Lower bound limit analysis of slabs with nonlinear yield criteria

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Abstract

A finite element formulation of the limit analysis of perfectly plastic slabs is given. An element with linear moment fields for which equilibrium is satisfied exactly is used in connection with an optimization algorithm taking into account the full nonlinearity of the yield criteria. Both load and material optimization problems are formulated and by means of the duality theory of linear programming the displacements are extracted from the dual variables. Numerical examples demonstrating the capabilities of the method and the effects of using a more refined representation of the yield criteria are given.

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1. Introduction

Limit state analysis of perfectly plastic slabs has been used in design for decades. Two basic methods can be applied, either the upper bound method or the lower bound method. With the upper bound method a geometrically possible collapse field for which the load carrying capacity is at a minimum is sought, whereas with the lower bound method one seeks a statically admissible stress field which maximizes the load carrying capacity. Both methods can be applied as hand calculation methods, prominent examples being Johansen's yield-line theory [1] and the strip method developed by Hillerborg [2], which were both developed with reference to slabs but can be generalized to arbitrary structures and loading conditions.

The limit analysis of perfectly plastic slabs by computer oriented methods has been treated in a number of ways. Both the lower bound method, see Refs. [3–5], and the upper bound method [6,7] have been applied in connection with the finite element concept of dividing the structure into a number of elements related to each other by equilibrium or compatibility conditions.

When using the lower bound method the restrictions consist of a set of equilibrium equations and a set of constraints preventing the violation of the yield criteria. If the number of stress variables exceeds the number of equilibrium conditions the structure is statically indeterminate, and a redistribution of stresses, corresponding to an optimization of the load carrying capacity, becomes possible. When using the upper bound method the internal work is minimized given a constant magnitude of the external work. Additionally, compatibility conditions relating the rotations of the elements to the displacements are imposed.

Both upper bound and lower bound formulations can be solved by means of linear programming (LP) techniques. With the lower bound method a linearization of the yield criteria is necessary, whereas the condition for
upper bound problems to remain linear is that the possible positions of the yield-lines are restricted to the edges in the assembled element mesh.

It is a noticeable fact that the dual principle of limit analysis has an analogy in the duality theory of LP. Within this theory there exists a primal/dual relationship, such that every primal problem has a dual counterpart. The primal and dual problems have identical solutions although the variables of each problem have different physical interpretations. In the context of limit analysis the formulation of upper or lower bound problems corresponds to the formulation of identical primal or dual problems. This can be utilized to extract the collapse mechanism from a lower bound solution.

A problem closely related to the load optimization problem is that of material optimization. Here the loads are given and the optimization problem then consists of finding the most favourable stress distribution with respect to the total amount of material needed to sustain the loads. This problem is especially relevant for concrete slabs where the strength is usually determined by four independent material parameters, namely the positive and negative yield moments in two orthogonal directions.

In the following a triangular slab element with discontinuous moment fields is used. As variables the three moment components are used. The element is formulated such that the variables are unique to each element and not to the nodes in the assembled mesh. Thus, at an internal node connecting a number of elements there will most likely be a discontinuity in the moment variables. These discontinuities are then countered by imposing equilibrium across element interfaces.

The optimization is performed with an algorithm taking into account the full nonlinearity of the constraints. This is especially useful in the case of material optimization where it is generally difficult to obtain a satisfactory linearization of the constraints, which contain both stress and material parameters. Moreover, in many cases a better representation of the yield conditions may improve the solutions relatively more than a mesh refinement. With the algorithm all intermediate solutions are strictly feasible, and the iteration procedure can thus be terminated at any given point to produce a lower bound solution.

2. Problem formulation

In the lower bound method the stress distribution is to be optimized in such a way that the load carrying capacity is at a maximum. The relation between the external load and the internal stress distribution is formulated as a set of linear equilibrium equations

$$\mathbf{H}\mathbf{\beta} = \mathbf{R}_c + \lambda\mathbf{R}$$

(1)

where \(\mathbf{\beta}\) is a vector containing the generalized stress parameters. The load consists of a constant part \(\mathbf{R}_c\) and a part \(\mathbf{R}\) proportional to a scalar load parameter \(\lambda\). Equilibrium is ensured through \(\mathbf{H}\). As with the finite element method applied to e.g. elastic problems, \(\mathbf{H}\) is assembled from local element contributions. The number of stress parameters will usually exceed the number of equilibrium conditions, corresponding to a statically indeterminate structure.
To prevent violation of the yield criteria a number of additional restrictions must be included. This leads to a number of constraints on the form

\[ f_i(\beta, C_d) \leq 0 \quad i = 1, 2, \ldots, n \]  

(2)

where \( C_d \) contains the material strengths. These constraints will for most cases be of a nonlinear nature, but can usually be linearized in a straightforward manner.

With these two sets of constraints, linear equilibrium equalities and nonlinear yield criteria inequalities, the optimization problem can be written as

\[
\begin{align*}
\text{maximize} & \quad \lambda \\
\text{subject to} & \quad H\beta = R_c + \lambda R \\
& \quad f_i(\beta, C_d) \leq 0, \quad i = 1, 2, \ldots, n
\end{align*}
\]  

(3)

In material optimization the loads are known, and the in some sense optimal material strengths are to be determined. This is done by considering the strength parameters in the yield conditions as variables. As in load optimization equilibrium must be ensured. The optimization problem can then be written as

\[
\begin{align*}
\text{minimize} & \quad a^T d \\
\text{subject to} & \quad H\beta = R \\
& \quad f_i(\beta, d) \leq 0, \quad i = 1, 2, \ldots, n
\end{align*}
\]  

(4)

where \( d \) are the variable material parameters, and \( a \) a vector of weights reflecting the cost of each material parameter. The material optimization problem can be extended to deal with more than one load case. This is realized by including additional sets of stress variables corresponding to the number of load cases.

Within the framework of LP the primal/dual concept can be utilized, and useful information retrieved by computing the optimal variables of both problems. Considering a linearized version of (3) we have

\[
\begin{align*}
\text{maximize} & \quad \begin{bmatrix} 0^T & 1 \end{bmatrix} \begin{bmatrix} \beta \\ \lambda \end{bmatrix} \\
\text{subject to} & \quad \begin{bmatrix} H & -R \\ C & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \lambda \end{bmatrix} \leq \begin{bmatrix} R_c \\ C_d \end{bmatrix}
\end{align*}
\]  

(5)

This problem is traditionally referred to as the dual load optimization problem. The primal problem is given by

\[
\begin{align*}
\text{minimize} & \quad -R_c^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} w \\ \theta \end{bmatrix} \\
\text{subject to} & \quad \begin{bmatrix} -C_d^T & H^T \\ R^T & C_d^T \end{bmatrix} \begin{bmatrix} w \\ \theta \end{bmatrix} \leq \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \theta \geq 0
\end{align*}
\]  

(6)

The dual variables are the stresses and the load parameter \( \lambda \). Following [5], the variables \( w \) and \( \theta \) of the primal problem can be interpreted as the displacements and the plastic strains, respectively. The primal problem is then seen to correspond to the upper bound formulation. For the discrete problem the notion of upper and lower bounds can, accordingly, be replaced with an equivalent primal/dual relationship. This makes it possible to extract the collapse mechanism from a problem originally formulated as lower bound.

Depending on the solution method the primal variables appear either as a byproduct of the iterations or can be computed without any significant extra cost.

Instead of solving the problem (5) directly a significant reduction in the number of variables can be achieved by operating on the statically indeterminate stress parameters only, as suggested in [8]. The stress variables \( \beta \) are split into two groups \( \beta_0 \) and \( \beta_1 \), such that the equilibrium equations can be written as

\[ H_0\beta_0 + H_1\beta_1 = R_c + \lambda R \]  

(7)

where \( H_0 \) is a square matrix of full rank. The matrix \( H_0 \) can be thought of as describing the statically determinate part of the structure, whereas the redistribution of stresses is governed by \( H_1 \). To avoid numerical errors the separation of the variables should generally be established using full pivoting.

The \( \beta_0 \)-variables in the yield condition inequalities are expressed by means of the free variables \( \beta_1 \). In the general case the transformation can be written as

\[ f(\beta_0, \beta_1, C_d) = f(\beta_0(\beta_1), \beta_1, C_d) \leq 0 \]  

(8)

where the function \( \beta_0(\beta_1) \) is defined by the separation performed in (7). In the linear case (5) we have

\[ f(\beta_0, \beta_1, C_d) = C_0\beta_0 + C_1\beta_1 - C_d \leq 0 \]  

(9)

The function \( \beta_0(\beta_1) \) is given by

\[ \beta_0(\beta_1) = H_0^{-1}(R_c + \lambda R - H_1\beta_1) \]  

(10)

which by substitution into (9) leads to the reduced problem

\[
\begin{align*}
\text{maximize} & \quad 0^T \begin{bmatrix} \beta_1 \\ \lambda \end{bmatrix} \\
\text{subject to} & \quad \begin{bmatrix} R^T & C^T \\ 0^T & C_d \end{bmatrix} \begin{bmatrix} \beta_1 \\ \lambda \end{bmatrix} \leq \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \lambda \geq 0
\end{align*}
\]  

(11)

where

\[ C = C_1 - C_0H_0^{-1}H_1 \]
\[ \tilde{C}_\lambda = C_0H_0^{-1}R \]
\[ \tilde{C}_d = C_d - C_0H_0^{-1}R_c \]  

(12)

As the ratio between the number of free variables and the total number of variables typically ranges from 0.1 to 0.5 a substantial reduction is possible. Furthermore, the reduction with full pivoting secures a more stable system which in many LP algorithms is essential. The method is independent of the type of element used, and
has been successfully applied for a variety of structures, see [9].

The primal upper bound problem can be reduced by performing a similar separation of the variables, or by utilizing the duality theory of LP. The problem then reads

$$\begin{align*}
\text{minimize} & \quad \bar{C}_0 \theta \\
\text{subject to} & \quad \begin{bmatrix} C' & C'' \end{bmatrix} \theta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{align*}$$ (13)

From the strains $\theta$ the displacements $w$ may be determined by solving the compatibility equations

$$-H^T w + C^T \theta = 0$$ (14)

In the case of material optimization with linear restrictions a separation of the variables equivalent to the one described above for load optimization may be performed, again leading to a system where only the free stress parameters appear.

3. Equilibrium equations

3.1. Plate bending equations

We consider an infinitesimal plate bending element in a Cartesian coordinate system as shown in Fig. 1.

The corresponding force components are bending moments $m_x$ and $m_y$, twisting moment $m_{xy} = m_{yx}$, and shear forces $q_x$ and $q_y$.

Moment equilibrium of an infinitesimal plate bending element gives two equations

$$\frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y} - q_x = 0$$ (15)

In the presence of a distributed transverse surface load $p$ the requirement for transverse equilibrium is

$$\frac{\partial q_y}{\partial x} + \frac{\partial q_x}{\partial y} + p = 0$$ (17)

From the equilibrium equations (15)–(17) it is seen that a quadratic variation of the moments corresponds to a transverse surface load of constant intensity. A linear variation corresponds to constant shear forces along the boundaries, i.e. external load can be represented as distributed loads of constant intensity along the boundaries of the element. In an element with a constant moment variation the only possibility of applying external load is as point loads carried by the torsional moments.

3.2. Equilibrium with triangular elements

The use of the equilibrium equations (15)–(17) in connection with triangular elements is most easily realized with the use of area coordinates. When using area coordinates a polynomial interpolation of a desired quantity, in this case the moments, is chosen. A linear function $f$ is represented as

$$f = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$$ (18)

where $f_j$ are the function values in the corners of the triangle. The area coordinates $\lambda_j = A_j/A$ are defined as shown in Fig. 2, with $A$ being the area of the triangle.

The first order derivatives of a function $f(x,y)$ with respect to $x$ is given as

Fig. 1. Infinitesimal plate bending element.

Fig. 2. Area coordinates.
\[ \frac{\partial f}{\partial x} = \sum_{j=1}^{3} \frac{\partial f}{\partial \lambda_j} = -\frac{1}{2A} \sum_{j=1}^{3} b_j \frac{\partial f}{\partial \lambda_j} \]  

(19)

and with respect to \( y \) as

\[ \frac{\partial f}{\partial y} = \sum_{j=1}^{3} \frac{\partial f}{\partial \lambda_j} = \frac{1}{2A} \sum_{j=1}^{3} a_j \frac{\partial f}{\partial \lambda_j} \]  

(20)

where \( a_j \) and \( b_j \) are defined in Fig. 2.

Using the differentiation rules (19) and (20), the equilibrium equations can be expressed in area coordinates as

\[ -\frac{1}{2A} \sum_{j=1}^{3} \left( b_j \frac{\partial m_x}{\partial \lambda_j} - a_j \frac{\partial m_y}{\partial \lambda_j} \right) - q_s = 0 \]  

(21)

\[ \frac{1}{2A} \sum_{j=1}^{3} \left( a_j \frac{\partial m_y}{\partial \lambda_j} - b_j \frac{\partial m_y}{\partial \lambda_j} \right) - q_s = 0 \]  

(22)

\[ -\frac{1}{2A} \sum_{j=1}^{3} \left( b_j \frac{\partial q_s}{\partial \lambda_j} - a_j \frac{\partial q_s}{\partial \lambda_j} \right) + p = 0 \]  

(23)

where the two first equations express moment equilibrium corresponding to (15) and (16), and the last is the transverse equilibrium equation corresponding to (17).

3.3. Boundary conditions

On a free unloaded edge the bending moments \( m_n \) must be equal to zero. Furthermore it seems reasonable to assume that twisting moments \( m_t \) as well as the transverse shear forces \( q_s \) also disappear. However, these two last condition are unduly excessive, and should be replaced with one boundary condition. This condition can be derived from statical considerations alone, or via the principle of virtual work. For a slab this takes the form

\[ \int_B \left( m_x \kappa_x + m_y \kappa_y + 2m_{xy} \kappa_{xy} \right) \mathrm{d}x \mathrm{d}y \]

\[ = \int_B p \mathrm{d}x \mathrm{d}y + \int_C q \mathrm{d}x \mathrm{d}s - \int_C m_x \frac{\partial w}{\partial n} \mathrm{d}s - \int_C m_y \frac{\partial w}{\partial s} \mathrm{d}s \]  

(24)

where the integration over \( B \) extends to the whole slab, and along \( C \) to the boundary. The \( s \)-coordinate defines the counterclockwise direction along the edges of the element. The last term on the right hand side may be evaluated from integration by parts. Considering the integration of \( m_t \) along a side \( k \) of a triangular element as shown in Fig. 3, i.e. from node \( k + 1 \) to node \( k - 1 \), we get

\[ \int_B \left( m_x \kappa_x + m_y \kappa_y + 2m_{xy} \kappa_{xy} \right) \mathrm{d}x \mathrm{d}y \]

\[ = \int_B p \mathrm{d}x \mathrm{d}y + \int_C q \mathrm{d}x \mathrm{d}s - \int_C m_x \frac{\partial w}{\partial n} \mathrm{d}s - \int_C m_y \frac{\partial w}{\partial s} \mathrm{d}s \]  

(25)

The principle of virtual work for the entire slab then becomes

\[ \int_B \left( m_x \kappa_x + m_y \kappa_y + 2m_{xy} \kappa_{xy} \right) \mathrm{d}x \mathrm{d}y \]

\[ = \int_B p \mathrm{d}x \mathrm{d}y + \int_C K \mathrm{d}x \mathrm{d}s - \int_C m_x \frac{\partial w}{\partial n} \mathrm{d}s - \int_C m_y \frac{\partial w}{\partial s} \mathrm{d}s + \sum_{j=1}^{3} R_j w' \]  

(26)

where the so-called Kirchhoff forces \( K \) are given by

\[ K = q + \frac{\partial m_s}{\partial s} \]  

(27)

and \( R_j \) are concentrated corner forces of a magnitude equal to the difference in twisting moments on either side of a corner, see Fig. 3.

3.4. Slab element

In the following the equilibrium relations for a slab element, see Fig. 4(a), with a linear moment variation are derived. The element is based on the same interpolation as the element used in [10] for analysis of plates subjected to in-plane forces. The moment field within each element is interpolated linearly between the corner values as

\[ m = \sum_{j=1}^{3} \lambda_j m_i \]  

(28)

With the corner moments \( m_i = [m_x^i, m_y^i, m_{xy}^i]^T \) the number of stress parameters per element is nine. Statically admissible moment discontinuities are accomplished by requiring continuity in the bending moments and Kirchhoff forces across element interfaces, see Fig. 4(b). The...
continuity conditions for two adjacent elements $A$ and $B$ are

$$\begin{align*}
(m_{n}^{1})_A &= (m_{n}^{2})_B \\
(m_{n}^{2})_A &= (m_{n}^{1})_B
\end{align*}$$

(29)

$$K_i + K_j = p_{AB}$$

(30)

where $p_{AB}$ is the external load along the connecting side of elements $A$ and $B$. Superscripts $i$ and $j$ refer to the side numbers in the triangles $A$ and $B$, respectively.

The bending and twisting moments along side $k$ of a triangular element can be written as

$$\begin{align*}
\begin{bmatrix} m_n^{k,1} \\ m_n^{k,2} \end{bmatrix} &= \frac{1}{l_k} \begin{bmatrix} -b_k^2 - a_k^2 & 2a_kb_k \\ a_kb_k & -a_kb_k \end{bmatrix} \begin{bmatrix} m_n^{k+\delta} \\ m_n^{k,\delta} \end{bmatrix} \\
&= h_k^{\delta} \begin{bmatrix} h_n^{k+\delta} \\ h_n^{k,\delta} \end{bmatrix}
\end{align*}$$

(31)

where on a given side $k$ the value of $\gamma$, 1 or 2, refers to the beginning and end nodes, respectively, and $l_k$ is the length of the side. Similarly the resulting shear force along an edge $k$ is given by

$$q_k^i = \frac{1}{l_k} (b_kq_x - a_kq_y)$$

(32)

By means of the equilibrium equations (21) and (22) the shear force can be expressed by the moments as

$$q_k^i = \frac{1}{2A_l} \sum_{j=1}^{3} (-b_kb_j - a_kb_j + a_kb_j) \begin{bmatrix} m_n^k \\ m_n^l \end{bmatrix}$$

(33)

In the general case the contribution $\partial m_i/\partial s$ to the Kirchhoff forces may be found using the relationship between the $s$-coordinate and the area coordinates $\lambda_j$. However, for a linearly varying moment field $\partial m_i/\partial s$ is determined directly as the slope of $m_i$ along an edge $k$

$$\begin{align*}
\left( \frac{\partial m_i}{\partial s} \right)^k &= \frac{m_n^{k,2} - m_n^{k,1}}{l_k} \\
&= \frac{1}{l_k} h_k^{\delta} \begin{bmatrix} m_n^{k,1} - m_n^{k+1} \\ m_n^{k+1} - m_n^{k,2} \\ m_n^{k,1} - m_n^{k+1} \end{bmatrix}
\end{align*}$$

(34)
From these two contributions, the equilibrium equations with respect to the Kirchhoff forces can be formulated.

The concentrated corner forces are determined as

\[ R^k = m_t^{k+1} - m_t^{k+2} \]

\[ = \begin{bmatrix} h_1^{k+1} - h_1^{k+2} \\ h_2^{k+1} - h_2^{k+2} \\ h_3^{k+1} - h_3^{k+2} \end{bmatrix} \] (35)

These contributions are included in the total equilibrium system by summation over all elements containing node \( k \)

\[ P^k = \sum_{\text{elements}} R^k \] (36)

where \( P^k \) is the external load in node \( k \).

In summary, the equilibrium and continuity equations across each element interface consist of one equation with respect to Kirchhoff forces and two equations ensuring continuity in the bending moments. In addition, equilibrium equations with respect to concentrated forces are enforced in the nodes of the assembled mesh.

### 3.4.1. Displacements

By solving the compatibility equations

\[-H^T w + C^T \theta = 0 \] (37)

the displacements \( w \) of the collapse mechanism are determined. These consist of three different quantities. From the equations ensuring equilibrium in the nodes of the assembled mesh, the corner displacements of the elements can be determined directly. The displacements corresponding to the equilibrium equations for the Kirchhoff forces are also determined but have to be scaled by a factor \( 1/l \). Likewise, from the equations concerning bending moments the rotations can be determined. These are scaled by a factor \( 2/l \). With regard to the last quantity it must be stressed that it is the external rotations through which external moments do work that are determined. The internal rotations are the strains contained in \( \theta \).

### 4. Solution algorithm

The simplex method is the most widely used algorithm in LP and has been applied extensively to structural optimization. The simplex method starts with a basic feasible solution, i.e. a point on the vertex of the boundary defined by a convex set of restricting planes. From this point an increment is taken in the direction which for a unit step increases the object function the most, thus arriving at an adjacent vertex. As the optimal solution is always found at a vertex, repeating the procedure a finite number of times will eventually lead to the optimal solution.

An entirely different approach to solving LP problems is contained in the interior point methods, so-called because the points generated by the algorithm lie within the interior of the feasible solution space. The modern application of interior point methods was initiated by the publication of Karmarkar’s algorithm in 1984 [11]. Soon after several simplified variants, the affine-scaling algorithms, were proposed, see e.g. [12].

Whereas different methods specifically designed to handle either the primal or the dual problem exist, also combined primal–dual methods are available. In these methods one operates on primal and dual variable simultaneously, the goal being to minimize the difference between the primal and dual solutions.

In the following the primal and dual affine-scaling algorithms are derived. These algorithms are then used to treat problems with nonlinear constraints, the idea being to add new linear restrictions as the iterations proceed and the nonlinear constraints are violated.

#### 4.1. The primal affine-scaling method

The idea behind affine-scaling methods is to move in the steepest descent direction, i.e. the direction which increases or decreases the objective the most. The initial point as well as all subsequent points generated by the algorithm are strictly feasible, i.e. they all lie within the interior of the feasible solution space. For many problems, including those of both load and material optimization, an initial point is usually easily found. For load optimization the initial solution could be obtained from an elastic analysis or, even simpler, by setting all stress variables and the scalar load parameter equal to zero. For problems where the determination of an initial feasible solution is not trivial several methods have been developed. One of the more common of these is the so-called big-\( M \) method. With this method artificial variables are added to make the initial solution feasible. The object function is then penalized by multiplication of the artificial variables by a large number \( M \), the idea being to drive the artificial variables towards zero. In the following it will be assumed that an initial feasible solution is available.

We consider the primal problem in the standard form

\[
\begin{align*}
\text{minimize} & \quad z = c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\] (38)

where \( A \) is an \( m \times n \) matrix, \( b \) a vector of \( m \) components, and \( c \) and \( x \) vectors of \( n \) components. Comparing with (13) the problem in standard form is seen to correspond to the upper bound load optimization problem.
For the problem (38) the direction which minimizes the objective the most, the steepest descent direction, is the direction $-c$.

In general the steepest descent direction can be regarded as the direction that solves the minimization of the linear approximation of a function $f$ within a unit sphere centered in $x^k$:

$$\begin{align*}
\text{minimize} & \quad f(x^k + p) \\
\text{subject to} & \quad \|p\| \leq 1
\end{align*}$$

with the solution being $p = -\nabla f(x^k)$.

Starting from a feasible point, we want to maintain feasibility in the next position, i.e. when moving in the direction $\Delta x$ we must assure that $\Delta A x = 0$. For this purpose an orthogonal projection of the steepest descent direction is used. Any vector $x$ can be decomposed in a unique way by

$$x = p + q$$

where $Ap = 0$ is the component in the null-space of $A$ and $q = A^T \lambda$, for some $m$-dimensional vector $\lambda$, is the component in the range space of $A$. The component $p$ can be expressed be means of the orthogonal projection matrix $P_A$

$$p = P_A x$$

where

$$P_A = I - A^T (AA^T)^{-1} A$$

The projected steepest descent direction

$$\Delta x = -P_A c$$

is the feasible direction that produces the fastest rate of decrease in the objective. A pseudo-algorithm can now be written as shown in Table 1.

The line search consists of determining the parameter $\alpha$ so that the new point $x^{k+1} = x^k + \alpha \Delta x$ satisfies the nonnegativity requirements.

A shortcoming in the method of the steepest descent is that the location of the current point is not taken into consideration, no directions are favoured, we simply move in the direction determined by the objective function. This means that if the current position is somewhere near the “center” of the feasible region, as shown in Fig. 5(a) by $\Delta x_1$, then the objective can be decreased significantly by moving in the projected steepest descent direction. If on the other hand the current position is near the boundary, as defined by the nonnegativity constraints, the improvement might be more moderate, as indicated by the increment $\Delta x_2$. In the affine-scaling methods this is countered by building in some bias such that whenever a movement is made, the proximity of each variable to boundary is taken into consideration. This is done by a scaling transformation of the current point $x^k$ to the point $e = (1, 1, \ldots, 1)^T$. With $X = \text{diag}(x^k)$ the transformed problem can be written as

$$\begin{align*}
\text{minimize} & \quad z = \bar{c}^T \bar{x} \\
\text{subject to} & \quad A \bar{x} = b \\
& \quad \bar{x} \geq 0 \quad (44)
\end{align*}$$

where

$$\bar{x} = X^{-1} x, \quad \bar{c} = Xc \quad \text{and} \quad \bar{A} = AX$$

A step is then taken along the steepest descent direction for the transformed problem, whereupon the resulting point is interpreted with respect to the original problem. As mentioned the steepest descent direction can be considered as the direction that solves the linear approximation of a function within a unit sphere. The scaling now affects the steepest descent direction in such a way that the minimization is performed within an ellipsoid centered in $x^0$ and with axes parallel to the coordinate axes, thus providing a more realistic reflection of the region of interest. The effect of the scaling transformation is depicted in Fig. 5(b). By applying the same
Table 2
Primal affine-scaling algorithm

<table>
<thead>
<tr>
<th>Operation</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial feasible solution</td>
<td>( x^{k-1} )</td>
</tr>
<tr>
<td>Scaling</td>
<td>( \bar{c} = Xc, \bar{A} = AX )</td>
</tr>
<tr>
<td>Increment in ( x )-space</td>
<td>( \Delta x = -P_d \bar{c} )</td>
</tr>
<tr>
<td>Line search</td>
<td>( x_{\text{max}} = \min \left( \frac{-s_k}{\Delta x} \right) )</td>
</tr>
<tr>
<td>Step length adjustment</td>
<td>( z = \gamma x_{\text{max}}, 0 &lt; \gamma &lt; 1 )</td>
</tr>
<tr>
<td>Update</td>
<td>( x^k = x^{k-1} + z \Delta x )</td>
</tr>
<tr>
<td>Transform to ( x )-space</td>
<td>( x^k = XX^k )</td>
</tr>
</tbody>
</table>

The necessary condition for feasibility is obtained by substitution of this expression into the original problem. This gives

\[
(I - A^T (AA^T)^{-1} A)(c - s) = 0 \tag{49}
\]

or

\[
P_d s = P_d c \tag{50}
\]

This problem can then be expressed as

\[
\text{maximize } w = b^T (AA^T)^{-1} A(c - s) \tag{51}
\]

subject to \( P_d s = P_d c \) \( s \geq 0 \)

or, by omitting the constant part in the object function, as

\[
\text{minimize } w = b^T s \tag{52}
\]

subject to \( P_d s = P_d c \) \( s \geq 0 \)

where

\[
b_d = A^T (AA^T)^{-1} b \text{ and } P_d = I - A^T (AA^T)^{-1} A \tag{53}
\]

This problem can then be solved using the same steepest descent and scaling procedure as for the primal problem. One small problem is that the orthogonal projection matrix \( P_d \) is usually singular, which means that the orthogonal projection matrix \( P_{d,s} \) for \( P_d \) does not exist as \( P_d \) must have full row rank in order for the inverse of the product \( P_d P_d^T \) to exist. This problem is overcome by using a slightly different orthogonal projection matrix

\[
Q_d = I - P_d = A^T (AA^T)^{-1} A \tag{54}
\]

where it is easily verified that \( P_d Q_d = 0 \).

The directions of the dual problem may now be found by transforming the original problem and then using the projection matrix \( Q_d \) in a manner equivalent to that of the primal method. Scaling the problem (47) by \( S = \text{diag}(s^k) \) we have

\[
\text{maximize } w = b^T y \tag{55}
\]

subject to \( A^T y + s = e \) \( s \geq 0 \)
where
\[ \dot{s} = S^{-1} s, \quad \dot{A} = AS^{-1} \quad \text{and} \quad \dot{c} = S^{-1} c \] (56)

The direction in \( s \)-space is then obtained by use of the projection matrix for the transformed problem
\[ \Delta s = -Q_s \bar{s} \]
\[ = -A^T (AA^T)^{-1} AA^T (AA^T)^{-1} b \]
\[ = -S^{-1} A^T (AS^{-2}A^T)^{-1} b \] (57)

Transforming this direction into \( s \)-space we have
\[ \Delta s = S\Delta s = -A^T (AS^{-2}A^T)^{-1} b \] (58)
while \( \Delta y \) is computed from the condition \( A^T \Delta y + \Delta s = 0 \) as
\[ \Delta y = -(AA^T)^{-1} A\Delta s = (AS^{-2}A^T)^{-1} b \] (59)

However, in implementing the algorithm it is more efficient to first compute \( \Delta y \) as shown in the last expression of (59), and then the direction of the dual slack as \( \Delta s = -A^T \Delta y \). Since this last computation will not involve any numerical difficulties, feasibility is easily maintained in each iteration. This is in contrast to the primal version of the algorithm where the condition of feasibility rests entirely on the accuracy of the computation of \( \Delta s \). It is also possible to compute estimates of the primal variables. By noticing that \( -AS^{-2}\Delta s = b \) an estimate of the primal variables can be obtained as
\[ x = -S^{-2} \Delta s \] (60)
which will converge towards the optimal primal solution. By calculating the duality gap
\[ c^T x - b^T y = x^T s \] (61)
it is possible to give an estimate on how close the solution is to the optimum.

Compared to the simplex method both the primal and dual affine-scaling methods have the advantage that the computation of the various directions only involves matrices whose sparsity patterns do not change from iteration to iteration, and hence need only be analyzed once.

4.3. Treatment of nonlinear constraints

The algorithms described in the previous sections are all exclusively concerned with linear problems. However, it is relatively straight-forward to include nonlinear constraints by means of a cutting-plane procedure. We consider a problem given by
\[ \text{maximize} \quad w = b^T y \]
\[ \text{subject to} \quad f(y) + s = c, \quad s \geq 0 \] (62)
where \( f(y) \) is a nonlinear convex function. Instead of performing an initial linearization and solving the resulting linear program, the constraints can be linearized as the iterations proceed.

The direction in which an increment should be taken in order to increase the objective the most is always defined regardless of the nature of the constraints, and as such new constraints can be included at any point in the iterations. The problem (62) can thus be solved using the following procedure. At a given point \( y^k \) the restrictions will consist of a number of linear restrictions and a number of nonlinear restrictions.

The point \( y^k \) is assumed to satisfy both set of constraints.

A direction which is feasible with respect to the linear constraints is then computed. In the linear case the maximum permissible step length is determined from the nonnegativity constraints on \( s \). With the inclusion of nonlinear constraints the line search now involves the full set of constraints, linear as well as nonlinear. Taking the maximum permissible step we arrive either on a part of the boundary defined by the nonnegativity constraints \( s \geq 0 \), or on a part which satisfies \( f(y^{k+1}) = 0 \). If the latter is the case, a linear restriction corresponding to the tangent plane of \( f \) in \( y^{k+1} \) is added, and the procedure is repeated. Thus, every iteration consists of computing a direction on the basis of the linear constraints, followed by the addition of new linear restrictions. The algorithm is illustrated in Fig. 6.

The procedure will be applicable for both the primal and dual affine-scaling methods, but in the following only the dual method is considered.

The combined primal–dual methods are usually very effective with respect to linear programs. However, in a scheme as described above they fail, the reason being that when a restriction is added, also an extra variable is added, primal or dual, depending on how the nonlinear constraints are defined. Because of this, maintaining feasibility from iteration to iteration becomes practically impossible.

Besides the purely primal or dual interior point methods, also the simplex method is applicable. If a
general purpose program is used in a scheme as outlined above, a restart facility is necessary in order to ensure an effective implementation. A program with this feature has been developed by Damkilde et al. [13].

4.3.1. Application to limit analysis

To permit direct application of the dual-affine-scaling algorithm the problem has to be in the standard form (47). With the reduced problem (11) this is easily realized by adding nonnegative slack variables $s$ to the left hand side, thus converting the inequality to an equality. The reduced problem in standard form is then

$$\begin{align*}
\text{maximize} & \quad \mathbf{0}^T \begin{bmatrix} \beta_1 \\ \lambda \end{bmatrix} \\
\text{subject to} & \quad \begin{bmatrix} \tilde{C} & \tilde{C}_d \end{bmatrix} \begin{bmatrix} \beta_1 \\ \lambda \end{bmatrix} + s = \tilde{C}_d
\end{align*}$$

(63)

In the following the yield criterion proposed by Nielsen [14] and Wolfensberger [15] for reinforced concrete slabs is used. The criterion is expressed as

$$\begin{align*}
- (m_{xx}^+ - m_x)(m_{yy}^+ - m_y) + m_{xy}^2 & \leq 0 \\
- (m_{xx}^- + m_x)(m_{yy}^- + m_y) + m_{xy}^2 & \leq 0 \\
m_{xx}^- & \leq m_x \leq m_{xx}^+ \\
m_{yy}^- & \leq m_y \leq m_{yy}^+
\end{align*}$$

(64)

where $m_{xx}^+$ and $m_{xx}^-$ are the positive yield moments in the $x$- and $y$-directions, respectively, and similarly $m_{yy}^-$ and $m_{yy}^+$ the negative yield moments in the two directions. The yield criterion consists of two intersecting cones as shown in Fig. 7.

The initial linear restrictions must constrain the problem sufficiently in order to define an initial solution. In the present this is done by limiting the bending moments as required in the two last expressions of (64) resulting in four planes. The torsional moments are restricted by two planes as shown below

$$m_{xy} \leq \frac{1}{2} \sqrt{(m_{xx}^+ + m_{xx}^-)(m_{yy}^+ + m_{yy}^-)}$$

$$m_{xy} \geq -\frac{1}{2} \sqrt{(m_{xx}^+ + m_{xx}^-)(m_{yy}^+ + m_{yy}^-)}$$

(65)

The algorithm starts with moving in the projected steepest descent direction arriving near the boundary as defined by the initial linear restrictions. In this point the original nonlinear yield conditions (64) are checked. If one or more of these are violated the stresses are returned in a direction towards the interior of the feasible region until all yield conditions are again fulfilled. At the intersection between the yield surface and the direction of return a linear restriction is then added, and the procedure is repeated, now with an extra linear restriction. The procedure is outlined in Table 4.

A transformation to original variables is performed in each iteration. This is not strictly necessary as the yield conditions could be expressed in free variables only. However, the transformation is relatively inexpensive and makes the implementation easier and more general.

5. Examples

In the following three examples are presented, demonstrating the capabilities of the slab element used in connection with both linear and nonlinear optimization. In the first example the limit load of a square clamped slab loaded by uniform pressure is determined. The solution is compared with the analytical solution and very good agreement is found. The second example also deals with the determination of the limit load of an isotropic slab, and the solution is here compared with several other solutions, lower bound as well as upper bound. The last example deals with the design of a slab subjected to known loads. A material optimization is performed to compute the four necessary yield moments. The solution obtained from using the full nonlinear yield criteria is compared to solutions where the restrictions have been linearized.

5.1. Example 1

We consider a square slab with side lengths $l$. The slab is clamped on all sides and the material is isotropic
with positive and negative yield moments \( m_p \) in both directions. The load is uniform with intensity \( p \). The exact solution was found by Fox [16] as

\[
p_e = 42.851 \frac{m_p}{I^2}
\]  

(66)

The stress distribution in the vicinity of the corners is rather complicated and the example is therefore well suited as a measure of the general performance of a given element. The example has been treated with the lower bound method by Krenk et al. [5] who used an element for which equilibrium was imposed in the nodes of the assembled mesh, i.e. the distributed forces along the edges of the element were transformed to nodal forces by means of the principle of virtual work. In the following a comparison between this element and the element described earlier is made. Two different meshes, shown in Fig. 8, were used. The symmetry is utilized so that only one eighth of the structure is modeled. The results obtained with the two meshes are shown in Fig. 9, where \( \delta \) is the deviation from the exact solution.

The accuracy is for both elements rather good. However, the limit loads computed with the present element for which exact equilibrium is fulfilled are all lower bound, whereas the opposite is the case with the element for which equilibrium may be violated locally. Furthermore, with the present element the results are more insensitive to the orientation of the mesh, which in part must be ascribed a better representation of the moments.

5.2. Example 2

The next example deals with a real structure with a relatively complicated geometry, as shown in Fig. 10. The load is uniform and the material is isotropic. The limit load was originally determined using a semi-automatic yield-line method, where the yield-line pattern shown in Fig. 10(a) was optimized. Later a fully automatic yield-line calculation gave a limit load of 89.6% of the one determined by the original hand calculation [9]. The yield-line pattern turned out to be very complex and not in particularly good agreement with the pattern shown in Fig. 10(a). In [5] the limit load was computed by means of the lower bound method with the same element as described in the previous example and the mesh shown in Fig. 10(b). Here a value of 83.5% of the original hand calculation result was obtained. Each yield surface was linearized using eight planes. With a linearization consisting of 16 planes a limit load of 88.3% of the original load was found. In the present work the load carrying capacity is determined as 88.8% of what was originally calculated. The solution is close to the best upper bound value, from which it can be concluded that even with relatively coarse meshes, the computed results will be satisfactory.

The displacement field of the collapse mechanism for the lower bound solution is shown in Fig. 10(c). As seen this displacement field is somewhat different from the

![Fig. 8. Element meshes (a) and (b).](image1)

![Fig. 9. Limit loads obtained with meshes (a) and (b).](image2)
yield-line pattern originally used, which explains the difference in the computed load carrying capacities.

5.3. Example 3

The last example deals with the design of the slab shown in Fig. 11. The structure has previously been analyzed in [17] using different meshes. In the finest mesh approximately 1150 constant moment triangles were used, while the coarsest contained 90 elements. The slab is designed to withstand a uniform load of 6.75 kN/m², with the four yield moments each being constant throughout the slab. Two different element meshes, (a) and (b), were used, containing 30 and 90 elements, respectively. The meshes are shown in Fig. 12(a) and (b).

To determine the influence of a linearization of the yield criteria, the optimization was performed with different linearizations as well as with a full nonlinear representation. In the linearizations the nonlinear constraints imposed at each node are represented by 8 or 16 hyper planes.

The results are shown in Table 5. In the last column the sum of the four different yield moments (e.g. for a slab with a fixed thickness, a measure of the total amount of reinforcement steel needed) have been normalized with respect to the result obtained for mesh (b) using a full nonlinear representation of the constraints. As seen from the table the results for the 90 element mesh in [17] are slightly better than the results obtained with the present element using a 30 element mesh. However, if the full nonlinear yield surface is used the situation is reversed. Comparing the finest meshes a similar observation can be made, again with the nonlinear representation of the yield surface having an important influence.

A closer look at the number of variables in the two cases reveals that the present element is actually superior to that used in [17]. With constant moment triangles the number of free variables is approximately $1.5E$ with $E$ being the number of elements in a mesh. The number of yield condition inequalities is $nE$, where $n$ is the number of hyper planes used to represent the yield surface. With the present element the number of free variables is
approximately $4E$ and the number of inequalities $3nE$. By inserting $E = 1150$ and $E = 90$, respectively, it is seen that both the number of variables as well as the number of restrictions are of a magnitude 5 lower in the present analysis than in [17].

From these results it can be concluded that with the present element, very accurate results can be obtained with a relatively coarse mesh when used in connection with an exact representation of the yield criteria. It is also observed how a better representation of the nonlinear constraints has a relatively greater influence on the results than does a mesh refinement.

In the case of isotropic reinforcement a limit load optimization using mesh (b) gives an load carrying capacity of $p = 0.148 \text{kN/m}^2 \times m_p$ corresponding to necessary yield moments of $m_p = 50.55 \text{kNm/m}$. This is a 66% increase in material cost compared to what was obtained with the optimization with four independent yield moments. A further reduction in material cost can be achieved by allowing for different yield moments throughout the slab. However, the practical difficulties with such a reinforcement layout may be considerable, and the number of independent yield moments subject to optimization should therefore be kept at a minimum.

6. Conclusion

A numerical procedure taking into account the nonlinearity of the yield criteria has been developed for limit analysis and material optimization using the lower bound method. The equilibrium equations for a triangular plate bending element have been derived using area coordinates. The element satisfies all internal equilibrium and continuity requirements resulting in true lower bound solutions. The duality theory of LP is utilized so that the collapse mode as well as stress distribution at collapse is computed. With the use of the element in combination with nonlinear optimization it is demonstrated how relatively coarse meshes are sufficient in order to obtain limit loads deviating only a few percent from the exact solutions. The most pronounced
advantage of being able to deal with nonlinear constraints appears in the case of material optimization where it is usually difficult to obtain a satisfactory linearization of the yield criteria. Furthermore, it is demonstrated how the optimal solution in this case is influenced more by the representation of the yield criteria than by the mesh size.

Table 5
Yield moments from material optimization in kNm/m

<table>
<thead>
<tr>
<th></th>
<th>$m_{px}^+$</th>
<th>$m_{py}^+$</th>
<th>$m_{px}^-$</th>
<th>$m_{py}^-$</th>
<th>Total</th>
<th>Total (rel.)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mesh (a), 30 elem.</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 h.p.l.</td>
<td>44.28</td>
<td>83.24</td>
<td>25.79</td>
<td>9.52</td>
<td>162.8</td>
<td>1.33</td>
</tr>
<tr>
<td>16 h.p.l.</td>
<td>40.32</td>
<td>76.79</td>
<td>11.68</td>
<td>7.47</td>
<td>136.3</td>
<td>1.12</td>
</tr>
<tr>
<td>Full nonlinear</td>
<td>37.69</td>
<td>75.60</td>
<td>11.36</td>
<td>6.37</td>
<td>131.0</td>
<td>1.07</td>
</tr>
<tr>
<td><strong>Mesh (b), 90 elem.</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 h.p.l.</td>
<td>39.53</td>
<td>76.18</td>
<td>22.55</td>
<td>5.67</td>
<td>143.9</td>
<td>1.18</td>
</tr>
<tr>
<td>16 h.p.l.</td>
<td>32.00</td>
<td>75.72</td>
<td>15.72</td>
<td>2.98</td>
<td>126.6</td>
<td>1.04</td>
</tr>
<tr>
<td>Full nonlinear</td>
<td>34.00</td>
<td>75.24</td>
<td>11.19</td>
<td>1.64</td>
<td>122.1</td>
<td>1.00</td>
</tr>
<tr>
<td><strong>Ref. [17], 90 elem.</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 h.p.l.</td>
<td>30.44</td>
<td>79.16</td>
<td>30.44</td>
<td>14.36</td>
<td>154.4</td>
<td>1.26</td>
</tr>
<tr>
<td>16 h.p.l.</td>
<td>27.94</td>
<td>75.85</td>
<td>21.00</td>
<td>7.16</td>
<td>126.6</td>
<td>1.08</td>
</tr>
<tr>
<td><strong>Ref. [17], 1150 elem.</strong></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>8 h.p.l.</td>
<td>29.81</td>
<td>80.82</td>
<td>25.72</td>
<td>3.87</td>
<td>140.2</td>
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<tr>
<td>16 h.p.l.</td>
<td>27.83</td>
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<td>14.56</td>
<td>3.26</td>
<td>126.6</td>
<td>1.02</td>
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References