Indices (exponents/powers)

An index is used as a shortcut to indicate multiplying the same number (called the base) by itself multiple times. In 2^3, 2 is the base, 3 is the index, and 2 is being raised to the third power (or cubed). For example:

2^5 means $2 \times 2 \times 2 \times 2 \times 2$ $\quad$ 5^3 means $5 \times 5 \times 5$ $\quad$ $(-3)^2$ means $(-3) \times (-3)$

Two special cases are the indices 0 and 1:

$$a^0 = 1 \qquad \text{and} \qquad a^1 = a$$

Also note that in the $(-3)^2$ example, multiplying two negatives gives a positive result. In general, taking a negative number to an even power will be positive, but to an odd power will be negative.

Multiplying and Dividing indices

If we multiply two indexed terms, we are actually adding more copies of the base to be multiplied. For example:

$$2^3 \times 2^4 = 2 \times 2 \times 2 \ \times \ 2 \times 2 \times 2 \times 2 \ = \ 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \ = 2^7$$

Dividing works much the same way:

$$2^5 \div 2^2 = \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2} = 2^3$$

We are effectively adding or subtracting the indices. This leads us to the following rules:

$$a^b \times a^c = a^{b+c}$$

$$a^b \div a^c = a^{b-c} \qquad \frac{a^b}{a^c} = a^{b-c}$$

What happens when we have an index on top of another index? For example:

$$(3^4)^3 = 3^4 \times 3^4 \times 3^4 = 3 \times 3 \times 3 \times 3 \ \times \ 3 \times 3 \times 3 \times 3 \ \times \ 3 \times 3 \times 3 \times 3 = 3^{12}$$

When we expand out the power, we get three lots of four 3's all multiplied together.

$$(a^b)^c = a^{bc} = (a^c)^b$$

Negative and Fractional Indices

Using the multiplying and dividing rules, we can figure out what a negative index means.

$$3^{-3} = 3^2 \div 3^5 = \frac{3 \times 3}{3 \times 3 \times 3 \times 3 \times 3} = \frac{1}{3 \times 3 \times 3} = \frac{1}{3^3}$$

A negative index turns out to be a count of how many of the base are on the bottom of the fraction. You might think of this as how many times we are dividing by the base, rather than multiplying by.

$$a^{-b} = \frac{1}{a^b}$$

With the way we have defined indices, a fraction as an index doesn't seem to make sense. How can the number of times we multiply a number by itself not be a whole number? So again lets try and use our existing rules to figure out what is going on.

$$\left(9^{\frac{1}{2}}\right)^2 = 9^{\left(\frac{1}{2} \times 2\right)} = 9^1 = 9$$

This shows that $9^{1/2}$ is a number which multiplies itself to give 9 (i.e. 3). Which means it is another way of writing $\sqrt{9}$.

It turns out that "to the power of $1/2$" means the square root, and "to the power of $1/3$" means the cube root, and so on. If we have an index of $2/3$, that is the same as $2 \times \left(\frac{1}{3}\right)$ so this means the square of the cube root, or the cube root of the square (either way will work as $2 \times \left(\frac{1}{3}\right) = \left(\frac{1}{3}\right) \times 2$). If we have an index of $3/2$, that would be the cube of the square root, or the square root of the cube.

$$a^{1/c} = \sqrt[c]{a}$$

$$a^{b/c} = \left(\sqrt[c]{a}\right)^b = \sqrt[c]{a^b}$$

Remember that you cannot have the square root (or any even root) of a negative number (although odd roots, such as cube roots, of a negative number are fine).

Exercises

1) $4^3 =$
2) $12^2 =$
3) $10^{-1} =$
4) $4^{-1} =$
5) $3^{-2} =$
6) $7^{-2} =$
7) $6^0 =$
8) $5^0 =$
9) $(-5)^2 =$
10) $(-1)^4 =$
11) $(-3)^3 =$
12) $a^5 \times a^9 =$
13) $m^2 \times m^3 =$
14) $c^6 \times c =$
15) $p^6 \div p^4 =$
16) $x^3 \div x^7 =$
17) $w^{10} \div w =$
18) $(x^3)^5 =$
19) $(2^2)^{-3} =$
20) $(x^3)^7/(x^5)^2 =$
21) $((x^3)^2)^5 =$
22) $144^{1/2} =$
23) $36^{1/2} =$
24) $1000^{1/3} =$
25) $8^{1/3} =$
26) $64^{2/3} =$
27) $8^{2/3} =$
28) $9^{3/2} =$

Answers

1) 64
2) 144
3) $\frac{1}{10}$
4) $\frac{1}{4}$
5) $\frac{1}{9}$
6) $\frac{1}{49}$
7) 1
8) 1
9) 25
10) 1
11) -27
12) a^{14}
13) m^5
14) c^7
15) p^2
16) $x^{-4} = \frac{1}{x^4}$
17) w^9
18) x^{15}
19) $2^{-6} = \frac{1}{64}$
20) x^{11}
21) x^{30}
22) 12
23) 6
24) 10
25) 2
26) 16
27) 4
28) 27