{sgn}(z), \quad n = |n| \text{sgn}(x),
\end{align*}
\] (8)
3. Shakedown theory

For linear elastic/perfectly plastic materials under cyclic loading three distinct modes of failure can be identified. These are commonly referred to as alternating plasticity (plastic non-shakedown), incremental collapse (ratcheting), and instantaneous collapse (plastic collapse). Roughly speaking, alternating plasticity is critical for problems where significant stress concentrations are present whereas for bending dominated problems failure tends to be by way of incremental or instantaneous collapse. Classical (or elastic) shakedown analysis deals with the prevention of all three types of failure. In the following, the elastic and plastic shakedown theorems are briefly discussed. For a more in-depth treatment, we refer to the papers of Zouain and Silveira (1999, 2001).

For the sake of convenience, we will only deal with the case where the load domain is defined by two points, one of them being zero. With such a load domain, we can determine the maximum amplitude of a single load set varying between zero and the maximum.

3.1. Elastic shakedown theorem

Consider a load domain defined by two points, one of them being zero. The elastic stresses corresponding to the non-zero point are denoted \( \gamma(x) \). Melan’s theorem then states that all of the above mentioned failure modes are prevented if there exists a residual stress field \( \rho(x) \) such that

\[
\nabla \cdot \rho(x) = 0, \quad x \in \Omega,
\n\mathbf{n} \cdot \rho(x) = 0, \quad x \in \partial \Omega_u,
\nF[\rho(x)] \leq 0, \quad x \in \Omega,
\nF[\rho(x) + \alpha \gamma(x)] \leq 0, \quad x \in \Omega,
\]

where \( F \) is the yield function, \( \Omega \) is the domain of interest, and \( \partial \Omega_u \) is the unsupported part of the boundary. If these conditions are fulfilled exactly the multiplier \( \alpha \) will be a lower bound on the true elastic shakedown multiplier. Thus, we seek to maximize \( \alpha \) subject to the above constraints.

3.2. Plastic shakedown theorem

The plastic shakedown theorem follows by simply neglecting the equilibrium constraints in (9) (Polizzotto, 1993; Zouain and Silveira, 1999, 2001). Thus, for any scalar \( \alpha \) which satisfies

\[
F[\rho(x)] \leq 0, \quad x \in \Omega,
F[\rho(x) + \alpha \gamma(x)] \leq 0, \quad x \in \Omega,
\]

collapse by alternating plasticity will not take place. Compared to (9) the plastic shakedown problem is much simpler. The residual stresses no longer need to be self-equilibrating and the yield condition can be checked independently at each point in the combined space-load domain. Obviously, since the plastic shakedown theorem appears as the special case of the elastic shakedown theorem where the equilibrium conditions are neglected, the resulting plastic shakedown multiplier will be an upper bound on the elastic shakedown multiplier, i.e.,

\[
\alpha_{AP} \geq \alpha_{SD},
\]

where \( \alpha_{AP} \) is the plastic shakedown multiplier, or the safety factor against alternating plasticity, and \( \alpha_{SD} \) is the elastic shakedown multiplier.

4. Shakedown analysis of Hertzian rolling and sliding contact problem

We now consider the application of the elastic and plastic shakedown theorems to the two-dimensional Hertzian contact problem described in Section 2. For this problem the elastic shakedown theorem reads
maximize \( \alpha \)
subject to \( \frac{\partial \rho_{xx}}{\partial x} = 0 \),
\[ F(\rho_{xx}) \leq 0, \]
\[ F(\rho_{xx} + \alpha \chi_{xx}, \alpha \chi_{xz}, \alpha \chi_{xz}) \leq 0. \]

The procedure used in the following is essentially identical to the method of conics proposed by Sharp and Booker (1984). The idea is to replace the above problem by a discrete one and subsequently attempt to maximize the shakedown multiplier while satisfying the constraints at each point. With reference to the grid shown in Fig. 1, a discrete form of (12) can be formulated as

\[
\begin{align*}
\text{maximize} & \quad \alpha \\
\text{subject to} & \quad F(\rho_{xx}^{i,j}) \leq 0 \\
& \quad F(\rho_{xx}^{i,j} + \alpha \chi_{xx}^{i,j}, \alpha \chi_{xz}^{i,j} ) \leq 0 \\
& \quad \rho_{xx}^{i,j} = \rho_{xx}^{i+1,j}, \rho_{xx}^{i+1,j} = \rho_{xx}^{i+2,j}, \ldots, \rho_{xx}^{N_x-1,j} = \rho_{xx}^{N_x,j}, j \in \mathcal{N}_z,
\end{align*}
\]

where \( \mathcal{N}_x = (1, N_x) \) and \( \mathcal{N}_z = (1, N_z) \) with \( N_x \) and \( N_z \) being the number of subdivisions in the \( x \) and \( z \) directions respectively. In the following, three different versions of this problem are solved. In classifying the resulting solutions, the terms ‘upper bound’ and ‘exact’ are used with reference to an infinitely fine grid.

4.1. Upper bounds of type 1

The simplest approach to estimating the solution to (13) is to ignore the first set of yield constraints and the equality constraints on the residual stresses. We thus have

\[
\begin{align*}
\text{maximize} & \quad \alpha \\
\text{subject to} & \quad F(\rho_{xx}^{i,j} + \alpha \chi_{xz}^{i,j}, \alpha \chi_{xz}^{i,j} ) \leq 0 \\
& \quad (i, j) \in \mathcal{N}_x \times \mathcal{N}_z.
\end{align*}
\]

This is the case solved by Yu (2005) and since the simplification involves dropping equilibrium as well as yield constraints, the problem can be seen as a weakened version of the plastic shakedown theorem (where all yield constraints are to be satisfied). As such, the resulting shakedown multiplier \( \alpha \) will be an upper bound estimate.

The problem is solved by computing the maximum multiplier at each point of the domain and then taking the minimum between the resulting \( N_x \times N_z \) multipliers (or the ones of these that are positive). The relevant optimization problem is given by

\[
\min_{\rho_{xx}^{i,j} \geq 0} \{ \max \chi^{i,j} \text{ s.t. } F((\rho_{xx} + \alpha \chi_{xx})^{i,j}, \alpha \chi_{xz}^{i,j}) \leq 0 \} \quad (i, j) \in \mathcal{N}_x \times \mathcal{N}_z.
\]

For each subproblem, the optimal residual stress \( \rho_{xx}^{i,j} \) is first found from the condition that

\[
\frac{\partial F}{\partial \rho_{xx}^{i,j}} = 0,
\]

with the solution

\[
\rho_{xx}^{i,j} = \left(-\alpha \chi_{xx} + \frac{2 - \cos^2 \phi}{\cos^2 \phi} \alpha \chi_{xz} - 2c \tan \phi \right)^{i,j}.
\]

This is inserted into the yield condition \( F = 0 \) which is solved for \( \chi^{i,j} \). This gives two solutions, the maximum of which is recorded:

\[
\chi^{i,j} = \max \left( \frac{\chi_{xz} \sin \phi \pm |\chi_{xz}| \cos \phi}{\chi_{zz}^2 \sin^2 \phi - \chi_{xz}^2 \cos^2 \phi} \right)^{i,j}.
\]

It can be shown that this is identical to the multiplier derived by Yu, 2005:
\[ \alpha_{i,j} = \frac{c}{|x_{i}z_{i}| + x_{i}z_{j} \tan \phi} \]  

\[ (19) \]

provided that \(|x_{i}z_{i}| + x_{i}z_{j} \tan \phi > 0\). Since, we are only interested in multipliers greater than zero, the above expression is generally valid for the problem considered here.

4.2. Upper bounds of type 2

It is clear that a potentially better estimate of the shakedown multiplier can be obtained if both yield constraints of the original plastic shakedown problem are considered. Using the same notation as above, the relevant optimization problem is then

\[ \min_{\alpha_{i,j} > 0} \{ \max \alpha_{i,j} \} \]

s.t. \[ F(\rho_{i,j}^{p}) \leq 0 \]

\[ F[(\rho_{xx}^{p} + x_{i}z_{i} \alpha_{i,j}^{+}, (x_{i}z_{j} \alpha_{i,j}^{+}, (x_{i}z_{j} \alpha_{i,j}^{+}) \leq 0), (i, j) \in \mathcal{N}_{x} \times \mathcal{N}_{z}. \]

This problem corresponds to application of the plastic shakedown theorem and can be solved in much the same way as the simplified problem discussed above.

First, it is easily verified that the condition \( F(\rho_{i,j}^{p}) \leq 0 \) limits the residual stresses to

\[ \rho_{xx}^{p} \leq \rho_{xx}^{+} \leq \rho_{xx}^{-}, \rho_{xx}^{+} = -2c \tan (\phi/2 \pm \pi/4). \]  

(21)

With this result in mind, the computations proceed as follows. At each point \((i, j)\) a multiplier is first computed by (19). The resulting \( \alpha_{i,j}^{+} \) is then used to compute the corresponding residual stress (17). If this falls in between the limits given by (21), the first yield constraint \( F(\rho_{i,j}^{p}) \) is not active and nothing is done. If, on the other hand, the bounds (21) are violated, the maximum permissible multiplier is to be found at the intersection between the two yield constraints. In this case the relevant multiplier is to be found by solution of

\[ F[(\rho_{xx}^{p} + x_{i}z_{i} \alpha_{i,j}^{+}, (x_{i}z_{j} \alpha_{i,j}^{+}, (x_{i}z_{j} \alpha_{i,j}^{+}) = 0 \]  

(22)

Closed-form solutions to these equations are again possible although they will not be given here.

4.3. Exact solutions

Finally, we consider the full lower bound elastic shakedown problem (130). This is solved in the following way. For each layer, a plastic shakedown problem of the type (20) is solved and the residual stress at the critical point \( \rho_{xx}^{p} \) is recorded. The yield conditions at each point within this layer are then checked. If these are all fulfilled, we proceed to the next layer. If not, we search within the layer for the residual stress which fulfills both yield constraints at every point and which maximizes the shakedown multiplier. In practice, this is done by searching for multipliers within the permissible range of residual stresses from \( \rho_{xx}^{p} \) to \( \rho_{xx}^{+} \). For each residual stress \( \rho_{xx}^{p} \) within this range, the corresponding multipliers are found by solving

\[ F[\rho_{xx}^{p} + x_{i}z_{i} \alpha_{i,j}^{+}, (x_{i}z_{j} \alpha_{i,j}^{+}, (x_{i}z_{j} \alpha_{i,j}^{+}) = 0, \quad i \in \mathcal{N}_{x}, \quad \text{for fixed } j. \]

(23)

For each such \( \rho_{xx}^{p} \), the minimum positive multiplier within the layer is recorded. When the minimum multipliers for a predefined number of residual stress \( \rho_{xx}^{p} \) in between \( \rho_{xx}^{-} \) and \( \rho_{xx}^{+} \) have been determined, the final multiplier for the layer is taken as the maximum of these. This procedure is repeated for each layer and finally, the minimum positive multiplier among all the layers in the domain is taken as the elastic shakedown multiplier. Alternatively, and arguably more conveniently, the elastic shakedown problem can be solved layer by layer using one of many readily available nonlinear programming routines. For each layer, this gives an optimal multiplier and associated residual stress and the final multiplier is then computed as the minimum of these optimal layer multipliers.

Regardless of exactly which procedure is used, we end up with an approximate solution to the full lower bound elastic shakedown problem. Furthermore, as the number of points in the grid becomes sufficiently large, we can expect the solution to converge to the exact solution.
In the strict classical sense, however, the solutions can neither be regarded as lower bounds nor exact solutions since the necessary conditions are only verified at a finite number of points.

4.3.1. Surface failure

When applying the lower bound elastic shakedown theorem, it is convenient to consider the situation where the critical point is located in the half-space separate from the situation where it is at the surface. At the surface, the elastic stresses are given by

\[
\begin{align*}
\tau_{xx} &= -2p_0 \mu \frac{x}{a} \sqrt{\left(\frac{x}{a}\right)^2 - 1} \\
\tau_{zz} &= 0 \\
\tau_{xz} &= 0 \\
\tau_{xx} &= -p_0 \sqrt{1 - \left(\frac{x}{a}\right)^2 + 2\mu x/a} \\
\tau_{zz} &= -p_0 \sqrt{1 - \left(\frac{x}{a}\right)^2} \\
\tau_{xz} &= -p_0 \mu \sqrt{1 - \left(\frac{x}{a}\right)^2} \\
\tau_{xx} &= -2p_0 \mu \frac{x}{a} \sqrt{\left(\frac{x}{a}\right)^2 - 1} \\
\tau_{zz} &= 0 \\
\tau_{xz} &= 0
\end{align*}
\]  

(24)

Following the procedure outlined in the previous section, a surface shakedown multiplier \( \alpha' \) can be determined. Thus, the problem to be solved is

\[
\begin{align*}
& \text{maximize} \quad \alpha \\
& \text{subject to} \quad F(p_{sx}) \leq 0, \\
& \quad F[p_{sx} + \alpha' \tau_{sx}(x), \alpha' \tau_{xx}(x), \alpha' \tau_{xz}(x)] \leq 0, \quad -\infty < x < \infty.
\end{align*}
\]

(27)

This problem can be solved by guessing the value(s) of \( x \) for which the yield constraints are active. If, subsequently, it can be shown that the yield constraints are also fulfilled for all other \( x \), the exact solution has been found. This solution is an upper bound estimate on the true shakedown multiplier which takes also subsurface failure into account.

It turns out that a good guess of such a solution consists of assuming that the second set of yield constraints in (27) are fulfilled simultaneously at \( x = -a \) and \( x = a \). Here, we have \( \tau_{xx} = 2\mu p_0 \) and \( \tau_{xx} = -2\mu p_0 \) respectively, whereas all other elastic stresses are equal to zero. If we ignore the requirement that \( F(p_{sx}) \leq 0 \), the problem to be solved is

\[
\begin{align*}
& \text{maximize} \quad \alpha' \\
& \text{subject to} \quad F(p_{sx} - \alpha' \mu p_0, 0, 0) \leq 0, \\
& \quad F(p_{sx} + \alpha' \mu p_0, 0, 0) \leq 0,
\end{align*}
\]

(28)

with the solution being

\[
\alpha' = \frac{1}{\mu \cos \phi} \frac{c}{p_0}, \quad p_{sx} = -2c \tan \phi.
\]

(29)

It is easily verified that the residual stress always lies between the bounds (21) so that \( F(p_{sx}) \leq 0 \) is always satisfied. It can further be verified (numerically) that the second yield constraint in (27) is fulfilled for all values of \( x \). Thus, if surface failure occurs, the associated shakedown limit is given by (29).

It is worth noting that the elastic limit at the surface is given by

\[
\alpha'_E = \frac{1 - \sin \phi}{\mu \cos \phi} \frac{c}{p_0}.
\]

(30)
where the critical point is located at \( x = -a \). Thus, in the case of surface failure involving a purely cohesive material, the shakedown limit is equal to the elastic limit.

### 5. Results

In the following, we present the results from the three different types of computations outlined above. For comparison purposes, the elastic limit is also given. The results can be seen in Tables 1–4. In Fig. 2, the various

#### Table 1

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<th>( \mu )</th>
<th>( \phi = 0^\circ )</th>
<th>5(^\circ)</th>
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Underlined multipliers are those for which the critical point is located at the surface.

#### Table 2

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Underlined multipliers correspond to surface failure.

#### Table 3

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Underlined multipliers correspond to surface failure.
multipliers, for different internal friction angles $\phi$, are shown as function of the surface-roller friction coefficient $\mu$.

Regarding the three different shakedown multipliers, the trend is that they are equal whenever the failure takes place beneath the surface. However, as surface failure becomes critical they start to diverge with the exact shakedown factors being significantly lower than the two upper bound multipliers, especially as $\mu$ and $\phi$ increase. The boundary between surface and subsurface failure, in terms of $\mu$ and $\phi$, is shown in Fig. 3. The trend observed in this figure follows that of the elastic limit multiplier where the critical point is also located beneath the surface only for small values of $\mu$ and $\phi$. These results demonstrate the necessity of accurately modeling the different layers of road pavements. Thus, if surface failure for a given pavement indeed is critical, the result must necessarily be viewed with some scepticism as the exact distribution of the

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Underlined multipliers correspond to surface failure—these are given by $\alpha = \alpha = 1/(\mu \cos \phi)/p_0$ in accordance with (29).

Fig. 3. Boundary between surface and subsurface failure.
load (for which the ideal Hertzian distribution is a crude approximation) then will be of paramount importance. Finally, we note that the multipliers derived by Yu (2005) as expected are in good agreement with the upper bounds of type 1.

6. Conclusions

A number of analytical shakedown solutions to the problem of a cohesive-frictional half-space subjected to rolling and sliding contact have been given. Two of these solutions constitute rigorous upper bounds whereas the last is exact, i.e., exact to within the number of points at which the necessary and sufficient shakedown conditions are verified. These solutions, which agree with those of Yu (2005) for cases involving subsurface failure, should prove useful as a means of verifying and benchmarking numerical procedures. For the purpose of practical road pavement design, however, the exact layering of the materials making up the pavement must necessarily be considered. For this purpose, especially if the elastic properties vary, numerical methods must generally be used. Finally, in this context of practical design, we stress that surface failure should be treated with some caution as the exact distribution of the tyre pressure in this case can be expected to have a significant impact on the shakedown multiplier.

References


