MATHS AND STATS

Solving quadratic equations

Quadratic equations are generally written in the form

$$ax^2 + bx + c = 0$$

where *a, b* and *c* are real numbers.

There are 3 common methods to solve such equations:

Method 1: factorisation

Type 1:

When $a = 1$, our equation is of the form $x^2 + \underline{b}x + \underline{c} = 0$

Use the expansion to factorise the equation

$$(x+p)(x+q) = x^2 + x\underline{(p+q)} + \underline{pq}$$

So we look for two numbers *p* and *q*, which add to give *b* and multiply to give *c*.

It is easiest to find the factors of *c* first (ie two numbers multiplying to give *c*)

Example

For $x^2 + \underline{3}x + \underline{2} = 0$, we want two numbers to multiply to 2 and add to 3.
The only factors of 2 are 2 and 1, and they add to give 3. Therefore,

$$x^2 + 3x + 2 = (x+1)(x+2) = 0$$

Now we use the fact that if two numbers multiply to give 0, one of them must be 0.

So either $(x+1) = 0$ or $(x+2) = 0$

$x = -1$ or $x = -2$.

If *c* is negative, then the two numbers we are looking for will have **different signs**. This means *b* will be the **difference** between them.

Example

For $x^2 - x - 6$, we find that 2 and 3 multiply to give 6 and differ by 1. The signs must be different to multiply to get -6, so our choices are -2 and 3, or -3 and 2.
The latter pair add to -1, ie the larger factor takes the sign of the *x*-term or *b*

$$x^2 - x - 6 = (x-3)(x+2) = 0$$

So $x = 3$ or $x = -2$.

Type 2:

If a does not equal 1, then our equation is of the form $ax^2 + bx + c = 0$

We need to do things a little differently. First we look for numbers which add to b, and multiply to give ac.

Example

Solve $2x^2 - x - 6 = 0$.

We need two numbers which multiply to -12 and so one number must be positive and the other negative. We choose our numbers from the following: (1 and 12), (2 and 6), or (3 and 4). They must also add to give -1.
3 and 4 differ by 1, so we use -4 and 3 because $-4 + 3 = -1$

Replace the middle term of the equation $-x$ with the equivalent expression $-4x + 3x$ (note -4 and 3 are the numbers we just found above).

$$2x^2 - 4x + 3x - 6 = 0$$

Now factorise in pairs:

$$2x(x-2) + 3(x-2) = 0$$
$$(2x+3)(x-2) = 0$$
$$2x + 3 = 0 \quad or \quad x - 2 = 0$$
$$x = -\frac{3}{2} \quad or \quad x = 2$$

Method 2: completing the square

We manipulate the quadratic equation so one side of it can be written as a square, i.e. $(x+p)^2$. We make use of the expansion $(x+p)^2 = x^2 + 2px + p^2$ Since we have $ax^2 + bx + c = 0$ we want a p such that $2p = b$. We also need to deal with the constant terms so we subtract c from both sides and add p^2 to both sides. Then we can factorise one side of the equation into a square. From here we can solve the equation for x simply by undoing the operations acting on x.

Example

Solve $x^2 + 6x - 7 = 0$

$$x^2 + 6x - 7 = 0$$

$$x^2 + 6x = 7$$ Put the constant on the other side

$$x^2 + 6x + 3^2 = 7 + 3^2$$ Halve b ie $\frac{6}{2}$ and add $\left(\frac{6}{2}\right)^2$ to both sides

$$(x+3)^2 = 16$$ Factorise the LHS into a perfect square

$$x + 3 = \pm\sqrt{16}$$ Squareroot both sides

$$x + 3 = 4 \quad \text{or} \quad -4$$

$$x = 1 \quad \text{or} \quad -7$$

Both of these methods are fairly easy for "nice" solutions, but when your *a, b,* and *c* are not whole numbers, or your solutions are surds, they are not so easy to do.
(*Completing the square* is also used when graphing circles and ellipses.)

Harder example

$$x^2 + 5x - 8 = 0$$
$$x^2 + 5x = 8$$
$$x^2 + 5x + \left(\frac{5}{2}\right)^2 = 8 + \left(\frac{5}{2}\right)^2$$
$$\left(x + \frac{5}{2}\right)^2 = \frac{57}{4}$$
$$x + \frac{5}{2} = \pm\sqrt{\frac{57}{4}}$$
$$x = -\frac{5}{2} \pm \frac{\sqrt{57}}{\sqrt{4}}$$
$$x = -\frac{5}{2} \pm \frac{\sqrt{57}}{2}$$

To complete the square halve b ie $\frac{5}{2}$ and add $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$ on both sides

Write as a square

Method 3: the quadratic formula

Given $ax^2 + bx + c = 0$

We use the formula for *x*: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This finds all solutions that exist for any quadratic, so is often the preferred method, even though it involves some computation. It gets easier with practise!

Example

Solve $4.2x \;\; + 2x - 5.1 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(4.2)(-5.1)}}{2(4.2)}$$
$$= \frac{-2 \pm \sqrt{89.68}}{8.4}$$
$$x \approx -0.238 \pm 1.127$$ … using a calculator
$$x \approx 0.889 \;\; or \;\; x = -1.365$$

Solving $ax^2 + bx + c = 0$ can be like asking where the graph $y = ax^2 + bx + c$ cuts the *x*-axis.

The graph of $y = ax^2 + bx + c$ is a parabola and the quadratic will have 0, 1 or 2 solutions (called roots or zeros) depending on the parabola's position in relation to the *x*-axis.
The number of solutions (roots or zeros) can be determined quickly from the sign of the number inside the square root, which is called the discriminant (or Delta): $\Delta = b^2 - 4ac$

Sign of $\Delta = b^2 - 4ac$	Number of solutions (roots or zeros)	Parabola feature
Δ > 0	2 – due to the $\pm\sqrt{\Delta}$ in the equation	cuts the *x*-axis twice
Δ = 0	1 – the square root gives ± 0	touches the *x*-axis
Δ < 0	0 real – can't take roots of negative numbers	not cross the *x*-axis

Exercises

Try solving these first questions with a variety of methods.

1) $x^2 + 5x + 4 = 0$
2) $x^2 + 10x + 21 = 0$
3) $x^2 + 4x - 21 = 0$
4) $3x^2 + 16x + 5 = 0$
5) $4x^2 + 2x - 2 = 0$

Some more difficult questions:

6) $x^2 - 5.6x + 31.36 = 0$
7) $5x^2 + 3x + 4 = 0$
8) $2x^2 + 8x + 5 = 0$
9) $-3x^2 - 6.5x + 2.2 = 0$

Solve these by completing the square

10) $x^2 - 8x + 11 = 0$
11) $x^2 - 6x + 3 = 0$
12) $x^2 + 2x - 7 = 0$
13) $2x^2 + 6x + 3 = 0$
    (Hint: Divide both sides by 2 first. Yes you will have to work with fractions.)

Answers

1. $x = -1 \quad or \quad x = -5$
2. $x = -7 \text{ or } x = -3$
3. $x = 3 \text{ or } x = -7$
4. $x = -15 \text{ or } x = -\frac{1}{3}$
5. $x = \frac{1}{2} \text{ or } x = -1$
6. No real solution
7. No real solution
8. $x = -0.775 \text{ or } x = -3.22$
9. $x = 0.298 \text{ or } x = -2.46$

10 $x = 4 + \sqrt{5} \approx 6.23 \text{ or } x = 4 - \sqrt{5} \approx 1.76$

11. $x = 3 + \sqrt{6} \approx 5.45 \text{ or } x = 3 - \sqrt{6} \approx 0.55$
12. $x = -1 + \sqrt{8} \approx 1.83 \text{ or } x = -1 - \sqrt{8} \approx -3.83$
13. $x = -\frac{3}{2} + \frac{\sqrt{3}}{2} \approx 0.63 \text{ or } x = -\frac{3}{2} - \frac{\sqrt{3}}{2} \approx -2.37$