MATHS AND STATS

Permutations and Combinations

A permutation is an arrangement with an order and the order is relevant. The permutation ABC is different to the permutation ACB.

A combination is a collection of things without an order or where the order is not relevant. The combination ABC is the same as the combination ACB.

PERMUTATIONS WITH REPETITION

Most examples can be approached in two different ways, by filling in boxes, or by using formulas.

If the ordering is relevant, repetitions are allowed and there are n objects to choose from, then there are n^r different arrangements of r objects possible.

Example

Choose a 4 digit PIN from the digits 0 to 9. Repetition is allowed.

(a) Think of the number of ways you can fill the four places by filling in boxes

Number of ways to do this $\boxed{10}\boxed{10}\boxed{10}\boxed{10}$

$$10 \times 10 \times 10 \times 10 = 10^4 = 10000$$

(b) Use the formula $n^r = 10^4 = 10000$

PERMUTATIONS WITHOUT REPETITION

Most examples can be approached in two different ways, by filling in boxes, or by using formulas.

If the ordering is relevant, repetitions are not allowed and there are n objects to choose from, then there are

$$_nP_r = \frac{n!}{(n-r)!}$$

different arrangements of r objects possible.

Some books use different notations.

$$P(n,r) = {_nP_r} = {^nP_r} = \frac{n!}{(n-r)!}$$

You have an $_nP_r$ button on your calculator.

Example

From a group of 10 people in the club choose 3 different people to be president, secretary and treasurer.

Number of ways to do this $\boxed{10}\boxed{9}\boxed{8} = 10 \times 9 \times 8 = 720$

OR $_nP_r = \frac{10!}{7!} = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1} = 10 \times 9 \times 8 = 720$

COMBINATIONS WITHOUT REPETITION

If the ordering is not relevant, repetitions are not allowed and there are n objects to choose from, then there are

$$_nC_r = \frac{n!}{r!(n-r)!}$$

different combinations of r objects possible.

Since we do not care about the order of the r objects $_nC_r$ is the number of ways to arrange r objects without repetition given n objects, $_nP_r$, divided by the number of ways of arranging those r objects, $r!$. That is $_nC_r = \frac{n!}{r!(n-r)!} = {_nP_r} \div r!$

Some books use different notations.

$$C(n,r) = {_nC_r} = {^nC_r} = \frac{n!}{r!(n-r)!}$$

You have an $_nC_r$ button on your calculator.

Example

From a group of 10 people, select 3 people to form a committee.

$$_{10}C_3 = \frac{10!}{7!\times3!} = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{(7\times6\times5\times4\times3\times2\times1)\times(3\times2\times1)} = \frac{10\times9\times8}{3\times2\times1} = 120$$

COMBINATIONS WITH REPETITION

If the ordering is not relevant, repetitions are allowed and there are n objects to choose from, then there are

$$_{n+r-1}C_r = \frac{(n+r-1)!}{r!\,(n-1)!}$$

different combinations of r objects possible.

You might find the notation

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!\,(n-1)!}$$

easier to work with.

Example

We have 4 different types of flour available to make our bread; rye, wheat, barley and soy. We need 3 cups of flour for the recipe. We can use any combination of the flours, from all 3 cups of the same type, to each cup being a different type. How many possible combinations are there?

Think of lining up our 4 jars of flour $\boxed{r}\ \boxed{w}\ \boxed{b}\ \boxed{s}$

We start from the rye jar and end at the soy jar (this requires 3 movements between jars) and we need to take 3 scoops. We can do these in any order. Let → indicate moving to the next jar and ■ mean taking a cup of flour.

The sequence → ■■ →→ ■ gives 2 cups wheat and 1 cup soy.
■ → ■ → ■ → gives 1 cup rye, 1 cup wheat, 1 cup barley.
So we can consider the question to be rephrased as "how many ways are there to arrange 3 ■ (representing the 3 cups of flour) and 3 → (the moves between the 4 flour jars)". Or equivalently how many ways can we choose 3 positions (for the 3 cups of flour) from the 6 positions available. The 6 positions are made from the 3 cups and the 4-1=3 moves between the 4 jars. So we have
$\binom{r+(n-1)}{r}$ or more commonly expressed as $\binom{n+r-1}{r}$
In this example $\binom{n+r-1}{r} = \binom{4+3-1}{3} = \binom{6}{3} = \frac{6!}{3!\times 3!} = 20$

OTHER PERMUTATIONS

1. Arranging n objects when *p* are alike, *q* are alike and *r* are alike.

Example

How many ways are there to arrange 15 folders on a bookshelf when 4 are blue, 6 are red and 5 are yellow?
If all 15 were different the answer would be 15! Here, as we can't tell the 4 blue folders apart we need to dive by the 4! arrangements of these blue folders, similarly for the red folders and the yellow folders. So there are
$\frac{n!}{p!\times q!\times r!} = \frac{15!}{4!\times 6!\times 5!} = 630630$ different arrangements of our folders.

2. Arranging n objects in a circle (clockwise and anticlockwise arrangements are considered different).

This is different to arranging in a line as there is no set place to start. Placing the first object can only be done in one way as all places are considered equivalent. After the first object is placed it becomes the same as placing the last $n-1$ objects in a line.
Number of ways = $(n-1)!$

Example

6 people are to be seated at a circular table at a restaurant. As the waiter always moves in a clockwise direction at this restaurant, clockwise and anticlockwise arrangements are different. How many ways are there of arranging the customers?
$(n-1)! = 5! = 120$

3. Arranging n objects in a circle when direction is not important.
As clockwise and anticlockwise are considered the same the formula becomes

$$\frac{(n-1)!}{2}$$

Example

6 people are to be seated at a circular table for a meeting. There is no designated head of the table so we are only interested in who is seated each side of a person. How many different arrangements are possible?

$$\frac{(n-1)!}{2} = \frac{5!}{2} = \frac{120}{2} = 60$$

EXERCISES

1. How many ways can you form a 4-person committee from 6 men and 5 women?
   (a) with no restrictions
   (b) the committee has 2 men and 2 women
   (c) the committee has only 1 woman

2. How many ways can you arrange 3 French books, 2 history books, and 5 maths books if
   (a) All books are different
   (b) All the French books look the same, all the history books look the same and all the maths books look the same.

3. How many number plates with 6 characters can be formed using letters and numbers from 0 to 9
   (a) With no restrictions
   (b) Without repetitions
   (c) If you have 3 letters followed by 3 numbers (repetition allowed)
   (d) If you have 2 letters, 2 numbers, then 2 letters and the 2-digit number must be odd.

4. How many ways can you choose a president and secretary from a club consisting of 16 members?
   (a) With no restrictions
   (b) With both positions held by men, if there are 7men and 9 women
   (c) If Dave is the president

5. How many ways can you pick a team of 6 players from a class of 25 students
   (a) With no restrictions
   (b) If Harry is not on the team
   (c) If Fred and Harry always fight so you can't have both of them on the team

6. How many ways can you arrange 8 people at a round table (clockwise and anticlockwise arrangements are considered different)
   (a) With no restrictions
   (b) If George and Harry must sit next to each other
   (c) If Mary refuses to sit next to George
   (d) If clockwise and anticlockwise arrangements are considered equivalent.

ANSWERS

1. (a) ${}_{11}C_4 = 330$
   (b) ${}_{6}C_2 \times {}_{5}C_2 = 150$
   (c) $5 \times {}_{6}C_3 = 100$

2. (a) $10! = {}_{10}P_{10} = 3628800$
   (b) $\frac{10!}{3!\times2!\times5!} = 2520$

3. (a) $36^6 = 2{,}176{,}782{,}336$
   (b) ${}_{36}P_6 = 1{,}402{,}410{,}240$
   (c) $26^3 \times 10^3 = 17{,}576{,}000$
   (d) $26^2 \times 10 \times 5 \times 26^2 = 22{,}848{,}800$

4. (a) ${}_{16}P_2 = 240$
   (b) ${}_{7}P_2 = 42$
   (c) $1 \times 15 = 15$

5. (a) ${}_{25}C_6 = 177100$
   (b) ${}_{24}C_6 = 134596$
   (c) $1 \times {}_{23}C_5 + 1 \times {}_{23}C_5 + {}_{23}C_6 = 168245$

6. (a) $7! = 5040$
   (b) $6! = 6! = 1440$
   (c) $5040 – 1440 = 3600$
   (d) $\frac{7!}{2} = 2520$

The following link is to a website that covers this material in more depth together with an online quiz.
http://www.mathsisfun.com/combinatorics/combinations-permutations.html