A semi-analytical finite element method for three-dimensional consolidation analysis

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Abstract

An efficient formulation, based on a semi-analytical finite element method, is described for elasto-plastic analyses of consolidation of an axi-symmetric soil body subjected to three-dimensional loading. Expressing the field quantities in the form of discrete Fourier series results in a set of modal equations that can be solved separately. This has the effect of considerably reducing the necessary storage and the cost of solving three-dimensional problems. The numerical method is applied to the problem of a laterally loaded pile in consolidating elasto-plastic soil. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

The equations of a non-linear consolidating soil are complex and in order to solve problems of any complexity it is usually necessary to resort to a numerical approach such as the finite element method. The conventional finite element method, which has proved to be an extremely powerful analytical tool for the solution of many engineering problems, is capable, at least in principle, of dealing with any two- or three-dimensional problems. Potentially, a three-dimensional finite element analysis could be used to analyse most complicated soil problems. However, analysis of such problems usually involves the solution of very large sets of algebraic equations, which is extremely time consuming and expensive. It is therefore desirable to search for an alternative technique that can reduce the computational labour.

In many physical problems loading may be three dimensional in nature but the geometry and material properties do not vary in one or more coordinate directions.

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In such cases it is possible to solve a set of algebraic equations arising from a substitute problem, not involving those particular coordinates, and to synthesise the true answer from a series of such simplified solutions. In general, this type of analysis has been termed the ‘semi-analytical method’ in finite element analysis [1].

A combination of a two dimensional finite element process and a continuous Fourier series in the third dimension has been shown to be an efficient way of analysing elastic and elasto-plastic behaviour. This method was first developed in the context of linear analysis by Wilson [2]. Extension of Wilson’s work to an axi-symmetric elasto-plastic body was first investigated by Meissner [3]. Winnicki and Zienkiewicz [4] used a visco-plastic formulation to tackle material non-linearity. Carter and Booker [5] applied the continuous Fourier series to provide an efficient analysis of the consolidation of axi-symmetric elastic bodies subjected to non-symmetric loading. Lai and Booker [6] used a discrete Fourier technique to analyse the non-linear behaviour of axi-symmetric solids under three-dimensional loading conditions. Hage-Chehade and Meimon [7] applied the continuous Fourier series to analyse the consolidation of axi-symmetric elasto-plastic bodies subjected to three-dimensional loading.

The continuous Fourier series approach is very useful for elastic analyses with relatively simple loading conditions. In such cases, only a few harmonics may be necessary to obtain an adequate representation of the field quantities. However, for analyses which incorporate elasto-plasticity there are difficulties associated with the calculation of the values of harmonic forces as well as with summing the large number of Fourier terms. Some of these difficulties were demonstrated by Hage-Chehade and Meimon [7].

The discrete Fourier series approach has several advantages over the continuous Fourier series approach. The discrete Fourier series terminates after a finite number of terms. The integration of pseudo forces occurring in plasticity and visco-plasticity presents no extra difficulty in the discrete Fourier series method, which thus seems to be preferable for problems involving material non-linearity.

In this paper, the discrete Fourier series in an axi-symmetric space is utilised to formulate an efficient finite element method for consolidation analysis of non-linear soil. The method is then applied to analyse the problem of a laterally loaded pile in consolidating elasto-plastic soil.

2. Formulation of a coupled finite element method based on discrete Fourier series

In an axi-symmetric body, it is possible to divide the body into $M$ identical wedges within a cylinder, provided that the geometry and material properties do not vary in the circumferential direction (Fig. 1). In this case the body exhibits a polar periodicity with period $M$. Therefore, any function $g$ of the discrete variable $j$, defined in the $M$ wedges satisfies

$$g_j = g_{j+km} \quad \text{for } k = 1, 2, \ldots$$
A periodic function like $g_j$ of the discrete variable $j$ can be represented in the discrete Fourier form as

$$g_j = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} G_k e^{ijk\alpha}$$

(1)

where $\alpha$ is equal to $2\pi/M$ and $G_k$ are the Fourier coefficients given by

$$G_k = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} g_j e^{-ijk\alpha}.$$  

(2)

With the above definition, the field quantities for a consolidation problem in soil mechanics may also be written in terms of their Fourier coefficients as

$$(u_r, u_z, u_\theta, q, f)_j = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} (U_r, U_z, U_\theta, Q, F)_k e^{ijk\alpha}$$

(3)

Fig. 1. Typical arrangement of wedges and elements in finite element idealisation.
where \((u_r, u_z, u_\theta)_j, q_j,\) and \(f_j\) denote nodal displacement components, excess pore pressures, and nodal forces applied to wedge \(j\), respectively. \((U_r, U_z, U_\theta, Q, F)_k\) are the \(k\)th Fourier coefficients of the nodal displacements, pore water pressures and applied nodal forces given by

\[
(U_r, U_z, U_\theta, Q, F)_k = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} (u_r, u_z, u_\theta, q, f)_j e^{-ijk\alpha}.
\]

A detailed formulation of the coupled finite element method for consolidation analysis is presented in the Appendix, where it is shown that application of the principle of virtual work to a saturated consolidating soil leads to the following equation

\[
dn^T S \Delta n = dn^T \Delta r
\]

where \(dn^T = (du^T, dq^T)^T\), \(\Delta n = (\Delta u, \Delta q)^T\), \(\Delta r = (f_R, \rho)^T\) and

\[
S = \begin{pmatrix}
    K & -L^T \\
    -L & -\Delta t \beta \Phi
\end{pmatrix}.
\]

In the above equations, \(u\) refers to nodal displacements, \(q\) represents nodal pore pressures, \(K\) is the stiffness matrix of the soil skeleton, \(L\) is the coupling matrix, \(t\) denotes time, \(\beta\) is an integration constant, \(\Phi\) is the flow matrix, \(f_R\) and \(f_\rho\) are the vectors of body forces and flow terms.

Since the body is divided into \(M\) identical wedges, the terms in the left hand side of Eq. (5) can be expressed as the summation of the contributions from each of the wedges, i.e.

\[
dn^T S \Delta n = \sum_{j=1}^{M} dn_j^T S_j \Delta n_j
\]

where \(S_j\) is the stiffness matrix of a typical wedge and \(n_j\) is the vector of nodal displacements and pore water pressures of the wedge, consisting of nodal variables on the two vertical cutting planes, \(w_j\) and \(w_{j+1}\), and variables at nodes within the wedge, \(w'_j\), viz.

\[
n_j = \begin{pmatrix}
w_j, w'_j, w_{j+1}
\end{pmatrix}^T.
\]
The matrix $S_j$ can be partitioned according to the above nodal subdivision, i.e.

$$S_j \cdot \Delta n_i = \begin{pmatrix} A_S & B_S & C_S \\ B_S^T & D_S & E_S \\ C_S^T & E_S^T & F_S \end{pmatrix} \begin{pmatrix} \Delta w_j \\
\Delta w_j' \\
\Delta w_{j+1} \end{pmatrix}. \quad (8)$$

Thus Eq. (6) may be written as:

$$\sum_{j=1}^{M} d n_j^T \cdot S_j \cdot \Delta n_j = \sum_{j=1}^{M} \begin{pmatrix} d w_j^T, d w_j'^T, d w_{j+1}^T \end{pmatrix} \begin{pmatrix} A_S & B_S & C_S \\ B_S^T & D_S & E_S \\ C_S^T & E_S^T & F_S \end{pmatrix} \begin{pmatrix} \Delta w_j \\
\Delta w_j' \\
\Delta w_{j+1} \end{pmatrix}. \quad (9)$$

Applying the discrete Fourier series representation to the nodal variables gives:

$$\begin{pmatrix} \Delta w_j \\
\Delta w_j' \\
\Delta w_{j+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{M}} \sum_{l=0}^{M-1} \Delta W_{l e^{j l \alpha}} \\
\frac{1}{\sqrt{M}} \sum_{l=0}^{M-1} \Delta W_{l e^{j l \alpha}} \\
\frac{1}{\sqrt{M}} \sum_{l=0}^{M-1} \Delta W_{l e^{j(1+l) \alpha}} \end{pmatrix}, \quad \begin{pmatrix} d w_j^T \\
d w_j'^T \\
d w_{j+1}^T \end{pmatrix} = \begin{pmatrix} 1 \sum_{k=0}^{M-1} d W_k^T e^{-j k \alpha} \\
1 \sum_{k=0}^{M-1} d W_k'^T e^{-j k \alpha} \\
1 \sum_{k=0}^{M-1} d W_{k+1}^T e^{-j k \alpha} \end{pmatrix}. \quad (10)$$

where $W$ and $W'$ are the Fourier coefficients of $w$ and $w'$, $W_k^T$ and $W_k'^T$ are the conjugate transposes of Fourier coefficients of $w$ and $w'$. Substituting relations (10) into Eq. (9) results in:

$$\sum_{j=1}^{M} d n_j^T \cdot S_j \cdot \Delta a_j = \frac{1}{M} \sum_{j=1}^{M} \sum_{k=0}^{M-1} \sum_{l=0}^{M-1} e^{-j l \alpha} (d W_k^T, d W_k'^T, d W_{k+1}^T e^{-j k \alpha})$$

$$\times \begin{pmatrix} A_S & B_S & C_S \\ B_S^T & D_S & E_S \\ C_S^T & E_S^T & F_S \end{pmatrix} \begin{pmatrix} \Delta W_l \\
\Delta W_l' \\
\Delta W_{l+1} \end{pmatrix} e^{j l \alpha}. \quad (11)$$

It can be shown that:

$$\frac{1}{M} \sum_{j=1}^{M} e^{j (l-k) \alpha} = \begin{cases} 1 & \text{if } l = k \\
0 & \text{if } l \neq k \end{cases}$$
Therefore, Eq. (11) is simplified to:

\[
\sum_{j=1}^{M} d_{n_j}^T S_j \Delta n_j = \sum_{k=0}^{M-1} \left( d W_k^T \cdot S_{1_k} \Delta W_k + d W_k^T \cdot S_{2_k} \Delta W_k' + d W_k^T' \cdot S_{3_k} \Delta W_k \right)
\]

or

\[
\sum_{j=1}^{M} d_{n_j}^T S_j \Delta n_j = \sum_{k=0}^{M-1} \left( d W_k^T, d W_k'^T \right) \left( \begin{array}{cc}
S_{1_k} & S_{2_k} \\
S_{3_k} & S_{4_k}
\end{array} \right) \left( \begin{array}{c}
\Delta W_k \\
\Delta W_k'
\end{array} \right)
\]

or in a compact form:

\[
d_{n}^T S \Delta n = \sum_{k=0}^{M-1} d_{N_k}^T \cdot S_k \cdot \Delta N_k
\]

In Eqs. (12) and (13), \( S_k \) and \( N_k \) are the \( k \)th Fourier coefficients of \( S \) and \( n \), \( N_k^T \) is the conjugate transpose of the \( k \)th Fourier coefficients of \( n \), and \( S_{1_k} \) to \( S_{4_k} \) are the \( k \)th Fourier coefficients of the partitioned \( S \), as defined below.

- \( S_{1_k} = A_S + C_S e^{i k \alpha} + C_S^T e^{-i k \alpha} + F_S \) (14a)
- \( S_{2_k} = B_S + E_S^T e^{-i k \alpha} \) (14b)
- \( S_{3_k} = B_S^T + E_S e^{i k \alpha} \) (14c)
- \( S_{4_k} = D_S \) (14d)

Utilizing the discrete Fourier series representation of the nodal variables and nodal loads, the terms in the right hand side of Eq. (5) can be expressed as

\[
d_{n}^T \Delta r = \sum_{k=0}^{M-1} \left( d W_k^T, d W_k'^T \right) \left( \begin{array}{c}
G_k \\
G_k'
\end{array} \right) = \sum_{k=0}^{M-1} d_{N_k}^T \cdot \Delta R_k
\]

where \( G_k \) and \( G'_k \) are the \( k \)th Fourier coefficients of the nodal values for nodes in the cutting plane and for nodes within the wedge, respectively, and \( R_k \) is the \( k \)th Fourier coefficient of \( r \).

Combining Eqs. (13) and (15) gives the equation of virtual work in a new form, which contains Fourier coefficients of the equation components, i.e.

\[
\sum_{k=0}^{M-1} d_{N_k}^T \cdot S_k \cdot \Delta N_k = \sum_{k=0}^{M-1} d_{N_k}^T \cdot \Delta R_k
\]
Eq. (16) is true for any arbitrary variations of virtual nodal values of $\Delta N_k^*$, thus:

\[
\sum_{k=0}^{M-1} S_k \cdot \Delta N_k = \sum_{k=0}^{M-1} \Delta R_k
\]

This equation defines $M$ sets of equations relating the load–deformation behaviour of the consolidating body in discrete Fourier space, i.e.

\[
S_k \cdot \Delta N_k = \Delta R_k \quad \text{for} \quad k = 0 \text{ to } M-1
\]

The modal stiffness matrices, $S_k$, given by Eq. (18) are, in general, non-symmetric and complex. A simple redefinition of the degrees of freedom allows them to be converted to a real and symmetric matrix. $A_S$ to $F_S$ can be decomposed into sub-matrices relating to each of their components of nodal variables, e.g.

\[
(A_S) \left( \begin{array}{c} \Delta u \\ \Delta q \end{array} \right) = \left( \begin{array}{ccc} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{array} \right) \left( \begin{array}{c} \Delta u_{r,z} \\ \Delta u_0 \\ \Delta q \end{array} \right) \]

Because of the geometry of an axi-symmetric body, each wedge has reflective symmetry about its bisecting plane. Therefore,

\[
C_S^T = J^T \cdot C_S \cdot J \quad (20a)
\]

\[
F_S = J^T \cdot A_S \cdot J \quad (20b)
\]

\[
E_S = J^T \cdot B_S^T \cdot J \quad (20c)
\]

\[
E_S^T = J^T \cdot B_S \cdot J \quad (20d)
\]

and

\[
D_S = \begin{pmatrix} D_{11} & 0 & D_{13} \\ 0 & D_{22} & 0 \\ D_{31} & 0 & D_{33} \end{pmatrix}
\]

where

\[
J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

and $I$ is identity matrix of the size of $u_r$ or $q$. 
Substitution of the relations (20) into Eq. (14a) gives:

\[ S_{1k} = A_S + C_S \cdot e^{jka} + J^T \cdot C_S \cdot e^{-jka} + J^T \cdot A_S \cdot J \]  

(21)

It thus follows that

\[ S_{1k} = 2 \begin{pmatrix} A_{11} + \eta C_{11} & i\mu C_{12} & A_{13} + \eta C_{13} \\ i\mu C_{21} & A_{22} + \eta C_{22} & i\mu C_{23} \\ A_{31} + \eta C_{31} & i\mu C_{32} & A_{33} + \eta C_{33} \end{pmatrix} \]  

(22)

in which \( \eta = \cos(ka) \) and \( \mu = \sin(ka) \). In the same way, the other components of matrix \( S_k \) can be expressed as:

\[ S_{2k} = \begin{pmatrix} B_{11}(1 + e^{-jka}) & B_{12}(1 - e^{-jka}) & B_{13}(1 + e^{-jka}) \\ B_{21}(1 - e^{-jka}) & B_{22}(1 + e^{-jka}) & B_{23}(1 - e^{-jka}) \\ B_{31}(1 + e^{-jka}) & B_{32}(1 - e^{-jka}) & B_{33}(1 + e^{-jka}) \end{pmatrix} \]  

(23)

\[ S_{3k} = \begin{pmatrix} B_{11}^T(1 + e^{jka}) & B_{12}^T(1 - e^{jka}) & B_{13}^T(1 + e^{jka}) \\ B_{21}^T(1 - e^{jka}) & B_{22}^T(1 + e^{jka}) & B_{23}^T(1 - e^{jka}) \\ B_{31}^T(1 + e^{jka}) & B_{32}^T(1 - e^{jka}) & B_{33}^T(1 + e^{jka}) \end{pmatrix} \]  

(24)

\[ S_{4k} = \begin{pmatrix} D_{11} & 0 & D_{13} \\ 0 & D_{22} & 0 \\ D_{31} & 0 & D_{33} \end{pmatrix} \]  

(25)

Substituting the above relationships into Eq. (18) yields

\[
\begin{pmatrix}
2(A_{11} + \eta \ C_{11}) & 2\mu C_{12} & 2(A_{13} + \eta \ C_{13}) \\
2\mu \ C_{21} & 2(A_{22} + \eta \ C_{22}) & 2\mu C_{23} \\
2(A_{31} + \eta \ C_{31}) & 2\mu C_{32} & 2(A_{33} + \eta \ C_{33}) \\
B_{11}^T(1 + e^{ika}) & B_{12}^T(1 - e^{ika}) & B_{13}^T(1 + e^{ika}) \\
B_{21}^T(1 - e^{ika}) & B_{22}^T(1 + e^{ika}) & B_{23}^T(1 - e^{ika}) \\
B_{31}^T(1 + e^{ika}) & B_{32}^T(1 - e^{ika}) & B_{33}^T(1 + e^{ika}) \\
(\Delta \ U_{r,z})_k & (\Delta \ U_{\theta})_k & (\Delta \ U_{r,z})_k \\
(\Delta \ O)_k & (\Delta \ U_{r,z})_k & (\Delta \ O')_k \\
(\Delta \ U_{\theta})_k & (\Delta \ U_{r,z})_k & (\Delta \ U_{\theta})_k \\
(\Delta \ Q)_k & (\Delta \ Q')_k & (\Delta \ Q')_k
\end{pmatrix}
\times
\begin{pmatrix}
(G_{r,z})_k \\
(G_{\theta})_k \\
(G_{r,z})_k \\
(G_{\theta})_k \\
(G_{r,z})'_k \\
(G_{\theta})'_k \\
(G_{r,z})'_k \\
(G_{\theta})''_k
\end{pmatrix}
\]  

(26)
A real and symmetric form of Eq. (26) can finally be obtained by multiplying the fourth, fifth and sixth rows by $e^{-ik_0^2}$, and multiplying the second and fifth rows by $i$, i.e.

$$
\begin{pmatrix}
2(A_{11} + \eta C_{11}) & 2\mu C_{12} & 2(A_{13} + \eta C_{13}) & 2\eta B_{11} & 2\mu B_{12} & 2\eta B_{13} \\
-2\mu C_{21} & 2(A_{22} + \eta C_{22}) & -2\mu C_{23} & -2\mu B_{21} & 2\eta B_{22} & -2\mu B_{23} \\
2(A_{31} + \eta C_{31}) & 2\mu C_{32} & 2(A_{33} + \eta C_{33}) & 2\eta B_{31} & 2\mu B_{32} & 2\eta B_{33} \\
2\eta B^T_{11} & -2\mu B^T_{12} & 2\eta B^T_{13} & D_{11} & 0 & D_{13} \\
2\mu B^T_{21} & 2\eta B^T_{22} & 2\mu B^T_{23} & 0 & D_{22} & 0 \\
2\eta B^T_{31} & -2\mu B^T_{32} & 2\eta B^T_{33} & D_{31} & 0 & D_{33}
\end{pmatrix}
$$

\begin{align}
&= \begin{pmatrix}
(\Delta U_{r,z})_k & i(\Delta U_{\theta})_k \\
(\Delta \theta) & e^{-ik_0^2(\Delta U_{r,z})}_k & e^{-ik_0^2(\Delta U_{\theta})}_k \\
e^{-ik_0^2(\Delta Q')}_k & e^{-ik_0^2(\Delta Q')}_k & e^{-ik_0^2(\Delta Q')}_k & e^{-ik_0^2(\Delta Q')}_k
\end{pmatrix}
\begin{pmatrix}
(G_{r,z})_k \\
(G_{\theta})_k \\
(G_p)_k \\
(G'_{r,z})_k \\
(G'_{\theta})_k \\
(G'_{p})_k
\end{pmatrix}
\end{align}

(27)

In Eq. (27), $A_{ij}$ to $D_{ij}$ are the components of the partitioned stiffness matrix defined by Eqs. (8) and (19), $\eta = \cos(ka)$ and $\mu = \sin(ka)$, $\eta' = \cos(ka/2)$ and $\mu' = \sin(ka/2)$, $(G_{r,z,\theta})_k$ and $(G_p)_k$ are the Fourier coefficients of $f_R$ and $f_p$ corresponding to the nodes on the cutting plane of the wedge, $(G'_{r,z,\theta})_k$ and $(G'_{p})_k$ are the Fourier coefficients of $f_R$ and $f_p$ corresponding to the nodes within the wedge, respectively. The modal equation for linear elements can be obtained by removing rows and columns related to mid-side nodes from Eq. (27).

3. Computation process and convergence

The solution to Eq. (27) can be obtained by separating the real and imaginary parts of the right hand side vector and solving for the real and imaginary parts of the Fourier coefficient of the nodal variables. Since the modal stiffness matrix is identical for both real and imaginary parts of the equation, it can be set up and factorised once. The real and imaginary components of the displacement coefficients can be calculated by a back-substitution process for real and imaginary parts of the load vector. Finally, the nodal variables can be calculated from their Fourier coefficients using Eq. (3).

In practice, it is only necessary to solve Eq. (27) for the first half of the discrete Fourier coefficients by taking advantage of the symmetry about one vertical plane. In a purely elastic material, the solution to the problem reduces to the solution of Eq. (27) for only the first two modes of the discrete Fourier coefficients, since the body forces and surface traction, which are expressed in term of $\eta$ and $\mu$, have zero values for all but the first two modes of the discrete Fourier coefficients.

A solution to Eq. (27) for an elasto-plastic material model requires a load increment process employing the ‘pseudo force’ method. This method, which has been used previously by Lai [8] in an iterative process and discussed in detail by Taiebat [9], incorporates a pseudo force in the load vector to accelerate the
convergence. The general equation of the pseudo force method for the solution of non-linear problems can be written as:

$$S_I \Delta n_i = r_i + \Delta r_i - \int B^T \sigma_i dV + (S_I - S_T) \Delta n_i$$  \hspace{1cm} (28)

where $S_I$ is the initial or elastic stiffness matrix, and $S_T$ is the tangent stiffness matrix. The integral expression is used to calculate the out-of-balance force. The pseudo force is also calculated by the last expression in Eq. (28) by approximating $\Delta n_i$ by $\Delta n_{i-1}$. This method is illustrated schematically in Fig. 2.

A solution to Eq. (27) gives the increments in the nodal variables over any time increment of $\Delta t$. If the nodal variables are known at time $t$, they can be found at time $(t + \Delta t)$ and so the solution can be marched forward in the time domain.

4. Illustrative example

To demonstrate the power of the newly developed algorithm in a consolidation analysis, the time dependent behaviour of a vertical pile embedded in a saturated elasto plastic soil and subjected to a lateral load applied at the mudline, is examined here. This problem was also studied by Carter and Booker [5] for a consolidating soil with a perfectly elastic skeleton.
A pile with diameter $D_p$ is embedded in a layer of saturated cohesionless soil which obeys the Mohr–Coulomb failure criterion. For this hypothetical problem, both associated flow rule and non-associated flow rule plasticity were considered for a purely frictional soil. The friction angle of the soil is $\phi' = 30^\circ$, and the dilation angle is $\psi = 30^\circ$ for the associated flow rule and $\psi = 0$ for the non-associated flow rule. The soil has a submerged unit weight of $\gamma_{\text{sub}} = 0.7\gamma_w$, where $\gamma_w$ is the unit weight of pore water, a Young’s modulus for fully drained conditions given by $E_s' = 3000\gamma_w$ and a Poisson’s ratio $\nu_s' = 0.30$. The initial value of the coefficient of lateral earth pressure is $K_o = 0.5$. The Young’s modulus of the pile material is $E_p = 1000 E_s'$. The problem was analysed assuming elastic as well as elasto-plastic models for the soil.

The dimensions of the problem are defined in Fig. 3, which also indicates a vertical cross-section of the finite element mesh used in the computations. The cylindrical mesh consists of 12 wedges. All elasto-plastic analyses have been carried out using quadratic hexahedron finite elements together with a reduced integration scheme [1].

To examine the time dependent consolidation behaviour of the pile, it is convenient to introduce a non-dimensional time factor $T_v$, defined as

$$T_v = \frac{k(1 - \nu_s')E_s't}{\gamma_w(1 - 2\nu_s')(1 + \nu_s')D_p^2}$$

![Fig. 3. Finite element mesh.](image-url)
where \( k \) is the coefficient of soil permeability and \( t \) represents time. It is also convenient to define a load rate parameter, \( \omega \), as

\[
\omega = \frac{\frac{d}{dT}(H/\gamma_w D^3_p)}{D^3_p}
\]

where \( H \) is the horizontal load applied to the head of the pile at ground level.

An elastic analysis of the problem was first conducted to evaluate the accuracy of the newly developed algorithm. Results of an analysis using the continuous Fourier series method suggested by Carter and Booker [5] have also been obtained with both isoparametric elements and mixed elements. The mixed elements are isoparametric with respect to displacements and sub-parametric with respect to pore pressure. Mixed elements are used sometimes in consolidation analysis in order to overcome the deficiency associated with the use of small time steps at the early stage of consolidation.

In the elastic analysis, a horizontal load, \( H \), was applied rapidly to the pile head. Thereafter the load was held constant with time. The predicted lateral displacements of the pile head in the direction of the applied load are plotted against dimensionless time, \( T_v \) (Fig. 4). The results of the analysis using the continuous Fourier method with isoparametric elements show discrepancies of about 3% at the early stage of consolidation. However, the results of the new method of analysis using the discrete Fourier approach are in close agreement with the results of the analysis using the continuous Fourier method and the mixed type of elements.
In a series of elasto-plastic analyses, the total lateral load was varied from \( H = 5\gamma_w \times D_p^3 \) to \( 55\gamma_w \times D_p^3 \). In each case the total load was applied during a total time of \( T_v = 0.00001 \), with a loading rate of \( \omega = 100,000 \). This loading rate was sufficiently high to approximate an initial undrained loading. Thereafter, the load was maintained constant with time and the analyses were continued, allowing excess pore pressures to dissipate, and thus for the soil to consolidate.

The time-dependent lateral displacements of the pile head predicted by the elasto-plastic analyses with both associated and non-associated flow rules are plotted in Fig. 5 for the case where the horizontal load is \( H = 15\gamma_w \times D_p^3 \). Also presented in Fig. 5 is the response of the pile in the elastic soil. A significant dependence of the response of the pile on the assumed soil model can be observed in Fig. 5. The largest displacement for the pile head is predicted by the elasto-plastic soil model with a non-associated flow rule. At the end of loading, the displacement of the pile head predicted using a non-associated flow rule with \( \psi = 0 \) is about twice that predicted using an associated flow rule. The stiffer behaviour of a pile in soil with an associated flow rule can be attributed to the dilative characteristic of the soil after failure. Expansion of the soil after failure increases confining pressures which in turn increase soil resistance, causing stiffer behaviour in comparison to the behaviour of soil with a non-associated flow rule.

The predicted load-displacement curves for the pile head, for cases where the pile deforms under fully drained conditions, are presented in Fig. 6. Cases are plotted for the elasto-plastic soil model with both associated and non-associated flow rules as well as for the elastic soil model. As explained previously, the soil with an associated flow rule and dilation angle of \( \psi = 30^\circ \) is stiffer than the soil with a non-associated flow rule and a dilation angle of \( \psi = 0 \).

The predicted responses of the pile in an elasto-plastic soil obeying an associated flow rule (\( \phi = \psi = 30^\circ \)) are plotted in Fig. 7. Lateral displacements of the pile

![Fig. 5. Comparison of the lateral displacements of the pile head in elastic and elasto-plastic soils.](image-url)
deforming under both fully drained state and rapid loading conditions followed by consolidation are presented for various horizontal load levels. The response of the pile during rapid loading is almost linear and close to the elastic response of the pile. During consolidation under a maintained load level, the lateral displacement of the pile increases, so that the displacement at the end of consolidation becomes approximately equal to the displacement predicted by the elasto-plastic analysis assuming fully drained conditions.

Fig. 6. Comparison of the pile response with different soil models, each deforming under fully drained conditions.

Fig. 7. Lateral displacement relationships for laterally loaded piles, Mohr-Coulomb soil model with associated flow rule, $\phi = \psi = 30^\circ$. 
The responses of the pile in elasto-plastic soil obeying a non-associated flow rule, \( \phi = 30^\circ \) and \( \psi = 0 \), deforming under both fully drained conditions and rapid loading conditions followed by consolidation are presented in Fig. 8. With this soil model, the response of the pile during rapid loading is very close to that corresponding to the fully drained conditions. For cases where the load is applied rapidly and then maintained, the displacement at the end of consolidation is greater than the displacement predicted by separate analyses assuming fully drained conditions. This behaviour indicates that for a soil that obeys the Mohr–Coulomb failure criterion and a non-associated flow rule, the load path has an important influence on the final displacement of the soil, as might have been expected.

Figs. 9 and 10 show the expansion of the yield zones around the piles after the rapid loading for both associated and non-associated flow rule plasticity. The plots are in the vertical plane of the applied horizontal load. Comparison of these figures shows that at the end of rapid loading, there is a clear difference between the plastic zones resulting from analyses using associated and non-associated flow rule plasticity models. In the case of using an associated flow rule, the plastic zones for all load levels are concentrated around the pile head and close to the soil surface. However, in the case of using a non-associated flow rule with \( \psi = 0 \), the plastic zone starts to expand around the pile tip at a horizontal load of approximately \( H = 25\gamma_w D_p^3 \), and finally it surrounds the entire pile.

The difference in pile response for different plasticity models can be attributed to the differences in the stress distributions predicted for the various soil models. This is illustrated by the distributions of radial effective stresses, \( \sigma'_r \), and excess pore water pressures, \( p \), predicted by the analyses for a horizontal load of \( H = 15\gamma_w D_p^3 \). These distributions are illustrated in Figs. 11–13. The plots are in the vertical plane of the applied horizontal load. The differences between the distribution of radial effective

![Fig. 8. Lateral displacement relationships for laterally loaded piles, Mohr–Coulomb soil model with non-associated flow rule, \( \phi = 30^\circ \), \( \psi = 0 \).](image-url)
Fig. 9. Expansion of the plastic zone under various load levels at the end of rapid loading, Mohr–Coulomb soil model with associated flow rule, $\phi = \psi = 30^\circ$.

Fig. 10. Expansion of the plastic zone under various load levels at the end of rapid loading, Mohr–Coulomb soil model with non-associated flow rule, $\phi = 30^\circ$, $\psi = 0$. 
Fig. 11. Distribution of the radial stresses at the end of rapid loading, $H = 15\gamma_w D_p^2$, Mohr–Coulomb soil model with associated flow rule, $\phi = \psi = 30^\circ$.

Fig. 12. Distribution of the radial stresses at the end of rapid loading, $H = 15\gamma_w D_p^2$, Mohr–Coulomb soil model with non-associated flow rule, $\phi = 30^\circ$, $\psi = 0$. 
stresses at the end of rapid loading are clear. Relatively high radial effective stresses can be detected close to the pile head and at the surface of the soil when an associated flow rule is adopted.

Distributions of excess pore pressures predicted for different soil models at the end of rapid loading are compared in Fig. 13. The highly dilative behaviour of the soil with an associated flow rule results in the expansion of the plastic soil close to the pile head. As a consequence, negative pore water pressures develop close to the soil surface during the period of loading. In general, the zone of negative pore water pressure predicted using an associated flow rule is greater than that predicted using a non-associated flow rule. As a result, at most times the effective stresses at any point in soil with an associated flow rule are generally greater than those predicted for a non-associated flow rule. This results in a stronger and stiffer response of the pile in the soil with the associated flow rule.

5. Computational efficiency

The efficiency of the semi-analytical method can be demonstrated by comparing the corresponding computational time required for the direct formulation of the finite element method and that for application of the discrete Fourier series. Using the usual solution techniques with solvers which take into account the equation bandwidth, the computational time required for any formulation is approximately

![Fig. 13. Excess pore water pressures at the end of rapid loading, $H = 15\gamma_u D_p^3$, Mohr–Coulomb soil model with associated and non-associated flow rule.](image)
proportional to the number of equations times the square of the bandwidth. Therefore, the ratio of the computational time required for the two formulations can be expressed as

\[ R_T = \frac{T_{\text{semi-analytical}}}{T_{\text{direct formulation}}} \approx \frac{144N_r^3N_z^3(2 + M)}{32N_r^3N_z^3N_d^3M} \approx \frac{9(2 + M)}{2N_r^2M} \]  

(29)

where \( N_r \) and \( N_z \) are the number of columns and rows of the finite element mesh and \( N_d \) is the number of degree of freedom. For the mesh presented in Fig. 3, with \( N_z = 18 \) and \( M = 12 \), Eq. (29) shows that the computational time for the semi-analytical method reduces to about 1.6% of the time required for the direct formulation. A solution to the elasto-plastic problem presented in this paper, using the semi-analytical method and a computer with Pentium-166 MHz processor, takes less than 2 min for each time increment.

6. Conclusions

An efficient formulation based on the semi-analytical finite element method has been presented for consolidation analysis of an axi-symmetric soil body subjected to three-dimensional loading. By representing field quantities in the form of discrete Fourier series, a three-dimensional problem is reduced to a number of two-dimensional problems, which has the effect of considerably reducing the necessary time for solving three-dimensional problems. Introduction of the discrete Fourier series into finite element consolidation analysis removes the need for a fully three-dimensional finite element analysis to study time dependent, non-linear, elasto-plastic, axi-symmetric problems.

The capability and accuracy of the method have been demonstrated through analysis of the problem of a laterally loaded pile. There appear to be no published solutions for laterally loaded pile foundations in consolidating elasto-plastic soil. However, application of this method to the problem indicates promising results. The results of consolidation analyses show that the response of the pile depends largely on the soil model used in the analyses. The results of the numerical analyses indicate a significant path-dependent behaviour for the pile in the elasto-plastic soil with a non-associated flow rule. For cases where the load is applied rapidly and then maintained, the lateral displacement at the end of consolidation is greater than the displacement predicted by separate analysis assuming fully drained conditions.

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Appendix. Formulation of a coupled finite element method for consolidation analysis

The finite element equations of a consolidating soil can be obtained in a variety of ways, all relying on some integral statement corresponding to the equilibrium condition and the continuity of pore water. The well known general theory of three-dimensional consolidation of Biot [10] is usually applied in deriving the equations. This theory considers the elastic deformation of a porous medium and the interaction of the solid and fluid phases. Several investigators have developed methods of solving Biot’s equations of consolidation by application of the finite element technique (e.g. [11–13]) based on spatial as well as temporal discretisation.

Coupled finite element equations for consolidation analysis can be obtained by using the principle of virtual work. For any virtual displacement, \( \delta \), the equilibrium equations and the stress boundary conditions are satisfied if

\[
\int \delta^T \sigma \cdot dV - \int \delta^T R \cdot dV = 0. \tag{A1}
\]

Similarly, for any virtual pore pressure, \( p \), the assumption of continuity of pore fluid is satisfied if

\[
\int \nabla p v \cdot dV - \int dp \frac{\partial v}{\partial t} \cdot dV = 0. \tag{A2}
\]

In the above equations, \( \varepsilon \) and \( \sigma \) refer to strain and stress, \( V \) indicates the volume, the vector \( R \) contains the components of body forces and surface tractions, \( p \) refers to excess pore water pressures in the soil, \( \delta \) refers to displacements, \( v \) are the components of superficial velocity of pore fluid flow, \( \varepsilon_v \) is volumetric strain, and \( t \) denotes time.

Introducing \( N_p \) as the shape functions for pore pressure and \( N_d \) as the shape functions for displacements, the variation of field variables can be approximated from nodal variables, i.e.

\[
\delta = N_d u \]

\[
p = N_p q
\]

where \( u \) and \( q \) are the nodal displacements and nodal pore pressures, respectively.

In order to be consistent with the traditional sign convention adopted in soil mechanics, it will be assumed that both compressive stress and strain are positive. Therefore, the strains are defined by

\[
\varepsilon = -B \cdot u \tag{A3}
\]

\[
\varepsilon_v = \varepsilon_r + \varepsilon_z + \varepsilon_\theta = e^T \cdot B \cdot u \tag{A4}
\]
and the gradients of the pore pressure by

\[ \nabla p = \left( \frac{\partial p}{\partial r}, \frac{\partial p}{\partial z}, \frac{\partial p}{\partial \theta} \right)^T = E.q. \]  \hspace{1cm} (A5) \]

Darcy’s law for the flow of pore fluid may be written as

\[ v = -\frac{k}{\gamma_w} \left( \frac{\partial p}{\partial r}, \frac{\partial p}{\partial z}, \frac{\partial p}{\partial \theta} \right)^T = -\frac{k}{\gamma_w}.E.q. \] \hspace{1cm} (A6) \]

In the above equations, \( B \) is the matrix of strain-displacement transformations, \( k \) is the matrix of permeability coefficients, \( E = \left( \partial N_p / \partial r, \partial N_p / \partial z, \partial N_p / \partial \theta \right)^T \), \( e = (1, 1, 1, 0, 0, 0)^T \), and \( \gamma_w \) is the unit weight of pore water.

Considering \( D \) as the constitutive matrix of an elastic solid skeleton, the stress-strain relation can be written in the form

\[ \sigma = D.e + e.p. \] \hspace{1cm} (A7) \]

Insertion of Eqs. (A3) and (A7) into the internal virtual work Eq. (A1) results in

\[ d\varepsilon^T.\sigma = -du^T.B^T.\sigma = -du^T.B^T(D\varepsilon + e.p) \]
\[ = du^T.B^T.D.B^T.u - du^T.B^T.e.N_p.q. \] \hspace{1cm} (A8) \]

Substituting Eq. (A8) into (A1) gives

\[ du^T(K.u-L^T.q) = du^T \int N_d.R.dV \] \hspace{1cm} (A9) \]

where

\[ K = \int B^T.D.B.dV \]
\[ L^T = \int B^T.e.N_p.dV. \]

Eq. (A9) can be written in incremental form as

\[ du^T(K.\Delta u-L^T.\Delta q) = du^T \int N_d.\Delta R.dV. \] \hspace{1cm} (A10) \]
The components of Eq. (A2) can also be written in the following forms

\[
\nabla dp. v = dq^T.E^T.v = -\frac{1}{\gamma_w}dq^T.E^T.k.E.q
\]

\[
dp. (\partial \varepsilon_v / \partial t) = dq.N_p.e^T.B.(\partial u/\partial t)
\]

As a result, Eq. (A2) becomes

\[
\int \nabla dp. v.dV - \int dp. \frac{\partial \varepsilon_v}{\partial t}.dV = -dq^T\left(\Phi.q + L.\frac{\partial u}{\partial t}\right) = 0
\]

(A11)

where

\[
\Phi = \frac{1}{\gamma_w}E^T.k.E.dV.
\]

Eq. (A11) can be integrated with respect to time by using an approximate single step integration rule, i.e.

\[
\int_{t}^{t+\Delta t} q.d\tau = \Delta t[(1 - \beta).q_t + \beta.q_{t+\Delta t}] = \Delta t(q_t + \beta.\Delta q)
\]

where \(\Delta t\) is the increment of time over which the integration is performed, \(q_t\) is the value of pore pressure at the beginning of the current increment, and \(\beta\) is a parameter which corresponds to the particular interpolation, with \(\beta = 0\) forward interpolation, \(\beta = 0.5\) linear interpolation, \(\beta = 1\) backward interpolation, \(\beta = 2/3\) parabolic interpolation, etc. By application of this integration rule, Eq. (A11) becomes

\[
dq^T(-L.\Delta u - \Delta t.\beta.\Phi.\Delta q) = dq^T(\Delta t.\Phi.q_t).
\]

(A12)

A combination of Eqs. (A10) and (A12) results in a coupled set of equations of virtual work for consolidation analysis, i.e.

\[
(du^T, dq^T)\begin{pmatrix} K & -L^T \\ -L & -\Delta t.\beta.\Phi \end{pmatrix}\begin{pmatrix} \Delta u \\ \Delta q \end{pmatrix} = (du^T, dq^T)\begin{pmatrix} f_R \\ f_p \end{pmatrix}
\]

(A13)

where

\[
f_R = \int N_d.\Delta R. dV
\]

\[
f_p = \Delta t.\Phi.q_t.
\]
Eq. (A13) is true for any arbitrary variations of virtual nodal values of displacements and pore pressures, thus

$\begin{pmatrix} K & -L^T \\ -L & -\Delta t, \beta, \Phi \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta q \end{pmatrix} = \begin{pmatrix} f_R \\ f_p \end{pmatrix}.$

(A14)

A solution to Eq. (A14) gives the increments in the nodal variables over any time increment $\Delta t$. If the nodal variables are known at time $t_1$, they can be found at time $t_2 = t_1 + \Delta t$ and so the solution can be marched forward in time in the usual way.

The conditions of stability and accuracy of the consolidation algorithm have been examined by many researchers, among them Booker and Small [14], Sandhu et al. [15], Vermeer and Verruijt [16] and Reed [17]. Booker and Small [14] have examined the algorithm for a time integration involving the parameter $\beta$, and concluded that the process is unconditionally stable for $\beta \geq 0.5$. Sandhu et al. [15] examined several integration algorithms in the time domain. They have shown that in temporal meshes involving drastic changes in the size of the time step, the error in pore pressure estimation is associated with the use of $\beta = 0.5$. In the current study, temporal integration with $\beta = 1$ (implicit Euler backward interpolation method) is used for all analyses.

If the stress–strain behaviour of the soil skeleton is non-linear, then the constitutive Eq. (A7) must be written in incremental form. It is not difficult to show that a set of governing equations, identical in form to (A4), will also apply to such a non-linear material.

References