The International Journal of Geomechanics
Volume 2, Number 2, 233–258 (2002)

Rigorous Similarity Solutions for Cavity Expansion in Cohesive-Frictional Soils

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ABSTRACT. *The problem of cavity expansion from zero radius has no characteristic length and therefore possesses a similarity solution, in which the cavity pressure remains constant and the continuing deformation is geometrically self-similar. In this case, the incremental velocity approach first used by Hill [7] to analyze cavity expansion in Tresca materials may be extended to derive a solution for limiting pressure of cavity expansion in Mohr–Coulomb materials. An analytical solution for cavity limit pressures in Mohr–Coulomb materials was suggested by Carter et al. [2]. However, the solution of Carter et al. may only be regarded as approximate since the convected part of the stress rate was neglected in their derivation. By including the convected part of the stress rate, Collins and Wang [4] later derived a semi-analytical similarity solution for cavity expansion in purely frictional soils. The solution of Collins and Wang [4] was, however, obtained from numerical integration as their solution could not be expressed in explicit form.*

In this article, a rigorous closed-form solution is derived for the expansion of cavities from zero initial radius in cohesive-frictional soils. The solution procedure adopted here follows the Hill incremental velocity method, which is different from that used by Collins and Wang [4]. In particular, the plastic radius c is used in this article as the time scale. Unlike the solution of Collins and Wang [4], it is shown that by using a series expansion the similarity solution can be expressed in closed form.

I. Introduction

Analysis of cavity expansion provides many useful solutions for a variety of practical problems in geotechnical engineering. The chief examples include the prediction of bearing capacity of driven piles, interpretation of pressuremeter and cone penetration tests, and analysis of deformation around tunnels. A unified presentation of cavity expansion theory and its detailed applications in geotechnical engineering has been made most recently by Yu [11]. While a complete review on various cavity expansion solutions can be found in Yu [11], we give below a brief summary of alternative methods that have been used to derive large strain analytical solutions for cavity expansion in Tresca and Mohr–Coulomb materials.

Key Words and Phrases. cavity expansion, plasticity, limit pressures, cohesive-frictional soils, analysis.

 DOI: 10.1061/(ASCE)1532-3641(2002)2:2(233)
ISSN 1532-3641

The large deformation analysis of cavity expansion problems in elastic-plastic materials is carried out using mainly two alternative solution procedures. The first approach is the total strain method originally proposed by Chadwick [3] for solving spherical cavity expansion problems in Tresca and associated Mohr–Coulomb materials. In this approach, the incremental form of the plastic flow rule is integrated directly to result in a relationship between total stresses and total strains. In order to use the stress-strain relation in its total form, some definition of finite strain is then adopted. Chadwick [3] suggests the use of natural (or logarithmic or Hencky) strains, which are simply the time integral of the deformation rates for the cavity problems in which no rotation of principal strains occurs. Of the many possible definitions of large strain, the Hencky definition is particularly attractive because of its simple physical interpretation for problems without rotation. The constitutive models used in the total strain approach are as follows: once yielding occurs, stress-strain relationships are expressed between the Eulerian stresses (force/current area) and the logarithmic (Hencky) strains. As the total strain approach assumes the strains (and stresses) depend on the current positions (i. e., radius) of soil particles and their initial positions, it may be termed as the Lagrangian method. Chadwick's approach has, in recent years, been extended by Bigoni and Laudiero [1], Yu [8], Yu and Houlsby [12], Yu [9, 10], and Yu and Houlsby [13] to study both spherical and cylindrical cavity expansion problems in dilatant, nonassociated Mohr–Coulomb materials. In particular, Yu [8] derived the first closed form solution for both cavity pressure-expansion curves and cavity limit pressure for the expansion and contraction of cavities in infinite soils using the nonassociated Mohr–Coulomb model. This solution for expansion and contraction of cavities in infinite soils has also been extended by Yu [9, 10] to include the effect of a finite external boundary.

The second approach is the incremental velocity method first used by Hill [7] to obtain cavity expansion solutions in Tresca materials. In Hill's method, the radius of the elastic-plastic interface is taken as a "time" scale to measure the progress of the deformation. By following the motion of each soil particle, the incremental form of the plastic flow rule is expressed as a differential equation in terms of the radius of the elastic-plastic interface (i. e., "time") and the velocity of soil particles. The resulting ordinary differential equation can then be integrated using the boundary condition for the velocity at the elastic-plastic interface from the known elastic displacement solution. As this method calculates strains (and therefore stresses) as a function of the current position (radius) and time, it may be termed as the Eulerian approach. Although Hill's approach was used for incompressible Tresca materials nearly 50 years ago, its application to the analysis of cavity expansion problems is dilatant, cohesive-frictional Mohr–Coulomb materials has only been attempted quite recently by Carter et al. [2] and Collins and Wang [4]. Using Hill's approach, no closed form solutions have been derived for cavity pressure-expansion curves in Mohr–Coulomb materials. The analytical solutions obtained by Carter et al. [2] and Collins and Wang [4] are only similarity solutions for cavity expansion from zero radius. This similarity solution can be viewed as an intermediate asymptote for the corresponding problem of the expansion of a cavity with a finite radius. It should be noted that the solution obtained by Carter et al. [2] is approximate only as the convected term of the stress rate has been neglected in their governing equation. The analysis of Collins and Wang [4] includes the convected term of the stress rate but only considers purely frictional materials and more importantly their solution is not expressed in closed form and requires numerical integration.

In light of the above review, this article has three objectives. The main objective is to present, for the first time, a rigorous closed form similarity solution for cavity expansion from zero radius in cohesive-frictional soils using Hill's incremental velocity approach. As noted by Hill [7] and Collins and Wang [4], the problem of cavity expansion from zero radius has no characteristic length and hence possesses a similarity solution, where the cavity pressure remains constant and the continuing deformation is geometrically self-similar. The second objective is to investigate the effect of the convected term of the stress rate on the cavity expansion solution. This can then be used to check the validity of the simplified cavity expansion solutions developed by Carter et al. [2]. The third objective is to compare the cavity expansion limit pressure solutions obtained using Hill's incremental velocity method (as developed in this article) and those of Chadwick's total strain approach (as developed by Yu [8] and Yu and Houlsby [12]) for nonassociated Mohr–Coulomb materials.

II. Cavity expansion solution for Tresca materials

Before a closed form similarity solution is presented for cavity expansion in Mohr–Coulomb materials, it is instructive first to review the cavity expansion solution for Tresca materials obtained using Hill's incremental velocity method. For simplicity, only a spherical cavity is considered in the following brief review.

A. Expansion of a spherical cavity in a finite medium

The large strain solution of stresses and displacements for the expansion of a spherical cavity in a Tresca material has been presented by Hill [7]. The following presentation includes non-zero external pressures (i. e., non-zero initial stresses before expansion) and therefore is a generalization of Hill's solution which was presented for zero initial stress conditions.

1. Stress analysis

Assume the current internal and external radii of a shell are denoted by a and b (see Figure 1). Initially the radii of the internal and external boundaries are a_0 and b_0, and a hydrostatic pressure p_0 acts throughout the soil. An internal pressure is increased monotonically from its initial value p_0 on the internal surface. An essential task of the analysis is to determine the variation of internal

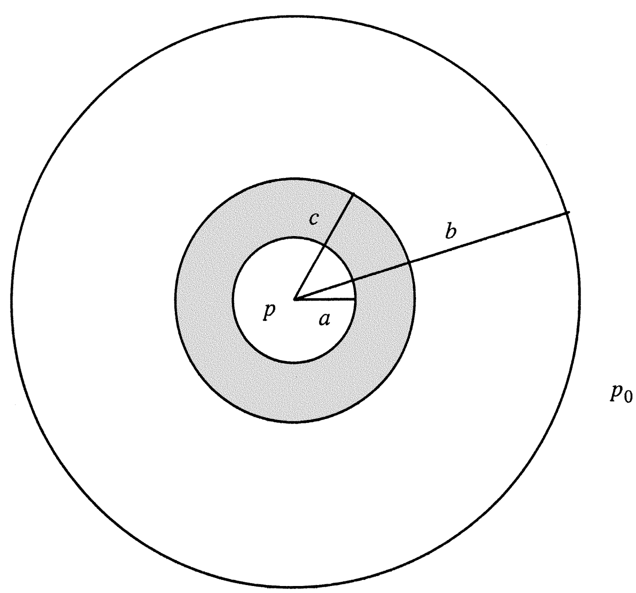

FIGURE 1 Expansion of a cavity with both internal and external pressures.

and external radii with the internal pressure p. As the internal pressure increases from p_0 the material will behave elastically, and the elastic solutions for both stresses and displacements can be shown to be:

$$\sigma_r = -p_0 - (p - p_0)\frac{\left(\frac{b_0}{r}\right)^3 - 1}{\left(\frac{b_0}{a_0}\right)^3 - 1} \tag{1}$$

$$\sigma_\theta = \sigma_\phi = -p_0 + (p - p_0)\frac{\frac{1}{2}\left(\frac{b_0}{r}\right)^3 + 1}{\left(\frac{b_0}{a_0}\right)^3 - 1}\,. \tag{2}$$

Note a tension positive notation is used in this article. The radial displacement is

$$u = r - r_0 = \frac{(p - p_0)}{E}\frac{(1-2v)r + \frac{(1+v)b_0^3}{2r^2}}{\left(\frac{b_0}{a_0}\right)^3 - 1}\,. \tag{3}$$

The Tresca yield criterion is expressed by maximum and minimum principal stresses as follows:

$$\sigma_1 - \sigma_3 = Y \tag{4}$$

where $Y = 2_{s_u}$ and s_u is known as the undrained shear strength.

It is easily seen that the yield condition (4) is first satisfied at the internal boundary. Substituting the elastic stress equations (1) and (2) into equation (4) and knowing that $\sigma_1 = \sigma_\theta$ and $\sigma_3 = \sigma_r$ the internal pressure required to cause yielding at the internal boundary is:

$$p = p_{1y} = p_0 + \frac{2Y}{3}\left[1 - \left(\frac{a_0}{b_0}\right)^3\right] \tag{5}$$

the displacements at both internal and external boundaries when plastic yielding commences at the internal boundary are

$$u|_{r=a_0} = \frac{Y_{a_0}}{E}\left[\frac{2(1-2v)a_0^3}{3b_0^3} + \frac{1+v}{3}\right] \tag{6}$$

$$u|_{r=b_0} = \frac{Y(1-v)a_0}{Eb_0^2}\,. \tag{7}$$

If the internal pressure is further increased, a plastic region will spread into the shell; the radius of the plastic region at any moment is denoted by c. In the outer elastic region, the stresses are still of the form:

$$\sigma_r = -A\left[\left(\frac{b_0}{r}\right)^3 - 1\right] - p_0 \tag{8}$$

$$\sigma_\theta = \sigma_\phi = A\left[\frac{1}{2}\left(\frac{b_0}{r}\right)^3 + 1\right] - p_0 \tag{9}$$

where A is a constant that can be determined by the condition that the material just on the elastic side of the plastic boundary must be on the point of yielding. Substitution of equations (8) and (9) in equation (4), A is found to be:

$$A = \frac{2Yc^3}{3b_0^3} . \tag{10}$$

The elastic stress distribution is therefore obtained by substituting (10) into (8) and (9):

$$\sigma_r = -\frac{2Yc^3}{3b_0^3}\left[\left(\frac{b_0}{r}\right)^3 - 1\right] - p_0 \tag{11}$$

$$\sigma_\theta = \sigma_\phi = \frac{2Yc^3}{3b_0^3}\left[\frac{1}{2}\left(\frac{b_0}{r}\right)^3 + 1\right] - p_0 . \tag{12}$$

The displacement in the elastic region is given by:

$$u = \frac{2Yc^3}{3Eb_0^3}\left[(1-2v)r + \frac{(1+v)b_0^3}{2r^2}\right] . \tag{13}$$

>From equations (11) to (13), it is seen that the solution in the elastic region is dependent only on the radius of elastic-plastic boundary c. In the plastic region, we have the equilibrium equation:

$$\frac{\partial \sigma_r}{\partial r} = \frac{2(\sigma_\theta - \sigma_r)}{r} . \tag{14}$$

Substituting the yield condition (4) into the equilibrium equation (14) results in:

$$\sigma_r = 2Y \ln r + B \tag{15}$$

where B is another constant that can be determined using the condition that the radial stress must be continuous across the elastic-plastic boundary. Equating (11) and (15) at $r = c$ leads to:

$$B = -2Y \ln c - \frac{2Y}{3}\left[1 - \left(\frac{c}{b_0}\right)^3\right] - p_0 . \tag{16}$$

By substituting (16) into (15) and (4), the following solution for stresses in the plastic region can be obtained:

$$\sigma_r = -2Y \ln\left(\frac{c}{r}\right) - \frac{2Y}{3}\left[1 - \left(\frac{c}{b_0}\right)^3\right] - p_0 \tag{17}$$

$$\sigma_\theta = Y - 2Y \ln\left(\frac{c}{r}\right) - \frac{2Y}{3}\left[1 - \left(\frac{c}{b_0}\right)^3\right] - p_0 . \tag{18}$$

By substituting $r = a$ in equation (17), the internal pressure needed to produce plastic flow to a radius c is found to be

$$p = 2Y \ln\left(\frac{c}{a}\right) + \frac{2Y}{3}\left[1 - \left(\frac{c}{b_0}\right)^3\right] + p_0 . \tag{19}$$

2. Displacement analysis

In calculating the displacement for any individual particle it is convenient to take the movement of the plastic boundary as the scale of “time” or progress of the expansion, because the parameter c appears in the formulae for the stresses. We can speak of the velocity V of a particle, meaning that the particle is displaced by an amount $V dc$ when the plastic boundary moves outward a further distance dc. V can be expressed directly in terms of the total displacement u, which is a function of both the current radius r and plastic radius c so that:

$$du = \frac{\partial u}{\partial c}\,dc + \frac{\partial u}{\partial r}\,dr = \left(\frac{\partial u}{\partial c} + V\frac{\partial u}{\partial r}\right) dc \tag{20}$$

where r and c are taken as the independent variables. Equating the above equation to $V dc$ we obtain the expression for the particle velocity:

$$V = \frac{\frac{\partial u}{\partial c}}{1-\frac{\partial u}{\partial r}}\ . \tag{21}$$

Now the compressibility equation in the plastic region is:

$$d\varepsilon_r + d\varepsilon_\theta + d\varepsilon_\phi = \frac{1-2v}{E}\left(d\sigma_r + d\sigma_\theta + d\sigma_\phi\right)\ . \tag{22}$$

To evaluate the increments of stress and strain, we must follow a given element. Thus:

$$d\varepsilon_r = \frac{\partial(du)}{\partial r} = \frac{\partial V}{\partial r}\,dc \tag{23}$$

$$d\varepsilon_\theta = d\varepsilon_\phi = \frac{du}{r} = \frac{V\,dc}{r} \tag{24}$$

$$d\sigma_r = \left(\frac{\partial \sigma_r}{\partial c} + V\frac{\partial \sigma_r}{\partial r}\right) dc \tag{25}$$

$$d\sigma_\theta = d\sigma_\phi = \left(\frac{\partial \sigma_\theta}{\partial c} + V\frac{\partial \sigma_\theta}{\partial r}\right) dc\ . \tag{26}$$

Hence the compressibility condition can be written as follows:

$$\frac{\partial V}{\partial r} + \frac{2V}{r} = \frac{1-2v}{E}\left(\frac{\partial}{\partial c} + V\frac{\partial}{\partial r}\right)(\sigma_r + 2\sigma_\theta)\ . \tag{27}$$

Substituting the expressions for stresses in the plastic region (17) and (18) into (27) leads to:

$$\frac{\partial V}{\partial_r} + \frac{2V}{r} = 6(1-2v)\frac{Y}{E}\left[\frac{V}{r} - \frac{1}{c}\left(1-\frac{c^3}{b_0^3}\right)\right]\ . \tag{28}$$

It is noted that the velocity is known on the plastic boundary from the solution for the displacement in the elastic region. Thus from (13) and (21),

$$V_{r=c} = \frac{Y}{E}\left[2(1-2v)\frac{c^3}{b_0^3} + (1+v)\right]\ . \tag{29}$$

With the above boundary condition, equation (28) can be integrated to give the following solution for the velocity V:

$$V = \frac{3(1-v)Yc^2}{Er^2} - \frac{2(2-2v)Y}{E}\left(1-\frac{c^3}{b_0^3}\right)\frac{r}{c}\ . \tag{30}$$

It should be noted that the above solution was obtained after second and higher powers of Y/E (which is typically very small for realistic values of Y and E) were ignored. For the cavity wall $r = a$, we have $V = da/dc$, so that

$$\frac{da}{dc} = \frac{3(1-v)Yc^2}{Ea^2} - \frac{2(2-2v)Y}{E}\left(1 - \frac{c^3}{b_0^3}\right)\frac{a}{c}\ . \tag{31}$$

After integration, we can express the cavity expansion in terms of the radius of the plastic boundary:

$$\left(\frac{a}{a_0}\right)^3 = 1 + \frac{3(1-v)Yc^3}{Ea_0^3} - \frac{2(1-2v)Y}{E} \times \left[3\ln\left(\frac{c}{a_0}\right) + 1 - \left(\frac{c}{b_0}\right)^3\right]\ . \tag{32}$$

B. Similarity solutions for cavity expansion in an infinite medium

For the special case when a spherical cavity is expanded from zero radius in an infinite medium, the stresses are functions of r/a only, and the ratio of plastic radius to the current cavity radius remains constant [7]. As a result, equation (31) can be directly used to give the plastic radius:

$$\frac{c}{a} = \left[\frac{E}{3(1-v)Y}\right]^{1/3}\ . \tag{33}$$

Substituting the above solution into equation (19) leads to the following solution for the constant internal cavity pressure:

$$P_{\lim} = \frac{2Y}{3}\left[1 + \ln\left(\frac{E}{3(1-v)Y}\right)\right] + p_0\ . \tag{34}$$

III. Similarity solutions for Mohr–Coulomb materials

A. Expansion of cavities from zero initial radius

This section deals with the special case of cavity expansion from zero initial radius in an infinite soil mass. As noted by Hill [7], this problem has no characteristic length and hence will possess a similarity solution, in which the cavity pressure is constant and the continuing deformation is geometrically self-similar. As a result, the incremental velocity approach used by Hill [7] to analyze cavity expansion in Tresca materials may also be used to obtain a solution for limiting pressures of the cavity expansion in Mohr–Coulomb materials. One of the first attempts in obtaining an analytical solution for cavity limit pressure in Mohr–Coulomb materials was made by Carter et al. [2]. However, the solution of Carter et al. is approximate only, as the convected part of the stress rate was ignored in their derivation. Later, by including the convected part of the stress rate, Collins and Wang [4] derived rigorous solutions for purely frictional soils. The solution of Collins and Wang [4] was however obtained by using numerical integration as their solution was not expressed in closed form.

By following Hill's solution procedure described in the previous section, a complete analytical solution for the expansion of cavities from zero initial radius in an infinite cohesive-frictional soil mass is derived. The solution procedure adopted in this article is different from that used by Collins and Wang [4] because the plastic radius c is used here as the time scale, it is shown that by using a series expansion the solution can be expressed in closed form.

B. Soil properties

The soil properties are defined by Young's modulus E and Poisson's ratio v, and the cohesion C, and angles of friction and dilation ϕ and ψ. The initial stress (assumed to be isotropic) is p_0. To simplify the presentation, the parameter k is used to distinguish between cylindrical analysis ($k = 1$) and spherical analysis ($k = 2$). The analysis presented here is for cohesive-frictional soils under fully drained conditions. All stress quantities are effective stresses.

To abbreviate the mathematics, it is convenient to define the following quantities, all of which are constants in any given analysis.

$$G = \frac{E}{2(1+v)} \tag{35}$$

$$M = \frac{E}{1 - v^2(2-k)} \tag{36}$$

$$Y = \frac{2C\cos\phi}{1-\sin\phi} \tag{37}$$

$$a = \frac{1+\sin\phi}{1-\sin\phi} \tag{38}$$

$$\beta = \frac{1+\sin\psi}{1-\sin\psi} \tag{39}$$

$$\delta = \frac{Y + (a-1)p_0}{2(k+a)G} \ . \tag{40}$$

At any time in any position in the soil around the cavity with radius a, the stresses must satisfy the following equation of equilibrium:

$$(\sigma_\theta - \sigma_r) = \frac{r}{k}\frac{\partial\sigma_r}{\partial r} \tag{41}$$

subject to the following two boundary conditions:

$$\sigma_r|_{r=a} = -p$$
$$\sigma_r|_{r=\infty} = -p_0 \ .$$

Using the tension positive notation, the Mohr–Coulomb yield condition during cavity expansion can be expressed in terms of maximum principal stress σ_θ and minimum principal stress σ_r:

$$a\sigma_\theta - \sigma_r = Y \ . \tag{42}$$

C. Elastic solution in the outer elastic zone

The stress-strain relations for soils in the outer elastic zone can be expressed as:

$$d\varepsilon_r = \frac{\partial(du)}{\partial r} = \frac{1}{M}\left[d\sigma_r - \frac{kv}{1-v(2-k)}d\sigma_\theta\right] \tag{43}$$

$$d\varepsilon_\theta = \frac{du}{r} = \frac{1}{M}\left[-\frac{v}{1-v(2-k)}d\sigma_r + [1-v(k-1)]d\sigma_\theta\right] \ . \tag{44}$$

The solution for the stresses and radial displacement can be shown to be:

$$\sigma_r = -p_0 - (p_{1y} - p_0)\left(\frac{c}{r}\right)^{1+k} \tag{45}$$

$$\sigma_\theta = -p_0 + \frac{p_{1y} - p_0}{k}\left(\frac{c}{r}\right)^{1+k} \tag{46}$$

$$u = \frac{p_{1y} - p_0}{2kG} r \left(\frac{c}{r}\right)^{1+k} \tag{47}$$

where

$$p_{1y} = \frac{k[Y + (a-1)p_0]}{k+a} + p_0 = 2kG\delta + p_0 \tag{48}$$

is the cavity pressure at first yield.

D. Stress solution in the plastic zone

1. Stresses in the plastic region

The stress components that satisfy the equilibrium equation (41) and the yield condition (42) are found to be:

$$\sigma_r = \frac{Y}{a-1} + Ar^{-\frac{k(a-1)}{a}} \tag{49}$$

$$\sigma_\theta = \frac{Y}{a-1} + \frac{A}{a} r^{-\frac{k(a-1)}{a}} \tag{50}$$

where A is a constant of integration.

2. Stresses in the elastic region

The stress components in the elastic region can be obtained from the equilibrium equation and elastic stress-strain equations as follows:

$$\sigma_r = -p_0 - Br^{-(1+k)} \tag{51}$$

$$\sigma_\theta = -p_0 + \frac{B}{k} r^{-(1+k)} \tag{52}$$

where B is the second constant of integration.

The continuity of stress components at the elastic-plastic interface can be used to determine the constants A and B:

$$A = -\frac{(1+k)a[Y + (a-1)p_0]}{(a-1)(k+a)} c^{\frac{k(a-1)}{a}} \tag{53}$$

$$B = \frac{k[Y + (a-1)p_0]}{k+a} c^{1+k} \,. \tag{54}$$

At the cavity wall we have $\sigma_r|_{r=a} = -p$ and this condition can be used to express the plastic radius c in terms of the current cavity radius and applied pressure

$$\frac{c}{a} = \left\{ \frac{(k+a)[Y + (a-1)p]}{a(1+k)[Y + (a-1)p_0]} \right\}^{\frac{a}{k(a-1)}} . \tag{55}$$

The stresses are now established in terms of a single unknown c. In the next subsection the displacements are examined, allowing determination of c and therefore the cavity pressure.

E. Elastic-plastic displacement analysis

Substituting equations (48) into (47) gives the displacement in the elastic zone:

$$u = \delta r \left(\frac{c}{r}\right)^{1+k} \tag{56}$$

with δ as defined earlier in this section. The determination of the displacement field in the plastic zone requires the use of a plastic flow rule that indicates the relative magnitude of plastic strains in different directions.

For the expansion of cavities, the nonassociated Mohr–Coulomb flow rule can be expressed as:

$$\frac{\dot{\varepsilon}_r^p}{\dot{\varepsilon}_\theta^p} = \frac{\dot{\varepsilon}_r - \dot{\varepsilon}_r^e}{\dot{\varepsilon}_\theta - \dot{\varepsilon}_\theta^e} = -\frac{k}{\beta} \tag{57}$$

where β is a simple function of dilation angle, defined previously. If $\beta = a$ then the flow rule for the soil is associated.

Substituting equations (43) and (44) into the plastic flow rule (57) results in:

$$\beta\, d\varepsilon_r + k\, d\varepsilon_\theta = \frac{1}{M}\left[\beta - \frac{kv}{1 - v(2-k)}\right] d\sigma_r + \frac{1}{M}\left[k(1-2v) + 2v - \frac{k\beta v}{1 - v(2-k)}\right] d\sigma_\theta \tag{58}$$

with M as defined at the beginning of this section. In the plastic zone, from the yield equation we have $d\sigma_\theta = \frac{1}{a} d\sigma_r$, and as a result equation (58) reduces to:

$$d\varepsilon_r + \frac{k}{\beta}\, d\varepsilon_\theta = \frac{\chi}{\beta}\, d\sigma_r \tag{59}$$

where

$$\chi = \frac{1}{M}\left[\beta - \frac{kv}{1 - v(2-k)}\right] + \frac{1}{Ma}\left[k(1-2v) + 2v - \frac{k\beta v}{1 - v(2-k)}\right] . \tag{60}$$

In calculating the displacement of any individual particle it is convenient to take the movement of the plastic boundary as the scale of “time” or progress of the expansion, because the parameter c appears in the formulae for the stresses. We can speak of the velocity V of a particle, meaning that the particle is displaced by an amount $V\,dc$ when the plastic boundary moves outward a further distance dc. V can be expressed directly in terms of the total displacement u, which is a function of both the current radius r and plastic radius c so that:

$$du = \frac{\partial u}{\partial c}\, dc + \frac{\partial u}{\partial r}\, dr = \left(\frac{\partial u}{\partial c} + V\frac{\partial u}{\partial r}\right) dc$$

where r and c are taken as the independent variables. Equating du to $V\,dc$ we obtain the expression for the particle velocity:

$$V = \frac{\frac{\partial u}{\partial c}}{1 - \frac{\partial u}{\partial r}} .$$

To evaluate the increments of stress and strain, we should follow a given material element and therefore:

$$d\varepsilon_r = \frac{\partial (du)}{\partial r} = \frac{\partial V}{\partial r}\, dc$$

$$d\varepsilon_\theta = \frac{du}{r} = \frac{V\,dc}{r}$$

$$d\sigma_r = \left(\frac{\partial \sigma_r}{\partial c} + V\frac{\partial \sigma_r}{\partial r}\right) dc .$$

Equation (59) can therefore be written in terms of velocity:

$$\frac{\partial V}{\partial r} + \frac{k}{\beta}\frac{V}{r} = \frac{\chi}{\beta}\left(\frac{\partial \sigma_r}{\partial c} + V\frac{\partial \sigma_r}{\partial r}\right) . \tag{61}$$

Substituting the expression for radial stress in the plastic zone (49) and (53) into equation (61) gives the following differential equation for the velocity:

$$\frac{\partial V}{\partial r} + P(r)V = Q(r) \tag{62}$$

in which

$$P(r) = \frac{k}{\beta r} - \frac{\chi q k(a-1)}{a\beta}\left(\frac{c}{r}\right)^{\frac{k(a-1)}{a}}\frac{1}{r} \tag{63}$$

$$Q(r) = \frac{s}{c}\left(\frac{c}{r}\right)^{\frac{k(a-1)}{a}} \tag{64}$$

and q and s are defined by:

$$q = \frac{(1+k)a[Y + (a-1)p_0]}{(a-1)(k+a)}$$

$$s = -\frac{\chi q^k (a-1)}{a\beta} .$$

It is noted that the velocity is known on the plastic boundary from the solution for the displacement in the elastic region. Thus from (56) we have:

$$V_{r=c} = \delta(1+k) . \tag{65}$$

With the above boundary condition, equation (62) can be integrated to give the following solution for the velocity V:

$$V = \exp\left[-\frac{\chi q}{\beta}\left(\frac{c}{r}\right)^{\frac{k(a-1)}{a}}\right]\left\{\sum_{n=0}^{\infty} A_n \left(\frac{c}{r}\right)^{\frac{k(a-1)(1+n)}{a}-1} + \left[\delta(1+k)\exp\left(\frac{\chi q}{\beta}\right) - \sum_{n=0}^{\infty} A_n\right]\left(\frac{c}{r}\right)^{\frac{k}{\beta}}\right\} \tag{66}$$

in which A_n is defined by:

$$A_n = \frac{1}{n!}\left(\frac{\chi q}{\beta}\right)^n \frac{a\beta s}{ka - k\beta(a-1)(1+n) + a\beta} \tag{67}$$

and n is an integer ranging from 0 to infinity.

At the cavity wall $r = a$, we have $V = da/dc$, so that

$$\frac{da}{dc} = \exp\left[-\frac{\chi q}{\beta}\left(\frac{c}{a}\right)^{\frac{k(a-1)}{a}}\right]\left\{\sum_{n=0}^{\infty} A_n \left(\frac{c}{a}\right)^{\frac{k(a-1)(1+n)}{a}-1} + \left[\delta(1+k)\exp\left(\frac{\chi q}{\beta}\right) - \sum_{n=0}^{\infty} A_n\right]\left(\frac{c}{a}\right)^{\frac{k}{\beta}}\right\} . \tag{68}$$

For the problem of cavity expansion from zero initial radius, the deformation can be assumed to be geometrically similar in the plastic zone so that the ratio of the radius of the elastic-plastic boundary to that of the cavity wall is a constant. Hence:

$$\frac{da}{dc} = \frac{a}{c} \tag{69}$$

with the above relation, equation (68) reduces to a nonlinear equation in c/a:

$$\frac{a}{c} = \exp\left[-\frac{\chi q}{\beta}\left(\frac{c}{a}\right)^{\frac{k(a-1)}{a}}\right]\left\{\sum_{n=0}^{\infty} A_n \left(\frac{c}{a}\right)^{\frac{k(a-1)(1+n)}{a}-1} + \left[\delta(1+k)\exp\left(\frac{\chi q}{\beta}\right) - \sum_{n=0}^{\infty} A_n\right]\left(\frac{c}{a}\right)^{\frac{k}{\beta}}\right\} \tag{70}$$

which can be easily solved for the value of c/a.

Once c/a is determined, equation (55) can be used to calculate the limit pressure p.

F. Neglecting the convected part of stress rate—the solution of Carter et al. [2]

If the convected part of the stress rate, $V\frac{\partial\sigma_r}{\partial r}$, is ignored, the governing equation (61) can be considerably simplified to:

$$\frac{\partial V}{\partial r} + \frac{k}{\beta}\frac{V}{r} = \frac{\chi}{\beta}\frac{\partial\sigma_r}{\partial c} \tag{71}$$

which can be further reduced to:

$$\frac{\partial V}{\partial r} + \frac{k}{\beta}\frac{V}{r} = Q(r) \tag{72}$$

in which $Q(r)$ is given by equation (64). With the boundary condition (65), the above equation can be solved to give the following solution for velocity V:

$$V = \gamma\left(\frac{c}{r}\right)^{\frac{k(a-1)}{a}-1} + \left[\delta(1+k) - \gamma\right]\left(\frac{c}{r}\right)^{\frac{k}{\beta}} \tag{73}$$

where

$$\gamma = \frac{a\beta s}{ka - k\beta(a-1) + a\beta}\ . \tag{74}$$

Following the same argument as used before, equation (73), can be applied at the cavity wall $r = a$ to obtain a nonlinear equation on c/a:

$$1 = \gamma \left(\frac{c}{a}\right)^{\frac{k(a-1)}{a}} + [\delta(1+k) - \gamma] \left(\frac{c}{a}\right)^{1+\frac{k}{\beta}} \ . \tag{75}$$

The above equation is the same as that obtained by Carter et al. [2].

IV. Results and discussion

Some selected numerical results for both plastic radius and cavity limit pressure will be presented. The aims of presenting these results are: (i) to compare the complete rigorous similarity solutions, defined by equation (70) with the approximate solution of Carter *et al.* [2], as defined by equation (75); (ii) to compare the solutions obtained from Chadwick's total strain method and those from Hill's incremental velocity approach.

Soil properties are defined by five parameters: Young's modulus E, Poisson's ratio v, cohesion C, and angles of friction and dilation ϕ and ψ. To cover a wide range of possible cases, the friction angle is varied from 20 to 50 degrees, and the dilation angle is varied from zero to the value of the friction angle. A constant value of 0.3 is used for Poisson's ratio. E/p_0 varies between 26 and 2600 (i. e., $2G/p_0$ is between 20 and 2000, as shown in Tables 1 to 8).

A. Plastic radius and limit pressure for purely frictional soils

First of all, we will focus on cavity expansion in purely frictional soils when soil cohesion $C = 0$. The results for the plastic radius and cavity limit pressures are presented in Tables 1 to 4 and Figures 2 to 3 for both cylindrical and spherical cavities. It is interesting to note from these tables that the solutions from Hill's incremental velocity approach (as obtained in this article) and those from Chadwick's total strain approach (as obtained by Yu and Houlsby, [12]) are practically identical. This was expected since both approaches are based on the same incremental plastic flow rule (i. e., equation (58) of this article is the same as equation (37) in Yu and Houlsby [12]). For Tresca materials, Chadwick [3] also showed that the solution from his approach was identical to that obtained by Hill [7] using an incremental velocity approach. However, if the convected part of stress rate is neglected when following Hill's solution method (as in Carter et al. [2]), errors would be introduced into the final cavity limit pressure solutions. Figures 2 to 3 indicate that the resulting errors depend on the angles of friction and dilation as well as soil stiffness properties. While the resulting error increases with increasing angles of friction and dilation, it tends to decrease with the soil stiffness. It is also noted that the error introduced by neglecting the convected part of the stress rate for a spherical cavity is significantly larger than that for a cylindrical cavity. For example, for all the soil properties considered, the maximum error is 11.25% for a cylindrical cavity and 28.89% for a spherical cavity.

B. Plastic radius and limit pressure for cohesive-frictional soils

In order to investigate the possible effect of soil cohesion on the similarity solutions, Figures 4 to 5 and Tables 5 to 8 present results for the case of cohesive-frictional soils with $C/p_0 = 1$. Inspection of the results suggests that the general conclusions made earlier for purely frictional soils are also valid for cohesive-frictional soils. It is noted that the resulting errors for a cohesive-frictional soil are generally larger than those for a purely frictional soil. The difference is more significant for soils with low values of friction angle. When the friction angle is very large (e. g., 50 degrees), the effect of cohesion on the resulting errors becomes very small.

TABLE 1

Results of Plastic Radius (c/a) for the Expansion of Cylindrical Cavities in Purely Frictional Soils

ϕ	Ψ	$2G/p_o$	Yu and Houlsby (1991)	This paper	Carter et al (1986)	Error (%)
20°	0°	20	4.76	4.74	4.64	2.15
		200	14.58	14.57	14.49	0.55
		2000	45.75	45.77	45.72	0.10
	10°	20	6.09	6.06	5.88	3.00
		200	22.47	22.46	22.27	0.83
		2000	85.74	85.86	85.70	0.19
	20°	20	7.67	7.64	7.33	4.07
		200	33.83	33.82	33.41	1.22
		2000	156.17	156.10	155.65	0.29
30°	0°	20	4.00	3.98	3.87	2.71
		200	12.12	12.11	12.02	0.78
		2000	37.90	37.92	37.84	0.21
	10°	20	4.93	4.90	4.71	3.88
		200	17.86	17.85	17.62	1.29
		2000	67.76	67.76	67.51	0.37
	20°	20	5.99	5.95	5.63	5.39
		200	25.64	25.62	25.11	2.00
		2000	116.81	116.80	116.06	0.63
	30°	20	7.13	7.08	6.59	6.98
		200	35.47	35.45	34.44	2.84
		2000	191.18	191.00	189.28	0.90
40°	0°	20	3.57	3.55	3.45	2.96
		200	10.74	10.74	10.63	1.04
		2000	33.49	33.50	33.40	0.29
	10°	20	4.30	4.27	4.08	4.50
		200	15.35	15.34	15.07	1.75
		2000	57.80	57.79	57.47	0.56
	20°	20	5.10	5.06	4.74	6.32
		200	21.30	21.28	20.70	2.73
		2000	95.74	95.67	94.76	0.95
	30°	20	5.94	5.88	5.40	8.20
		200	28.47	28.44	27.32	3.95
		2000	150.08	150.03	147.56	1.65
	40°	20	6.76	6.69	6.02	10.07
		200	36.46	36.42	34.49	5.29
		2000	220.44	220.42	214.93	2.49
50°	0°	20	3.31	3.28	3.18	3.14
		200	9.89	9.88	9.76	1.17
		2000	30.74	30.74	30.63	0.36
	10°	20	3.92	3.89	3.69	4.94
		200	13.81	13.80	13.51	2.11
		2000	51.67	51.66	51.27	0.75
	20°	20	4.58	4.53	4.21	6.96
		200	18.71	18.69	18.06	3.37
		2000	83.04	83.02	81.87	1.38
	30°	20	5.24	5.18	4.71	9.07
		200	24.40	24.37	23.18	4.90
		2000	125.95	125.93	122.99	2.33
	40°	20	5.88	5.81	5.17	11.10
		200	30.51	30.47	28.47	6.57
		2000	178.87	178.64	172.57	3.40
	50°	20	6.46	6.38	5.56	12.85
		200	36.54	36.51	33.58	8.01
		2000	237.22	237.19	226.50	4.46

TABLE 2

Results of Plastic Radius (c/a) for the Expansion of Spherical Cavities in Purely Frictional Soils

ϕ	Ψ	$2G/p_o$	Yu and Houlsby (1991)	This paper	Carter et al (1986)	Error (%)
20°	0°	20	2.64	2.63	2.57	2.17
		200	5.50	5.49	5.46	0.60
		2000	11.74	11.74	11.72	0.20
	10°	20	3.24	3.23	3.12	3.44
		200	7.97	7.97	7.87	1.23
		2000	20.39	20.39	20.32	0.37
	20°	20	4.00	3.99	3.78	5.34
		200	11.68	11.68	11.42	2.18
		2000	36.15	36.15	35.86	0.80
30°	0°	20	2.33	2.32	2.25	2.68
		200	4.79	4.78	4.74	0.92
		2000	10.16	10.16	10.13	0.28
	10°	20	2.75	2.73	2.61	4.48
		200	6.58	6.58	6.45	2.02
		2000	16.59	16.59	16.46	0.78
	20°	20	3.24	3.22	3.00	6.77
		200	9.01	9.00	8.67	3.71
		2000	26.96	26.96	26.46	1.85
	30°	20	3.79	3.77	3.40	9.81
		200	12.06	12.05	11.29	6.29
		2000	42.25	42.25	40.62	3.86
40°	0°	20	2.14	2.12	2.06	2.78
		200	4.36	4.35	4.30	1.20
		2000	9.21	9.20	9.16	0.46
	10°	20	2.46	2.44	2.32	4.96
		200	5.77	5.76	5.61	2.67
		2000	14.31	14.31	14.13	1.26
	20°	20	2.82	2.80	2.58	7.86
		200	7.52	7.51	7.13	5.08
		2000	21.72	21.72	21.05	3.08
	30°	20	3.20	3.17	2.82	11.00
		200	9.52	9.51	8.72	8.26
		2000	31.21	31.21	29.30	6.12
	40°	20	3.57	3.54	3.04	14.10
		200	11.60	11.59	10.22	11.83
		2000	41.69	41.69	37.59	9.84
50°	0°	20	2.01	1.99	1.93	2.91
		200	4.07	4.07	4.01	1.43
		2000	8.57	8.57	8.52	0.60
	10°	20	2.27	2.25	2.13	5.42
		200	5.24	5.24	5.05	3.62
		2000	12.83	12.83	12.60	1.76
	20°	20	2.56	2.53	2.32	8.42
		200	6.60	6.60	6.20	6.09
		2000	18.49	18.49	17.72	4.18
	30°	20	2.84	2.82	2.49	11.68
		200	8.06	8.05	7.29	9.54
		2000	25.00	25.00	23.08	7.67
	40°	20	3.12	3.09	2.63	14.93
		200	9.47	9.47	8.21	13.30
		2000	31.48	31.48	27.74	11.87
	50°	20	3.37	3.33	2.74	17.82
		200	10.74	10.73	8.94	16.72
		2000	37.21	37.21	31.35	15.76

TABLE 3

Results of Liquid Pressure (p/p_o) for the Expansion of Cylindrical Cavities in Purely Frictional Soils

ϕ	Ψ	$2G/p_o$	Yu and Houlsby (1991)	This paper	Carter et al (1986)	Error (%)
20°	0°	20	2.97	2.97	2.93	1.25
		200	5.26	5.26	5.24	0.32
		2000	9.42	9.42	9.42	0.03
	10°	20	3.37	3.36	3.31	1.49
		200	6.56	6.55	6.53	0.35
		2000	12.98	12.98	12.97	0.07
	20°	20	3.79	3.78	3.70	2.01
		200	8.08	8.08	8.03	0.68
		2000	17.61	17.61	17.58	0.16
30°	0°	20	3.78	3.77	3.70	1.91
		200	7.91	7.91	7.87	0.53
		2000	16.93	16.93	16.91	0.15
	10°	20	4.35	4.33	4.21	2.68
		200	10.25	10.25	10.16	0.93
		2000	24.93	24.93	24.87	0.26
	20°	20	4.95	4.92	4.75	3.54
		200	13.04	13.04	12.86	1.38
		2000	35.84	35.84	35.69	0.43
	30°	20	5.56	5.53	5.27	4.70
		200	16.19	16.18	15.88	1.87
		2000	49.78	49.74	49.44	0.60
40°	0°	20	4.45	4.43	4.32	2.39
		200	10.53	10.53	10.44	0.84
		2000	25.64	25.64	25.69	0.21
	10°	20	5.15	5.12	4.94	3.61
		200	13.92	13.92	13.73	1.40
		2000	39.30	39.29	39.12	0.44
	20°	20	5.88	5.84	5.55	4.93
		200	17.99	17.98	17.59	2.15
		2000	58.33	58.30	57.86	0.76
	30°	20	6.62	6.57	6.15	6.45
		200	22.58	22.56	21.86	3.11
		2000	82.92	82.90	81.82	1.30
	40°	20	7.33	7.27	6.69	7.99
		200	27.40	27.38	26.24	4.18
		2000	112.02	112.02	109.82	1.97
50°	0°	20	4.98	4.95	4.82	2.75
		200	12.89	12.88	12.75	1.02
		2000	34.48	34.48	34.37	0.31
	10°	20	5.78	5.73	5.49	4.29
		200	17.23	17.22	16.90	1.84
		2000	54.10	54.10	53.75	0.66
	20°	20	6.61	6.55	6.15	6.06
		200	22.42	22.40	21.74	2.93
		2000	81.66	81.64	80.66	1.21
	30°	20	7.43	7.36	6.78	7.92
		200	28.22	28.19	26.99	4.27
		2000	117.20	117.19	114.81	2.03
	40°	20	8.21	8.13	7.34	9.71
		200	34.25	34.22	32.26	5.72
		2000	158.90	158.71	154.02	2.96
	50°	20	8.91	8.82	7.83	11.25
		200	40.06	40.03	37.23	6.99
		2000	202.99	202.97	195.07	3.89

TABLE 4

Results of Liquid Pressure (p/p_o) for the Expansion of Spherical Cavities in Purely Frictional Soils

ϕ	Ψ	$2G/p_o$	Yu and Houlsby (1991)	This paper	Carter et al (1986)	Error (%)
20°	0°	20	4.07	4.06	3.97	2.24
		200	8.60	8.60	8.54	0.67
		2000	18.65	18.65	18.61	0.20
	10°	20	5.03	5.01	4.83	3.61
		200	12.57	12.57	12.41	1.28
		2000	32.75	32.74	32.62	0.37
	20°	20	6.23	6.20	5.87	5.32
		200	18.56	18.55	18.14	2.24
		2000	58.70	58.70	58.22	0.81
30°	0°	20	5.55	5.51	5.32	3.48
		200	14.52	14.51	14.31	1.35
		2000	39.63	39.63	39.46	0.42
	10°	20	6.93	6.88	6.48	5.89
		200	22.21	22.19	21.59	2.69
		2000	76.17	76.16	75.36	1.05
	20°	20	6.64	8.57	7.80	9.03
		200	33.73	33.71	32.04	4.97
		2000	145.51	145.51	141.91	2.47
	30°	20	10.64	10.55	9.20	12.78
		200	49.75	49.72	45.59	8.30
		2000	264.88	264.82	251.27	5.12
40°	0°	20	6.85	6.78	6.48	4.40
		200	20.91	20.90	20.49	1.96
		2000	67.46	67.45	66.96	0.72
	10°	20	8.56	8.46	7.80	7.84
		200	32.44	32.41	31.04	4.22
		2000	134.64	134.62	131.92	2.01
	20°	20	10.59	10.46	9.21	11.99
		200	49.16	49.10	45.25	7.85
		2000	258.61	258.59	246.21	4.79
	30°	20	12.89	12.73	10.61	16.67
		200	71.11	71.04	62.03	12.68
		2000	455.95	455.92	412.96	9.42
	40°	20	15.33	15.14	11.92	21.27
		200	96.83	96.76	79.45	17.89
		2000	717.54	717.50	610.02	14.98
50°	0°	20	7.94	7.83	7.44	5.03
		200	27.09	27.05	26.38	2.49
		2000	98.52	98.50	97.49	1.03
	10°	20	9.86	9.70	8.81	9.17
		200	41.98	41.93	39.57	5.61
		2000	198.54	198.51	192.46	3.05
	20°	20	12.09	11.88	10.20	14.14
		200	62.73	62.65	56.17	10.34
		2000	374.28	374.23	347.49	7.15
	30°	20	14.54	14.29	11.52	19.37
		200	88.60	88.50	74.36	15.97
		2000	631.74	631.69	549.95	12.94
	40°	20	17.06	16.78	12.67	24.48
		200	117.30	117.19	91.48	21.94
		2000	942.44	942.33	756.69	19.70
	50°	20	19.47	19.16	13.63	28.89
		200	145.75	145.63	106.01	27.21
		2000	1259.35	1259.29	935.18	25.74

TABLE 5

Results of Plastic Radius (c/a) for the Expansion of Cylindrical Cavities in Cohesive Frictional Soils

ϕ	Ψ	$2G/p_o$	This paper	Carter et al (1986)	Error (%)
20°	0°	20	2.55	2.45	3.92
		200	7.61	7.52	1.25
		2000	23.71	23.64	0.29
	10°	20	2.96	2.81	5.20
		200	10.53	10.33	1.82
		2000	39.71	39.53	0.45
	20°	20	3.40	3.17	6.68
		200	14.28	13.92	2.50
		2000	64.74	64.30	0.68
30°	0°	20	2.49	2.40	3.89
		200	7.41	7.31	1.42
		2000	23.00	22.92	0.37
	10°	20	2.86	2.71	5.49
		200	10.08	9.86	2.19
		2000	37.76	37.51	0.65
	20°	20	3.25	3.02	7.22
		200	13.40	12.98	3.19
		2000	60.00	59.37	1.05
	30°	20	3.65	3.32	9.04
		200	17.31	16.57	4.31
		2000	91.14	89.69	1.59
40°	0°	20	2.47	2.37	3.85
		200	7.33	7.21	1.52
		2000	22.69	22.59	0.45
	10°	20	2.82	2.66	5.61
		200	9.85	9.60	2.50
		2000	36.67	36.36	0.84
	20°	20	3.18	2.94	7.61
		200	12.91	12.42	3.75
		2000	57.09	56.27	1.44
	30°	20	3.54	3.20	9.62
		200	16.41	15.55	5.20
		2000	84.58	82.71	2.21
	40°	20	3.89	3.45	11.46
		200	20.13	18.77	6.75
		2000	118.42	114.58	3.25
50°	0°	20	2.48	2.38	3.72
		200	7.35	7.23	1.58
		2000	22.72	22.61	0.50
	10°	20	2.82	2.66	5.72
		200	9.80	9.54	2.71
		2000	36.33	35.97	1.00
	20°	20	3.17	2.92	7.85
		200	12.73	12.19	4.19
		2000	55.72	54.73	1.77
	30°	20	3.52	3.17	10.03
		200	15.99	15.05	5.88
		2000	81.03	78.74	2.83
	40°	20	3.85	3.39	12.01
		200	19.39	17.90	7.67
		2000	111.14	106.51	4.16
	50°	20	4.14	3.57	13.82
		200	22.66	20.54	9.38
		2000	143.25	135.33	5.52

TABLE 6

Results of Plastic Radius (c/a) for the Expansion of Spherical Cavities in Cohesive Frictional Soils

ϕ	Ψ	$2G/p_o$	This paper	Carter et al (1986)	Error (%)
20°	0°	20	1.77	1.70	3.62
		200	3.58	3.54	1.37
		2000	7.58	7.55	0.37
	10°	20	1.99	1.89	5.32
		200	4.72	4.61	2.33
		2000	11.86	11.78	0.74
	20°	20	2.26	2.09	7.40
		200	6.24	6.01	3.72
		2000	18.82	18.55	1.42
30°	0°	20	1.71	1.65	3.50
		200	3.47	3.41	1.59
		2000	7.30	7.26	0.52
	10°	20	1.91	1.80	5.55
		200	4.45	4.32	2.94
		2000	11.04	10.91	1.20
	20°	20	2.13	1.96	8.04
		200	5.69	5.40	4.99
		2000	16.61	16.19	2.54
	30°	20	2.36	2.10	10.89
		200	7.15	6.60	7.71
		2000	24.23	23.07	4.79
40°	0°	20	1.68	1.62	3.34
		200	3.39	3.33	1.74
		2000	7.12	7.07	0.66
	10°	20	1.85	1.75	5.67
		200	4.27	4.12	3.44
		2000	10.46	10.28	1.67
	20°	20	2.04	1.87	8.42
		200	5.31	4.99	5.99
		2000	15.04	14.49	3.66
	30°	20	2.24	1.98	11.35
		200	6.47	5.87	9.23
		2000	20.66	19.26	6.77
	40°	20	2.43	2.08	14.44
		200	7.64	6.68	12.62
		2000	26.74	23.95	10.43
50°	0°	20	1.66	1.61	3.25
		200	3.35	3.29	1.79
		2000	7.02	6.97	0.75
	10°	20	1.82	1.72	5.71
		200	4.16	4.00	3.80
		2000	10.08	9.87	2.08
	20°	20	1.99	1.82	8.63
		200	5.08	4.74	6.71
		2000	14.00	13.36	4.62
	30°	20	2.17	1.91	11.73
		200	6.04	5.43	10.11
		2000	18.41	16.92	8.10
	40°	20	2.33	1.99	14.79
		200	6.97	6.01	13.75
		2000	22.78	20.01	12.16
	50°	20	2.48	2.05	17.52
		200	7.81	6.48	17.05
		2000	26.64	22.40	15.93

TABLE 7

Results of Limit Pressures (p/p_o) for the Expansion of Cylindrical Cavities in Cohesive Frictional Soils

ϕ	Ψ	$2G/p_o$	This paper	Carter et al (1986)	Error (%)
20°	0°	20	5.36	5.20	3.02
		200	11.41	11.32	0.80
		2000	22.50	22.47	0.16
	10°	20	6.00	5.76	3.95
		200	13.95	13.79	1.12
		2000	30.10	30.02	0.26
	20°	20	6.64	6.31	4.90
		200	16.75	16.50	1.50
		2000	39.39	39.24	0.37
30°	0°	20	5.80	5.61	3.38
		200	13.84	13.70	1.07
		2000	31.41	31.33	0.26
	10°	20	6.53	6.22	4.67
		200	17.39	17.10	1.62
		2000	44.39	44.19	0.45
	20°	20	7.27	6.83	6.04
		200	21.39	20.90	2.30
		2000	61.08	60.63	0.73
	30°	20	7.98	7.39	7.44
		200	25.69	24.90	3.09
		2000	81.26	80.37	1.09
40°	0°	20	6.11	5.89	3.60
		200	15.91	15.71	1.28
		2000	40.24	40.10	0.36
	10°	20	6.90	6.55	5.20
		200	20.37	19.94	2.08
		2000	59.13	58.74	0.67
	20°	20	7.71	7.18	6.92
		200	25.46	24.67	3.09
		2000	84.11	83.15	1.15
	30°	20	8.50	7.76	8.68
		200	30.96	29.64	4.25
		2000	114.83	112.82	1.75
	40°	20	9.23	8.29	10.26
		200	36.54	34.53	5.50
		2000	149.80	145.94	2.57
50°	0°	20	6.29	6.06	3.69
		200	17.48	17.23	1.43
		2000	47.96	47.75	0.44
	10°	20	7.14	6.74	5.56
		200	22.69	22.13	2.45
		2000	72.47	71.83	0.88
	20°	20	8.00	7.39	7.57
		200	28.67	27.60	3.75
		2000	105.40	103.75	1.56
	30°	20	8.83	7.99	9.59
		200	35.13	33.29	5.25
		2000	146.19	142.57	2.47
	40°	20	9.61	8.52	11.41
		200	41.68	38.84	6.83
		2000	192.55	185.54	3.64
	50°	20	10.30	8.95	13.10
		200	47.85	43.85	8.34
		2000	240.18	228.58	4.83

TABLE 8

Results of Limit Pressures (p/p_o) for the Expansion of Spherical Cavities in Cohesive Frictional Soils

ϕ	Ψ	$2G/p_o$	This paper	Carter et al (1986)	Error (%)
20°	0°	20	7.39	7.02	5.05
		200	18.11	17.82	1.62
		2000	42.02	41.84	0.41
	10°	20	8.72	8.10	7.11
		200	24.83	24.18	2.63
		2000	67.91	67.37	0.79
	20°	20	10.26	9.28	9.61
		200	33.95	32.55	4.10
		2000	110.35	108.70	1.50
30°	0°	20	8.36	7.88	5.64
		200	24.07	23.53	2.26
		2000	67.91	67.42	0.72
	10°	20	9.92	9.06	8.61
		200	34.25	32.84	4.10
		2000	119.16	117.22	1.63
	20°	20	11.72	10.30	12.15
		200	48.17	44.88	6.83
		2000	206.60	199.58	3.40
	30°	20	13.72	11.53	16.02
		200	66.01	59.13	10.42
		2000	343.10	321.25	6.37
40°	0°	20	9.11	8.57	5.92
		200	29.80	28.97	2.79
		2000	97.66	96.65	1.04
	10°	20	10.84	9.79	9.67
		200	43.23	40.86	5.48
		2000	179.34	174.63	2.62
	20°	20	12.81	11.01	14.04
		200	61.35	55.58	9.39
		2000	317.55	299.46	5.70
	30°	20	14.97	12.19	18.59
		200	83.86	71.90	14.25
		2000	522.77	468.31	10.42
	40°	20	17.20	13.22	23.17
		200	109.26	88.23	19.25
		2000	783.33	659.03	15.87
50°	0°	20	9.66	9.08	6.01
		200	34.76	33.66	3.16
		2000	127.44	125.76	1.32
	10°	20	11.50	10.30	10.39
		200	50.94	47.57	6.62
		2000	239.59	230.95	3.61
	20°	20	13.58	11.49	15.40
		200	72.32	64.00	11.50
		2000	424.13	390.66	7.89
	30°	20	15.83	12.57	20.55
		200	98.02	81.33	17.03
		2000	682.51	589.29	13.66
	40°	20	18.12	13.51	25.41
		200	125.94	97.23	22.80
		2000	987.44	788.28	20.17
	50°	20	20.30	14.29	29.63
		200	153.38	110.66	27.85
		2000	1296.39	959.04	26.02

2G

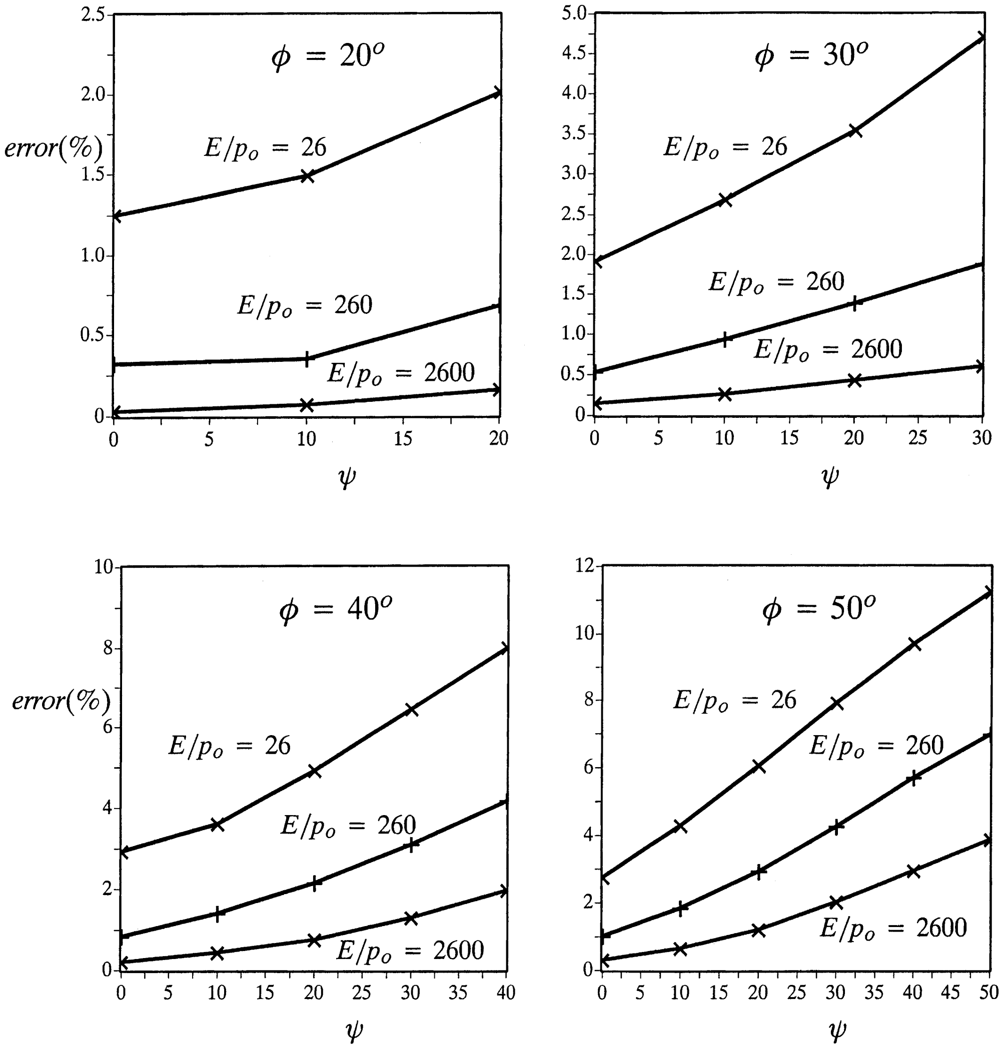

FIGURE 2 Errors of limit pressure caused by ignoring the convected part of stress rate for cylindrical cavity expansion in purely frictional soils ($C = 0$).

V. Conclusions

The following conclusions can be drawn from the results obtained in this article:

1. Cavity expansion from zero initial radius in an infinite Mohr–Coulomb material possesses a similarity solution, in which the cavity pressure remains a constant and the continuing deformation is geometrically self-similar. This constant cavity pressure is equal to the limiting pressure achieved at very large strains for the expansion of a finite cavity. By following Hill's incremental velocity approach, a rigorous closed form similarity solution for cavity expansion from zero radius is cohesive frictional soils has been derived in this article.

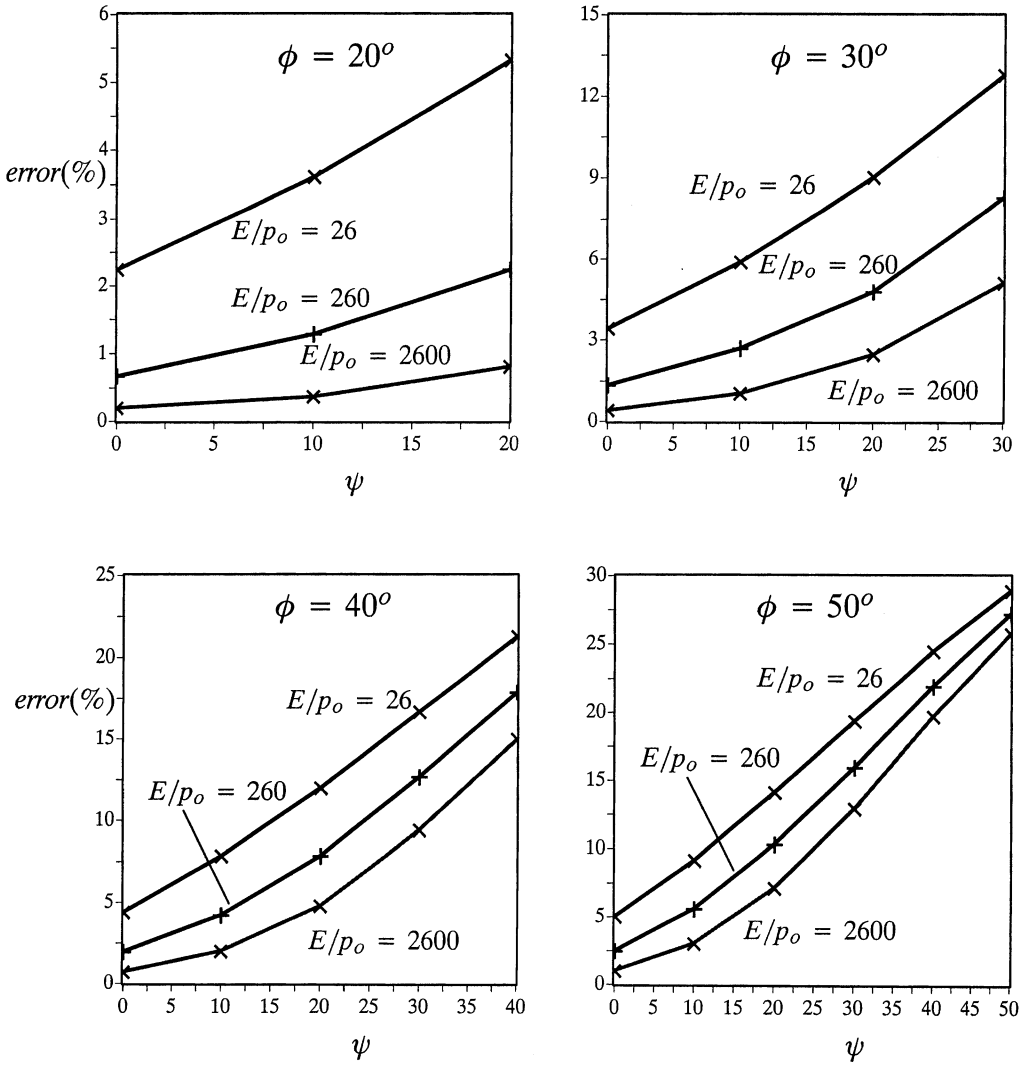

FIGURE 3 Errors of limit pressure caused by ignoring the convected part of stress rate for spherical cavity expansion in purely frictional soils ($C = 0$).

2. By neglecting the convected part of the stress rate in the governing equations, the approximate limit solution of Carter et al. [2] can be recovered from the rigorous similarity solution developed in this article. The errors introduced by this assumption are small for low values of frictional and dilation angles but will increase to as much as 30% when friction and dilation angles become large. The difference is also dependent on cavity type and stiffness properties of the soil. It is important however to note that neglecting the convected part of the stress rate tends to give lower value of cavity limit pressures, and therefore would be conservative if used to estimate bearing capacity of deep foundations in practice. It should also be pointed out that for realistic values of stiffness and friction and dilation angles, the maximum error caused by neglecting the convected part of the stress rate on the cavity limit pressure will be a few percent for a cylindrical cavity and some 10% for a spherical cavity.

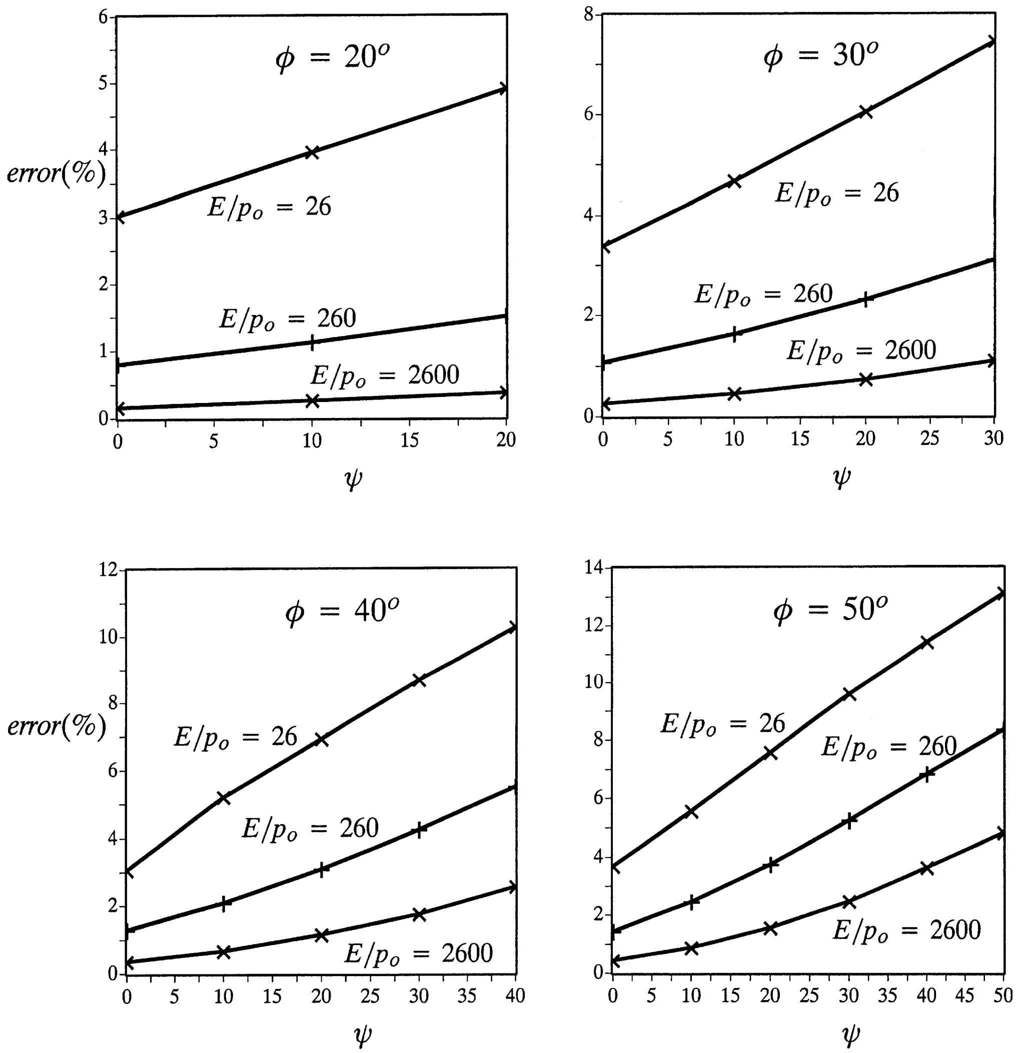

FIGURE 4 Errors of limit pressure caused by ignoring the convected part of stress rate for cylindrical cavity expansion in cohesive-frictional soils ($C/p_0 = 1$).

3. Numerical results suggest that the cavity limit solutions obtained by Yu [8] and Yu and Houlsby [12] using Chadwick's total strain method are practically identical to the solutions obtained in this article by following Hill's incremental velocity approach. However, the main advantage of Chadwick's total strain method is that it can be used to derive analytical solutions for cavity expansion curves, as shown in Yu and Houlsby [12].

Acknowledgments

The authors wish to thank Professor I. F. Collins of Auckland University for stimulating discussions on cavity expansion theory. Mr. Doan Khong prepared some of the tables and figures presented in this article.

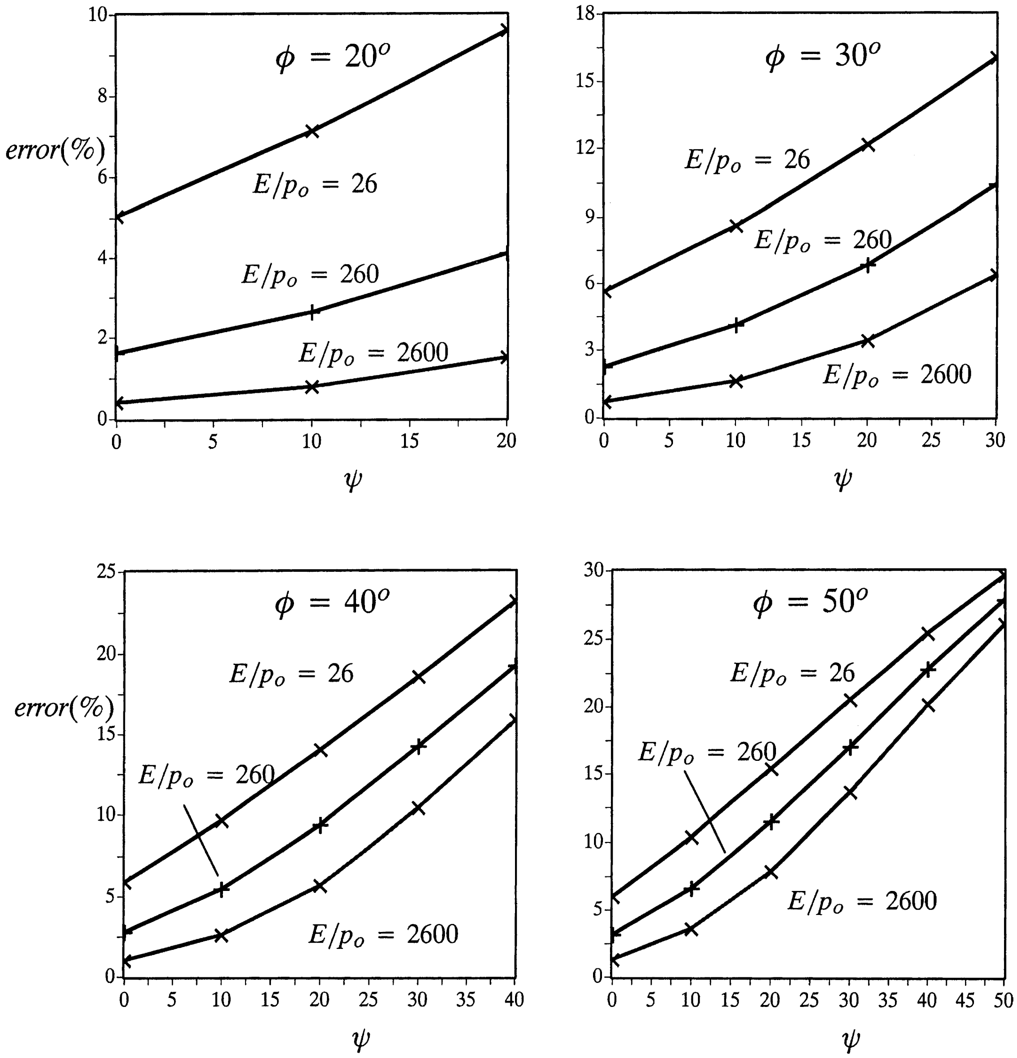

FIGURE 5 Errors of limit pressure caused by ignoring the convected part of stress rate for spherical cavity expansion in cohesive-frictional soils ($C/p_0 = 1$).

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