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A THEORY OF FINITE ELASTIC CONSOLIDATION

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Abstract—Presented in this paper is a general theory describing the consolidation of a porous elastic soil. The formulation allows for the occurrence of finite geometry changes and finite elastic strains during the consolidation process. The governing equations have been cast in a rate form and the laws which determine deformation and pore fluid flow, i.e. Hooke's law and Darcy's law, are presented in a frame indifferent manner. A numerical technique is described that provides an approximate solution to the governing equations. The theory and the solution technique are illustrated by several examples of practical interest.

1. INTRODUCTION

Predicting the behaviour of a foundation resting on a saturated clay is an important problem in foundation engineering.

Saturated clay consists of two phases, a compressible solid phase (the soil skeleton) and a liquid phase (the water filling the pores). When the foundation is first loaded the soil skeleton tends to compress and so excess pore pressures develop and the foundation undergoes an initial settlement. The pore water then tends to flow from regions of higher excess pore pressure to regions of lower excess pore pressure. As this dissipation of the excess pore pressure occurs the foundation settles and ultimately reaches a final settlement.

The process of consolidation described above was first investigated by Terzaghi[1] for one-dimensional conditions. Subsequently, Biot[2, 3] extended Terzaghi's theory to three dimensional situations. However, exact solutions to problems involving the consolidation of a soil mass are not easy to obtain. This is not surprising when it is considered that the equations of consolidation combine the complexities of an elastic problem with those of a diffusion process. For this reason exact solutions have been found only to problems in which the body under consideration has a particularly simple geometry and is subjected to simple boundary conditions (see, for example, Mandel[4], McNamee and Gibson[5, 6], Gibson and McNamee[7] and Gibson, Schiffman and Pu[8]). In most practical problems it is necessary to employ numerical techniques to integrate the equations of Biot's theory. For the case of a soil with an elastic skeleton numerical approaches have been developed by various authors[9–13].

In the formulations of Terzaghi and Biot the authors restricted their attention to conditions of infinitesimal strain and thus the theory they developed is only strictly applicable to situations in which the geometry of the problem varies only slightly during loading. Gibson, England and Hussey[14] recognised this limitation and developed a one-dimensional theory which accounted for such finite deformation. More recently, Mesri and Rokhsar[15] have included some account for finite strain in their numerical treatment of one-dimensional consolidation. Smiles and Poulos[16] examined the one-dimensional problem with no restriction on the magnitude of strain, and an allowance for the variation in flow parameters with variation in void ratio, in an endeavor to explain the phenomenon of secondary consolidation.

In this paper the theory of finite consolidation is generalised from one to three-dimensional conditions. The basic equations are approximated by the finite element method and several example problems are solved.

2. GOVERNING EQUATIONS

There have been several investigations of the finite deformation of soil[17–19]. These treatments have regarded the soil as a single phase material and have ignored the interaction between the solid and fluid phase and, therefore, can only be used to predict settlements under either totally drained or totally undrained conditions.

In this paper the methods developed in Ref. [17] will be extended to obtain an incremental analysis of both the soil skeleton and the pore water while taking into account the coupling of the two processes.

2.1 *Effective stress-strain behaviour*

Suppose that at some time t_0 a consolidating soil occupies a region in space V_0 bounded by a surface S_0. Part of this surface S_{0T} is subject to specified applied tractions T_{0i} while on the complementary portion S_{0D} the displacements are assumed to be zero. Portion of this surface S_{0P} is free to drain, the complementary portion S_{0I} is assumed to be impermeable† as shown in Fig. 1. At some later time t the body will have moved to a region in space V bounded by a surface S. The traction specified, displacement specified, permeable and impermeable portions of S will be denoted S_T, S_D, S_P and S_I respectively.

Using a Cartesian reference frame, consider a specified material point of the skeleton which occupies a position $a_i (i = 1, 2, 3)$ at some time t_0; at a later time t this point will have moved to the position x_i, where

$$x_i = a_i + u_i, \quad i = 1, 2, 3. \tag{1}$$

The quantity u_i represents the displacement of the solid particle and is measured relative to the position of the body at time t_0.

In an Eulerian description the instantaneous rate of deformation may be described by the velocity gradient

$$\frac{\partial v_{si}}{\partial x_j} = e_{ij} + \omega_{ij} \tag{2}$$

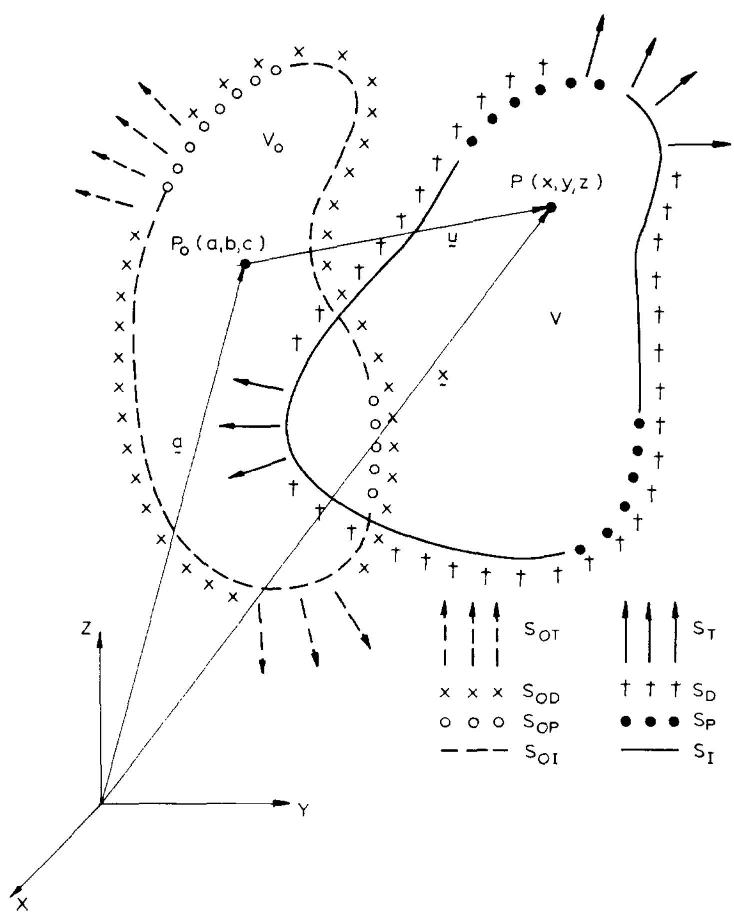

Fig. 1. Deformation mapping.

†The extension to more complicated boundary conditions both elastic and hydraulic is straightforward and will not be given here.

where

$$\frac{\partial}{\partial t}u_i(a_k, t) = \frac{\mathrm{d}}{\mathrm{d}t}u_i(x_k, t), \quad k = 1, 2, 3 \tag{3}$$

is the velocity of the soil skeleton.

The symmetric deformation rate tensor e_{ij} and the skew-symmetric spin tensor ω_{ij} are given as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_{si}}{\partial x_j} + \frac{\partial v_{sj}}{\partial x_i}\right) \tag{4}$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial v_{si}}{\partial x_j} - \frac{\partial v_{sj}}{\partial x_i}\right). \tag{5}$$

In any constitutive law that expresses the stress rate as a function of the deformation rate, the definition of stress rate employed should strictly be frame indifferent[20, 21]. In this paper the stress rate used is that due to Jaumann[22] which is defined by

$$\hat{\sigma}_{ij} = \mathring{\sigma}_{ij} - \sigma_{ik}\omega_{ki} - \sigma_{jk}\omega_{kj} \tag{6}$$

where σ_{ij} denotes the Cauchy total stress field at time t with tensile stresses reckoned positive. The superior dot is used to represent material differentiation with respect to time and unless otherwise stated repeated indices will imply summation.

A general linear relationship between the objective stress rate and the deformation rate (i.e. the effective stress-strain law) can be written in the form

$$\hat{\sigma}_{ij} + \mathring{p}\delta_{ij} = D_{ijkl}e_{kl} \tag{7}$$

where p is the pore pressure at x_i at time t with compressive pore pressures positive; δ_{ij} is the Kronecker delta; and D_{ijkl} are the elastic constants for the drained behaviour of the soil which may of course depend upon both position and time. By the use of a piecewise linearity of the "constants" D_{ijkl} any combination of experimentally observed and intuitively assumed skeleton behaviour is readily incorporated into the numerical technique described later.

An alternative form of eqn (7) useful in subsequent developments is

$$\mathring{\sigma}_{ij} = D_{ijkl}e_{kl} + \sigma_{ik}\omega_{ki} + \sigma_{jk}\omega_{kj} - \mathring{p}\delta_{ij}. \tag{8}$$

2.2 *Fluid flow behaviour*

It will be assumed that the movement of the fluid through the soil is governed by Darcy's law but, as noted by Gibson *et al.*[14], some care is necessary in formulating this in a consistent form. Thus if the fluid has an actual velocity v_{fi}, then the superficial velocity of the fluid relative to the skeleton is $\alpha(v_{fi} - v_{si})$, where α is the soil porosity in the neighbourhood of x_i at time t. This superficial velocity is proportional to the hydraulic gradient, i.e.

$$\alpha(v_{fi} - v_{si}) = -k_{ij}\frac{\partial h}{\partial x_j}. \tag{9}$$

where $h = (p/\gamma_f) + x_k b_k$; γ_f, is the unit weight of pore fluid; k_{ij}, are the permeability coefficients which may depend upon position and time; and b_i, are the components of a vector indicating the direction of gravity.

Some care is also necessary in the definition of the permeability coefficients and this matter is discussed in the appendix.

2.3 *Mass flow for solid and fluid phases*

Consider a physical element of the soil skeleton which has unit weight and porosity (γ_s, α) at time t. Conservation of mass leads to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\gamma_s}{g}(1-\alpha)\right\}+\theta\frac{\gamma_s}{g}(1-\alpha)=0 \tag{10}$$

where g is the acceleration due to gravity, and $\theta = e_{ii}$ is the rate of volume strain.

If it is supposed that the material of the soil skeleton is much less compressible than the soil consisting of both the solid and fluid phases, then γ_s is constant, so that

$$\theta = \mathring{\alpha}/(1-\alpha). \tag{11}$$

Similarly, considering the mass flow of the fluid into and out of a specified physical element, with velocity v_{fi} and unit weight γ_f at time t then it is found that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\gamma_f}{g}\alpha\right\}+\theta\frac{\gamma_f}{g}\alpha=-\frac{\partial}{\partial x_i}\left\{\frac{\gamma_f}{g}\alpha(v_{fi}-v_{si})\right\}. \tag{12}$$

Again assuming that the fluid is much less compressible than the two phase soil, then

$$\mathring{\alpha}+\alpha\theta=-\frac{\partial}{\partial x_i}\{\alpha(v_{fi}-v_{si})\}. \tag{13}$$

Equations (11) and (13) may be combined to obtain an expression for the overall volume behaviour of the soil

$$\theta=-\frac{\partial}{\partial x_i}\{\alpha(v_{fi}-v_{si})\}. \tag{14}$$

2.4 *Virtual work expressions*

The total stress distribution within the soil must always satisfy the conditions of equilibrium, so that at time t

$$\frac{\partial}{\partial x_j}\{\sigma_{ij}\}+F_i=0 \tag{15}$$

where $F_i=\{\gamma_s(1-\alpha)+\gamma_f\alpha\}b_i$ is the body force vector.

For our purposes a more convenient form of eqn (15) incorporating the stress boundary conditions is the equation of virtual work

$$\int_V \mathrm{d}e_{ij}\sigma_{ji}\,\mathrm{d}V=\int_V \mathrm{d}v_{si}F_i\,\mathrm{d}V+\int_{S_T}\mathrm{d}v_{si}T_i\,\mathrm{d}S \tag{16a}$$

where the stress field δ_{ij} is in equilibrium with the tractions T_i and body forces F_i, while the virtual velocities $\mathrm{d}v_{si}$ are compatible with the virtual deformation rates $\mathrm{d}e_{ij}$ and the velocity boundary conditions on S_D.

When the rate law (8) is introduced into eqn (16a) it becomes

$$\int_V \mathrm{d}e_{ij}\left\{\int_{t_0}^{t}(D_{jikl}e_{kl}+\sigma_{ik}\omega_{ki}+\sigma_{jk}\omega_{kj})\,\mathrm{d}t-(p-p_0)\delta_{ji}\right\}\mathrm{d}V=R_i \tag{16b}$$

where $R_i=\int_V \mathrm{d}v_{si}F_i\,\mathrm{d}V+\int_{S_T}\mathrm{d}v_{si}T_i\,\mathrm{d}S-\int_V \mathrm{d}e_{ij}\sigma_{0ij}\,\mathrm{d}V$ and σ_{0ij}, p_0 are the total stress and pore pressure distributions within V_0 at time t_0.

Similarly the volume behaviour, eqn (14), can be replaced by the integral formulation

$$\int_V\left\{\alpha(v_{fi}-v_{si})\frac{\partial}{\partial x_i}\{\mathrm{d}p\}-\theta\,\mathrm{d}p\right\}\mathrm{d}V=0$$

which on introduction of Darcy's law (9) becomes

$$\int_V \left(\frac{\partial}{\partial x_j} \{h\} k_{ji} \frac{\partial}{\partial x_i} \{\mathrm{d}p\} + \theta \,\mathrm{d}p \right) \mathrm{d}V = 0 \tag{17b}$$

where virtual pore pressure changes are consistent with the boundary conditions on S_p.

3. APPROXIMATE SOLUTION

Equations (16) and (17) are exact expressions governing the finite consolidation behaviour of an elastic soil. It is not, in general, possible to find rigorous solutions to these equations; however, numerical solutions may be found using the finite element technique to perform the spatial integrations and a marching process to perform the time integration.

In developing the numerical formulation it is convenient to rewrite several of the equations developed in the previous section in the following alternative notation.

Adopting the Cartesian reference frame (x, y, z), let the field quantities u_i, v_{si}, v_{fi} be the components of vectors $\mathbf{u}$, $\mathbf{v}_s$, $\mathbf{v}_f$ respectively, where

$$\mathbf{u}^T = (u_x, u_y, u_z)$$
$$\mathbf{v}_s{}^T = (v_{sx}, v_{sy}, v_{sz})$$
$$\mathbf{v}_f{}^T = (v_{fx}, v_{fy}, v_{fz}).$$

For convenience we have replaced subscripts $i = 1, 2, 3$ by x, y, z respectively. Since σ_{ij} is symmetric we define $\boldsymbol{\sigma}$ a vector of stress components as

$$\boldsymbol{\sigma}^T = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}).$$

Utilising the symmetry and skew symmetry of e_{ij} and ω_{ij} we define vectors of deformation rate $\mathbf{e}$ and $\boldsymbol{\omega}$ as

$$\mathbf{e} = \partial \mathbf{v}_s$$
$$\boldsymbol{\omega} = \boldsymbol{\xi} \mathbf{v}_s$$

$$\text{with } \partial^T = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial y & 0 & \partial/\partial z \\ 0 & \partial/\partial y & 0 & \partial/\partial x & \partial/\partial z & 0 \\ 0 & 0 & \partial/\partial z & 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}$$

and

$$\boldsymbol{\xi} = \begin{bmatrix} \partial/\partial y & -\partial/\partial x & 0 \\ 0 & \partial/\partial z & -\partial/\partial y \\ -\partial/\partial z & 0 & \partial/\partial x \end{bmatrix}$$

For a soil with an elastic skeleton the rate law becomes

$$\mathring{\boldsymbol{\sigma}} = \mathbf{P}\mathbf{d} - \mathring{p}\boldsymbol{\eta} \tag{18}$$

where

$$\mathbf{d}^T = (\mathbf{e}^T, \boldsymbol{\omega}^T)$$
$$\boldsymbol{\eta}^T = (1, 1, 1, 0, 0, 0)$$

$$\mathbf{P} = \left[\begin{array}{c|ccc} & \sigma_{xy} & 0 & -\sigma_{zx} \\ \mathbf{D} & -\sigma_{xy} & \sigma_{yz} & 0 \\ & 0 & -\sigma_{yz} & \sigma_{zx} \\ \hline & \frac{1}{2}(\sigma_{yy} - \sigma_{xx}) & \frac{1}{2}\sigma_{zx} & -\frac{1}{2}\sigma_{yz} \\ \mathbf{O} & -\frac{1}{2}\sigma_{zx} & \frac{1}{2}(\sigma_{zz} - \sigma_{yy}) & \frac{1}{2}\sigma_{xy} \\ & \frac{1}{2}\sigma_{yz} & -\frac{1}{2}\sigma_{xy} & \frac{1}{2}(\sigma_{xx} - \sigma_{zz}) \end{array} \right]$$

and $\mathbf{D}$ is the matrix of elastic constants for the drained behaviour of the soil skeleton.

Darcy's law takes the form

$$\alpha(\mathbf{v}_f - \mathbf{v}_s) = -\mathbf{K}\nabla h \tag{19}$$

where $\mathbf{K}$ denotes the matrix of permeability coefficients corresponding to k_{ij}.

The governing eqns (16) and (17) when expressed in this notation are

$$\int_V \mathrm{d}\mathbf{e}^T\left\{\int_{t_0}^{t} \mathbf{P}\mathbf{d}\,\mathrm{d}t - (p - p_0)\boldsymbol{\eta}\right\}\mathrm{d}V = \mathbf{R} \tag{20}$$

and

$$\int_V \{(\nabla h)^T\,\mathbf{K}^T\nabla(\mathrm{d}p) + \theta\,\mathrm{d}p\}\,\mathrm{d}V = 0 \tag{21}$$

where

$$\mathbf{R} = \int_V \mathrm{d}\mathbf{v}_s{}^T\,\mathbf{F}\,\mathrm{d}V + \int_{S_T} \mathrm{d}\mathbf{v}_s{}^T\mathbf{T}\,\mathrm{d}S - \int_V \mathrm{d}\mathbf{e}^T\boldsymbol{\sigma}_0\,\mathrm{d}V$$

with $\mathbf{T}$, $\mathbf{F}$, $\boldsymbol{\sigma}_0$ corresponding to T_i, F_i and σ_{0ij} respectively.

The variational problem described by eqns (20) and (21) can be solved approximately as follows:

(i) Suppose that the deforming body is represented by a number of finite elements and that the continuous displacement and pore pressure fields can be adequately described by their values at the connecting nodes $1, 2, \ldots, N$ and let

$$\Delta\boldsymbol{\delta}^T = (\mathbf{u}_1{}^T, \mathbf{u}_2{}^T, \ldots, \mathbf{u}_N{}^T) = \boldsymbol{\delta}^T(t) - \boldsymbol{\delta}^T(t_0)$$
$$\mathbf{q}^T = (p_1, p_2, \ldots, p_N) = \mathbf{q}^T(t).$$

The subscripts in the above definitions refer to values at a particular node and note that $\mathbf{q} = \mathbf{q}(t)$ represents the nodal pore pressure at time t while $\boldsymbol{\delta} = \boldsymbol{\delta}(t)$ represents the total nodal displacement in the time interval $(0, t)$.

(ii) Suppose that the continuous fields $\mathbf{v}_s$ and p can be adequately approximated in terms of nodal values, so that

$$\mathbf{v}_s = \mathbf{A}\Delta\mathring{\boldsymbol{\delta}} = \mathbf{A}\mathring{\boldsymbol{\delta}} \tag{22a}$$

$$p = \mathbf{a}^T\mathbf{q} \tag{22b}$$

where the form of $\mathbf{A}$ and $\mathbf{a}$ depend upon the particular type of element used and will in general depend upon its current position.

(iii) In terms of the nodal quantities, the velocity and pore pressure gradients may be written

$$\mathbf{d} = \mathbf{B}\mathring{\boldsymbol{\delta}} \tag{23a}$$

$$\mathbf{e} = \mathbf{C}\mathring{\boldsymbol{\delta}} \tag{23b}$$

$$\theta = \mathbf{N}^T\mathring{\boldsymbol{\delta}} \tag{23c}$$

$$\nabla p = \mathbf{E}\mathbf{q} \tag{23d}$$

$$\nabla h = \mathbf{E}\mathbf{q}/\gamma_f + \mathbf{i}_g \tag{23e}$$

where

$$\mathbf{B} = \begin{pmatrix}\partial\\ \xi\end{pmatrix}\mathbf{A},\quad \mathbf{C} = \partial\mathbf{A},\quad \mathbf{N}^T = \boldsymbol{\eta}^T\mathbf{C}$$
$$\mathbf{E}^T = (\partial\mathbf{a}/\partial x,\ \partial\mathbf{a}/\partial y,\ \partial\mathbf{a}/\partial z)$$

and $\mathbf{i}_g$ is the vector containing the terms b_i.

(iv) Equations (20) and (21) can now be approximated by

$$\mathrm{d}\mathring{\boldsymbol{\delta}}^T \int_V \mathbf{C}^T \left\{ \int_{t_0}^{t} \mathbf{P}\mathbf{B}\mathring{\boldsymbol{\delta}}\, \mathrm{d}t \right\} \mathrm{d}V - \mathrm{d}\mathring{\boldsymbol{\delta}}^T \mathbf{L}^T \Delta\mathbf{q} = \mathrm{d}\mathring{\boldsymbol{\delta}}^T \mathbf{m} \tag{24a}$$

$$-\mathrm{d}\mathbf{q}^T \mathbf{L}\mathring{\boldsymbol{\delta}} - \mathrm{d}\mathbf{q}^T \boldsymbol{\Phi}\mathbf{q} = \mathrm{d}\mathbf{q}^T \mathbf{n} \tag{24b}$$

where

$$\Delta\mathbf{q} = \mathbf{q}(t) - \mathbf{q}(t_0)$$

$$\mathbf{L}^T = \int_V N\mathbf{a}^T\, \mathrm{d}V$$

$$\boldsymbol{\Phi} = \frac{1}{\gamma_f} \int_V \mathbf{E}^T \mathbf{K}^T \mathbf{E}\, \mathrm{d}V$$

$$\mathbf{m} = \int_V (\mathbf{A}^T \mathbf{F} - \mathbf{C}^T \boldsymbol{\sigma}_0)\, \mathrm{d}V + \int_{S_T} \mathbf{A}^T \mathbf{T}\, \mathrm{d}S$$

$$\mathbf{n} = \int_V \mathbf{E}^T \mathbf{K}^T \mathbf{i}_g\, \mathrm{d}V.$$

Equations (24a, b) apply for any arbitrary $\mathrm{d}\mathring{\boldsymbol{\delta}}^T$ and $\mathrm{d}\mathbf{q}^T$, hence the set of approximating equations becomes

$$\int_V \mathbf{C}^T \left\{ \int_{t_0}^{t} \mathbf{P}\mathbf{B}\mathring{\boldsymbol{\delta}}\, \mathrm{d}t \right\} \mathrm{d}V - \mathbf{L}^T \Delta\mathbf{q} = \mathbf{m} \tag{25a}$$

$$-\mathbf{L}\mathring{\boldsymbol{\delta}} - \boldsymbol{\Phi}\mathbf{q} = \mathbf{n}. \tag{25b}$$

3.1 *Numerical method*

Equations (25) are a set of differential-integral equations and they may be integrated from t_0 to t obtain the following approximation

$$\begin{bmatrix} \bar{\mathbf{Q}} & -\bar{\mathbf{L}}^T \\ -\bar{\mathbf{L}} & -\beta\Delta t\bar{\boldsymbol{\Phi}} \end{bmatrix} \begin{pmatrix} \Delta\boldsymbol{\delta} \\ \Delta\mathbf{q} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{m}} \\ \bar{\mathbf{n}}\Delta t + \bar{\boldsymbol{\Phi}}\mathbf{q}_0\Delta t \end{pmatrix} \tag{26}$$

where

$$\bar{\mathbf{Q}} = \int_{\bar{V}} (\bar{\mathbf{C}}^T \bar{\mathbf{P}}\bar{\mathbf{B}})\, \mathrm{d}V, \quad \Delta t = t - t_0, \quad \mathbf{q}_0 = \mathbf{q}(t_0).$$

The superior bar denotes that the quantity is evaluated for some average or representative value, spatial integrations being performed over some representative configuration. It can be seen from eqn (26) that if the solution is known at time t_0 it can be marched forward to obtain the solution at $t_0 + \Delta t$, however it should be noted that since $\bar{\mathbf{Q}}$, $\bar{\mathbf{L}}$, $\bar{\boldsymbol{\Phi}}$, $\bar{\mathbf{m}}$, $\bar{\mathbf{n}}$ may all contain average quantities it may be necessary to solve eqns (26) iteratively for each time step.

The parameter β corresponds to the approximation

$$\int_{t_0}^{t} \boldsymbol{\Phi}\mathbf{q}\, \mathrm{d}t \approx \bar{\boldsymbol{\Phi}}(\mathbf{q}_0 + \beta\Delta\mathbf{q})\Delta t.$$

In order to ensure stability of the marching process it is necessary to choose $\beta \geq \frac{1}{2}$[13].

4. EXAMPLES

Both one and two-dimensional examples are presented to illustrate the theory. In all cases the soil is an isotropic, elastic two phase continuum. It is assumed uniform and homogeneous with regard to both deformation and flow properties. In all cases gravity acts in the direction of the applied load.

For the plane strain case the matrix **D** of eqn (18) is taken as

$$\mathbf{D}=\begin{bmatrix}\Lambda+2G & \Lambda & 0\\ \Lambda & \Lambda+2G & 0\\ 0 & 0 & G\end{bmatrix} \tag{27}$$

where Λ and G are the Lamé parameters of the classical theory.

4.1 *One-dimensional finite consolidation*

The problem of one-dimensional consolidation is analysed for the situation in which the traction q is applied instantaneously to the soil surface at $t = 0$ and thereafter held constant and drainage occurs only at this surface. Under these conditions a solution for the settlement as a function of time t can be seen from eqn (26) to depend upon the following parameters: q/E'; $\gamma_f H/E'$; ν'; e_0 and S_g where: E' and ν' are the drained Young's modulus and Poisson's ratio respectively for the soil; H is the initial depth of the layer; γ_f is the unit weight of the pore fluid; e_0 is the uniform initial void ratio of the soil; k is the soil permeability; and S_g is the specific gravity of the solid particles.

For the results presented the following material properties were chosen: $e_0 = 10$; $\nu' = 0.3$; $S_g = 2.65$. Figures 2 and 3 show some solutions for the degree of settlement, U as a function of the parameters q/E' and $\gamma_f H/E'$ and the dimensionless time $T = c_v t/H^2$, where c_v is the usual one dimensional consolidation coefficient. The curves indicate that shallow stiff layers exhibit a consolidation behaviour more like the Terzaghi prediction than do deeper, less stiff layers. For

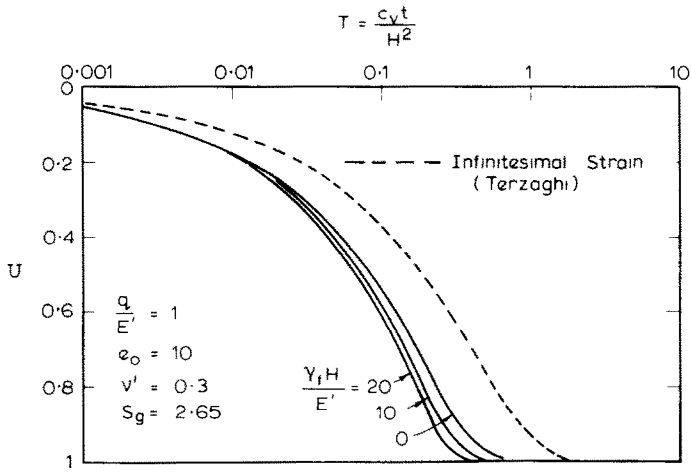

Fig. 2. 1 − *D* finite consolidation—$\gamma_f H/E'$ effect.

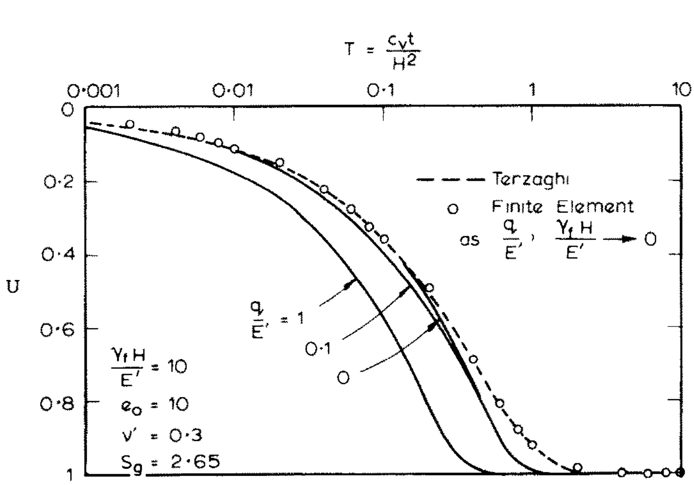

Fig. 3. 1 − *D* finite consolidation—q/E' effect.

any given soil a further departure from the classical behaviour is observed as the magnitude of the consolidating pressure is increased. Note that to obtain the Terzaghi solution it is necessary for *both of the parameters* $\gamma_f H/E'$ and q/E' to approach zero. Numerical agreement with this solution is shown in Fig. 3.

4.2 *Two-dimensional finite consolidation—a rigid footing*

The plane strain problem of the finite consilidation of a rigid, permeable strip footing resting on the surface of a saturated clay layer is described in the inset to Fig. 4. As with the one-dimensional problem the solution for the settlement, ρ with time can be shown to depend upon, amongst others, the parameters $Q/2BG$ and $\gamma_f B/G$. G is the shear modulus for the soil, B is the footing half width and Q is the total applied load per unit length of footing.

To approximate an instantaneous loading the vertical force on the footing was increased linearly with time from zero at $T = 0$ to its ultimate value at $T = 0.0001$ over a number of steps. To model a rigid footing the load was applied as a series of nodal forces to several very stiff (compared to the soil) footing elements.

Some results for the footing settlement ρ as a function of the dimensionless time factor $T = c_v t/D^2$, where D is the layer depth at $t = 0$, are given in Fig. 4 for the case $\gamma_f\, B/G = 0.1$, $E_0 = 10$ and $\nu' = 0.3$. The curves show that the settlement behaviour is more unlike the small strain prediction for larger values of the parameter $Q/2BG$ (i.e. as either the load is increased and/or less stiff soils are loaded). According to the finite theory a settlement equal to the layer depth would be approached as $Q/2BG$ approaches an infinite value. This is not the case for infinitesimal theory where physically impossible settlements are predicted at finite load levels, see for example, the curve for $Q/2BG = 10$ of Fig. 4.

Figure 5 shows the configuration of the finite element mesh at various times for the case of $Q/2BG = 5$. For large values of the parameter $Q/2BG$ severe distortion of elements occurs, particularly near the edge of the footing. This may render the calculation unrealistic, especially if the void ratio becomes zero in any element.

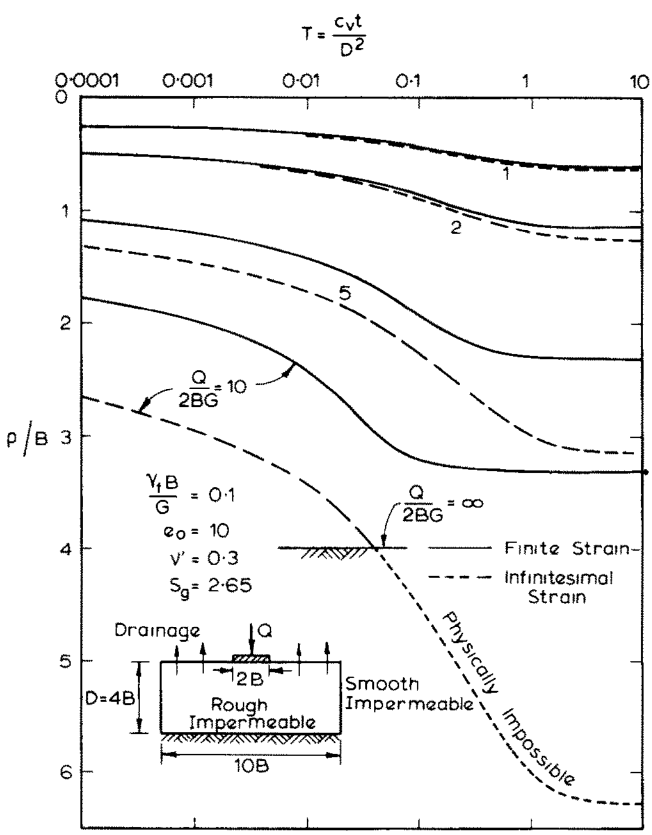

Fig. 4. Two dimensional finite consolidation.

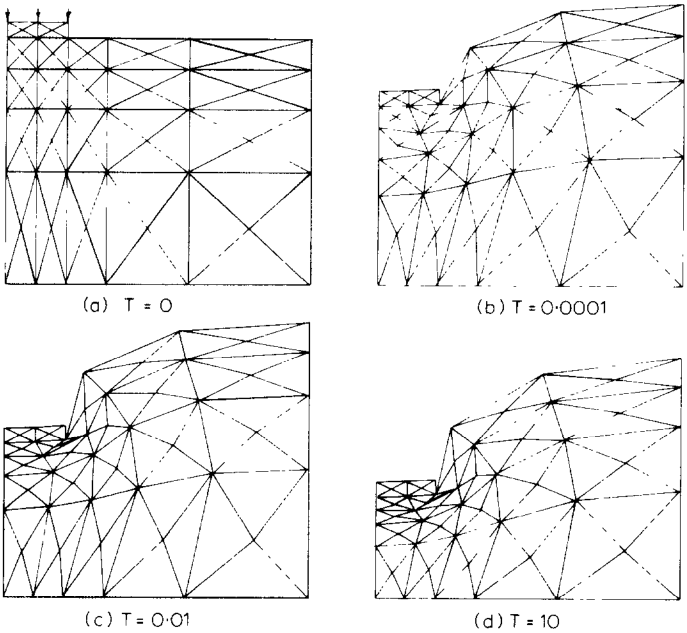

Fig. 5. 2 − *D* finite consolidation—mesh geometries.

5. CONCLUSIONS

A consistent formulation for the consolidation of a soil which incorporates the effects due to significant changes of geometry has been proposed. In developing the theory attention has been restricted to the case of a soil with an elastic skeleton. However, using the same approach it is possible to extend this theory to analyse the behaviour of a soil with an inelastic skeleton. In general, real soil may be modelled as one for which the elastic moduli and soil permeabilities vary with stress level and void ratio and which includes the possibility of plastic yielding. The consolidation of a soil with such an inelastic skeleton forms the subject of future work.

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APPENDIX

The permeability matrix

We present here the derivation of matrix $\mathbf{K}$ of eqn (19) for the general case of three dimensional consolidation.

Consider an element of soil with centre $P_0(a, b, c)$ which deforms from an initial position A at time t_0 to an adjacent position B with centre $P(x, y, z)$ in a time interval dt, as shown schematically in Fig. 6. Initially the flow properties are characterised by

$$\mathbf{K}_0 = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{xy} & k_{yy} & k_{yz} \\ k_{xz} & k_{yz} & k_{zz} \end{bmatrix}$$

so that Dancy's law at A may be written as

$$\alpha_0(\mathbf{v}_{f0} - \mathbf{v}_{s0}) = -\mathbf{K}_0\nabla_0 h_0 \tag{A1}$$

where h_0, α_0 are the head and porosity in the vicinity of P_0 respectively, and $(\mathbf{v}_{f0} - \mathbf{v}_{s0})$ is the vector of velocity components (measured with respect to x, y, z axes) of the fluid phase relative to the solids at P_0. The quantity $\nabla_0 h_0$ is given by

$$\nabla_0 h_0 = \left(\frac{\partial h_0}{\partial a}, \frac{\partial h_0}{\partial b}, \frac{\partial h_0}{\partial c}\right)^T. \tag{A2}$$

The question now arises as to the form of Darcy's law when the element of soil is in position B. One reasonable assumption is that the form of any flow anisotropy is intrinsic to the element so that

$$\alpha(\mathbf{v}'_f - \mathbf{v}'_s) = -\mathbf{K}\nabla' h \tag{A3}$$

where h, α represent the same quantities as before only measured at P at time $t_0 + dt$, and

$$\nabla' h = \left(\frac{\partial h}{\partial \xi}, \frac{\partial h}{\partial \eta}, \frac{\partial h}{\partial \zeta}\right)^T. \tag{A4}$$

The vector $(\mathbf{v}'_f - \mathbf{v}'_s)$ contains the components of the relative velocity at P but measured with respect to rotated axes (ξ, η, ζ). The relationship between $(\mathbf{v}'_f - \mathbf{v}'_s)$ and $(\mathbf{v}_f - \mathbf{v}_s)$, the relative velocity vector at P measured with respect to (x, y, z), is given by

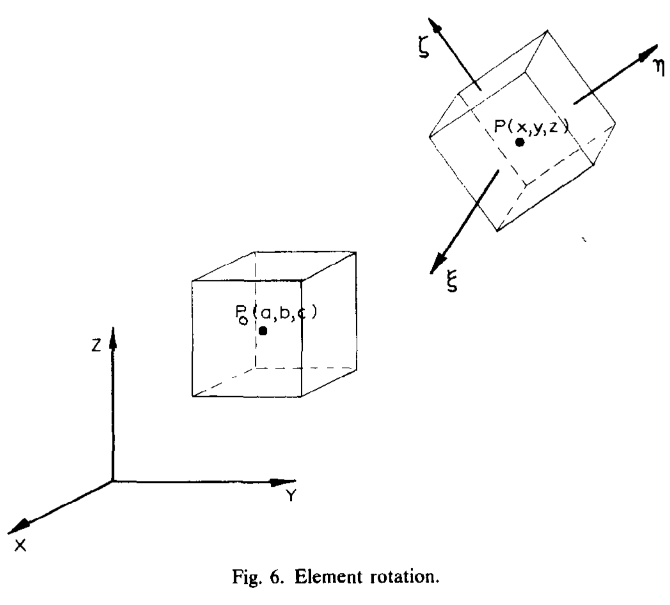

Fig. 6. Element rotation.

$$(\mathbf{v}_f' - \mathbf{v}_s') = \mathbf{R}(\mathbf{v}_f - \mathbf{v}_s) \tag{A5}$$

where $\mathbf{R}$ is the matrix which corresponds to the appropriate rotation of coordinate axes, (e.g. for plane deformations

$$\mathbf{R} = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix}$$

where γ is the angle of rotation between (x, y) and (ξ, η)).

Considering (x, y, z) as the independent variables then

$$\nabla' h = \mathbf{R}\nabla H \tag{A6}$$

where

$$\nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right)^T.$$

Equation (A3) may thus be transformed to give Darcy's law at B in the form

$$\alpha(\mathbf{v}_f - \mathbf{v}_s) = -\mathbf{K}\nabla h \tag{A7}$$

where

$$\mathbf{K} = \mathbf{R}^T\mathbf{K}_0\mathbf{R}.$$

$\mathbf{K}$ thus remains symmetric during the rotation. It is interesting to note that, according to the above assumption, as an element rotates the form of Darcy's law (referred to the initial set of axes) changes. Thus a material which is anisotropic, but whose initial anisotropy is homogeneous develops an inhomogeneity of anisotropy as different elements rotate by different amounts. Of course this does not occur (according to this formulation) if the material is initially isotropic and in such a case

$$\alpha(\mathbf{v}_f - \mathbf{v}_s) = -k\nabla h \tag{A8}$$

where k is the isotropic permeability.