LOWER BOUND LIMIT ANALYSIS FOR JOINTED ROCKS USING
THE HOEK-BROWN YIELD CRITERION

by A. V. Lyamin1, H.S. Yu2, S.W. Sloan3, M. Z. Hossain4

1 PhD Student, 2 Senior Lecturer, 3 Professor, 4 PhD Student, Department of Civil, Surveying and
Environmental Engineering, The University of Newcastle, Australia

ABSTRACT

This paper describes a technique of lower bound limit analysis for jointed rocks using the Hoek-Brown yield criterion. The lower bound collapse load is computed from a statically admissible stress field which satisfies equilibrium, stress boundary conditions and yield criterion. As manual linearization of a non-linear yield criterion is time consuming, restricts the search domain for the maximum lower bound load and produces a large number of inequality constraints, a new method called dynamic linearization is presented in this paper to overcome these shortcomings. The distinct feature of this method is that it linearizes the yield surface automatically as part of the optimisation process and does that only in the vicinity of the intersection points between the search vector and the nearest bounding yield surface. A conventional active set algorithm is used to solve the resulting optimisation problem. To verify the efficiency of the proposed formulation a rigid strip footing is analysed using at first the Hoek-Brown criterion. This criterion is then manually piecewise linearized to produce a number of yield envelopes which are incorporated in the existing lower bound formulation with the Mohr-Coulomb criterion to produce another lower bound collapse pressure. The comparison of results from these analyses justifies the effective utilisation of the dynamically linearized Hoek-Brown criterion for jointed rocks where the empirical parameters and rock strength dependency are included in the formulation.

1 INTRODUCTION

The lower bound theorem of plasticity theory is a powerful tool for analysing the stability of problems. The theory states that the collapse load calculated from a statically admissible stress field is a lower bound to the actual collapse load. A statically admissible stress field is one which satisfies the stress boundary condition, equilibrium and the yield condition (the stresses must lie inside or on the yield surface in stress space). Although the lower bound theorem is a particularly useful tool for the analysis of stability, it is often difficult to apply to practical problems involving complicated loading and complex geometry. Sloan (1988(a)) employed the formulation of Bottero et al. (1980) and solved the resulting linear programming problems using a steepest edge active set algorithm. This algorithm is ideally suited to the solution of lower bound optimisation problems, which typically involve several thousand unbounded variables and an even larger number of constraints. Advantages of the lower bound technique, which follow from the finite element type of formulation, include the ability to deal with complex loadings and complicated geometries. Linearization of the yield criterion restricts the search domain for finding the maximum lower bound resulting in a lower prediction and also prior linearization of the non-linear yield criterion produces a large number of inequality constraints on the nodal stresses. Therefore, it is the authors wish to incorporate a new approach called dynamic linearization to overcome these factors. The distinct feature of the dynamic linearization is that the inequalities are linearized automatically in the vicinity of the intersection points of the search vector with the nearest bounding yield surface and these planes are added to the active set of linear constraints.

This paper presents the application of lower bound limit analysis to rock mechanics using the Hoek-Brown yield criterion. This criterion includes a non-linear dependency of shear strength on normal stress which makes it more realistic than the Mohr-Coulomb yield criterion.
2 THE HOEK-BROWN YIELD CRITERION

Jointed rocks show distinct behaviour from the intact rocks. The behaviour of the engineering structures are highly dependent on the jointed rock properties. Empirical relationships between the principal stresses or between the shear and normal stresses at failure are proposed because of the difficulty involved in modelling the fracture propagation and failure in rock. The well known Hoek-Brown yield criterion was derived from an extensive set of field and laboratory data (Hoek & Brown, 1980; Hoek, 1983).

Assuming tensile stresses are positive, the original Hoek-Brown criterion in terms of the principal stresses can be expressed as

\[ \sigma_1 = \sigma_3 + \sqrt{-\frac{m\sigma_1\sigma_3 + s\sigma_3^2}{C}} \]

where \( \sigma_1 \) and \( \sigma_3 \) are the major and minor principal stresses at failure and \( \sigma_c \) is the uniaxial compressive strength of the intact rock material in the specimen. The constants \( m \) and \( s \) are empirical parameters which depend upon the properties of the rock and upon the extent to which it has been broken before being subjected to loading.

The principal stresses can be expressed in terms of the cartesian stresses under the plane strain condition as

\[
\begin{align*}
\sigma_1 &= \frac{1}{2} \left( \sigma_x + \sigma_y \right) + \sqrt{\left( \sigma_x - \sigma_y \right)^2 + 4\tau_{xy}^2} \\
\sigma_3 &= \frac{1}{2} \left( \sigma_x + \sigma_y \right) - \sqrt{\left( \sigma_x - \sigma_y \right)^2 + 4\tau_{xy}^2}
\end{align*}
\]

Substituting equation (2) into equation (1) we get the Hoek-Brown yield criterion in terms of the Cartesian stresses as

\[ f = (\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 + \frac{m\sigma_c}{2} \left( \sigma_x + \sigma_y \right) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} - s\sigma_c^2 = 0 \]

This yield criterion is included in the lower bound finite element formulation using the dynamic linearization procedure.

3 FINITE ELEMENT FORMULATION OF LOWER BOUND THEOREM

The finite element lower bound formulation used in this paper follows that of Sloan (1988(a)). For the sake of convenience, some portions of that paper is repeated in this section.

3.1 TRIANGULAR STRESS ELEMENT

The triangular element used to model the stress field under conditions of plane strain is shown in Figure 1. The variation of the stress throughout each element is assumed to be linear and each node is associated with three unknown stresses \( \sigma_x, \sigma_y \) and \( \tau_{xy} \). Each stress varies through an element according to

\[ \sigma_x = \sum_{i=1} N_i \sigma_{xi} ; \quad \sigma_y = \sum_{i=1} N_i \sigma_{yi} ; \quad \tau_{xy} = \sum_{i=1} N_i \tau_{xyi} \]

where \( \sigma_{xi}, \sigma_{yi} \) and \( \tau_{xyi} \) are the nodal stresses and \( N_i \) are linear shape functions. These shape functions are:
Figure 1. 3-Noded linear stress triangle

\[ N_1 = \left( \xi_1 + \eta_1 + \zeta_1 \right)/2A \]
\[ N_2 = \left( \xi_2 + \eta_2 + \zeta_2 \right)/2A \]
\[ N_3 = \left( \xi_3 + \eta_3 + \zeta_3 \right)/2A \]

where
\[ \xi_1 = x_2y_3 - x_3y_2 ; \eta_1 = y_2 - y_3 ; \zeta_1 = x_3 - x_2 \]
\[ \xi_2 = x_3y_1 - x_1y_3 ; \eta_2 = y_3 - y_1 ; \zeta_2 = x_1 - x_3 \]
\[ \xi_3 = x_1y_2 - x_2y_1 ; \eta_3 = y_1 - y_2 ; \zeta_3 = x_3 - x_1 \]

and \( 2A = | \eta_1 \zeta_2 - \eta_2 \zeta_1 | \) is twice the element area. Statically admissible stress discontinuities are permitted at shared edges between adjacent triangles. If \( E \) denotes the number of triangles in the mesh, then there are \( 3E \) nodes and \( 9E \) unknown stresses.

### 3.2 ELEMENT EQUILIBRIUM

In order to satisfy equilibrium, the stresses throughout each triangular element must obey the equations

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 ; \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} = \gamma \]

where tensile stresses are taken as positive, \( \gamma \) is the unit weight and a right handed cartesian coordinate system is adopted. Using equations (4) and (5) and substituting these into the above equation, the equilibrium constraints on the nodal stresses are obtained as

\[ \mathbf{A}^e \mathbf{\sigma}^e = \mathbf{b}_{\text{equil}}^e \]

where

\[ \mathbf{A}^e_{\text{equil}} = \frac{1}{2A^e} \begin{bmatrix} \eta_1 & \zeta_1 & \eta_2 & \zeta_2 & \eta_3 & \zeta_3 \\ 0 & \zeta_1 & \eta_1 & \zeta_2 & \eta_2 & \zeta_3 \\ 0 & \zeta_1 & \eta_1 & \zeta_2 & \eta_2 & \zeta_3 \end{bmatrix} \]

\[ \{ \mathbf{\sigma}^e \}^T = \{ \sigma_{x1}^e \ \sigma_{x1}^e \ \sigma_{y1}^e \ \sigma_{y1}^e \ \sigma_{xy2}^e \ \sigma_{xy2}^e \ \sigma_{xy3}^e \ \sigma_{xy3}^e \ \sigma_{xy3}^e \ \sigma_{xy3}^e \ \sigma_{xy3}^e \ \sigma_{xy3}^e \ \sigma_{xy3}^e \} \]

\[ \{ \mathbf{b}_{\text{equil}}^e \}^T = \{ 0 \ \gamma \} ; \]
and \( A^e \) is area of element. Thus the equilibrium condition for each triangular element generates two equality constraints on the nodal stresses.

### 3.3 DISCONTINUITY EQUILIBRIUM

In order to permit statically admissible discontinuities at the edges of adjacent triangles, additional constraints are necessary to enforce on the nodal stresses. A statically admissible discontinuity permits the tangential stress to be discontinuous, but requires that continuity of the corresponding shear and normal components is preserved. With reference to Figure 2, the normal and shear stresses acting on a plane inclined at an angle \( \theta \) to the x-axis (measured positive anticlockwise) are given by

\[
\begin{align*}
\sigma_n &= \sin^2 \theta \sigma_x + \cos^2 \theta \sigma_y - \sin 2 \theta \tau_{xy} \\
\tau &= -\frac{1}{2} \sin 2 \theta \sigma_x + \frac{1}{2} \sin 2 \theta \sigma_y + \cos 2 \theta \tau_{xy}
\end{align*}
\]

(8)

Figure 2. Resolution of stresses into normal and shear components acting on a plane

Figure 3 illustrates two triangles, \( a \) and \( b \), which share a side \( d \) defined by the nodal pairs (1,2) and (3,4). Equilibrium of the discontinuity requires that at every point along \( d \)

\[
\sigma_n^a = \sigma_n^b ; \quad \tau^a = \tau^b
\]

Since the stresses vary linearly along each element edge, this condition is equivalent to enforcing the constraints

\[
\sigma_{n1}^a = \sigma_{n2}^b ; \quad \sigma_{n3}^a = \sigma_{n4}^b ; \quad \tau_{1}^a = \tau_{2}^b ; \quad \tau_{3}^a = \tau_{4}^b
\]
Substituting equation (8), these equations are summarised by the matrix equation

\[ A_d^{\text{equil}} \alpha = b_d^{\text{equil}} \]  

(9)

where

\[ A_d^{\text{equil}} = \begin{bmatrix} T - T & 0 & 0 \\ 0 & 0 & T - T \end{bmatrix} \]

\[ T = \begin{bmatrix} \sin^2\theta_d & \cos^2\theta_d - \sin2\theta_d \\ -\frac{1}{2} \sin 2 \theta_d & \frac{1}{2} \sin 2 \theta_d \cos 2 \theta_d \end{bmatrix} \]

\[ (\alpha)^T = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \tau_{12} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \tau_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \tau_{34} \\ \tau_{21} & \tau_{32} & \tau_{34} & \tau_{44} \end{bmatrix} \]

\[ b_d^{\text{equil}} = \{0 \ 0 \ 0 \} \]

Hence the equilibrium condition for each statically admissible discontinuity along an element edge generates four equality constraints on the nodal stresses.

### 3.4 BOUNDARY CONDITIONS

Many problems in geotechnical engineering generates stress boundary conditions of the form

\[ \sigma_n = q = \text{constant}; \quad \tau = t = \text{constant} \]

Since each of the stress components \( \sigma_x, \sigma_y \) and \( \tau \), vary linearly along the edge of each triangle, it is possible to cater for a slightly more general type of boundary condition of the form (Figure 4)

\[ \sigma_n = q_1 + (q_2 - q_1)\xi; \quad \tau = t_1 + (t_2 - t_1)\xi \]

(10)

where \( l \) is the edge of triangle \( e \) where boundary tractions are specified (defined by nodes 1 and 2); \( \xi \) is a local coordinate along \( l \); \( q_1, q_2 \) are normal stresses specified at nodes 1 and 2; \( t_1, t_2 \) are shear stresses specified at nodes 1 and 2.

The boundary conditions defined by equation (10) are satisfied exactly by requiring that
Letting $\theta_i$ denote the angle of $l$ to the x-axis and using equation (8), the stress boundary conditions give rise to the constraints

$$ A_i^{\text{bound}} \sigma' = b_i^{\text{bound}} $$

where

$$ A_i^{\text{bound}} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} $$

$$ T = \begin{bmatrix} \sin^2 \theta_i & \cos^2 \theta_i - \sin 2\theta_i \\ \frac{1}{2} \sin 2\theta_i & \frac{1}{2} \sin 2\theta_i \cos 2\theta_i \end{bmatrix} $$

$$ [\sigma']^T = [\sigma_{s1} \; \sigma_{s2} \; \tau_{y1} \; \tau_{y2} \; \sigma_{s1} \; \sigma_{s2} \; \tau_{y1} \; \tau_{y2}] $$

$$ [b_i^{\text{bound}}]^T = [q_1 \; t_1 \; q_2 \; t_2] $$

Thus each edge $l$, where boundary tractions are prescribed, generates a maximum of four equality constraints on the nodal stresses.

### 3.5 OBJECTIVE FUNCTION

For most plane strain geotechnical problems in limit analysis, we wish to find a statically admissible stress field which maximizes an integral of the form

$$ Q = h \int \sigma_n ds $$

where $Q$ is the collapse load, $h$ is the out-of-plane thickness and $\sigma_n$ is the normal stress acting over some part of the boundary $s$. Figure 5 illustrates an edge of a triangle, defined by nodes 1 and 2 over which $\sigma_n$ is to be maximized. Since

$$ Q = \frac{Lh}{2} (\sigma_{s1} + \sigma_{s2}) = \text{normal load} $$

$$ h = \text{out-of-plane thickness} $$

![Figure 5. Load in a direction normal to a boundary edge](image)

the stresses are assumed to vary linearly throughout each element, $Q$ is given by

$$ Q = \frac{Lh}{2} (\sigma_{s1} + \sigma_{s2}) $$
where \( L \) is the length of the edge \( s \) and \( \sigma_{n1}, \sigma_{n2} \) are the normal stresses at nodes 1 and 2 respectively of triangle \( e \).

If \( \theta \) denote the inclination of \( s \) to the \( x \)-axis, assuming unit thickness in the out-of-plane direction and substituting equation (8) we obtain

\[
Q = \{c^s\}^T \sigma^s
\]

(12)

where

\[
\{c^s\}^T = \frac{L}{2} \left[ \sin^2 \theta, \cos^2 \theta, - \sin 2\theta, \sin \theta \cos \theta, \sin \theta \cos \theta, - \sin 2\theta \right]
\]

\[
\{\sigma^s\}^T = \left[ \sigma_{n1}^s, \sigma_{n2}^r, r_{y1}^s, r_{y2}^s, r_{y1}^s, r_{y2}^s \right]
\]

where \( c^s \) is known as the vector of objective function coefficients and has units of area. Since it has been assumed that \( \sigma_n \) is positive for tensile loading, it is necessary to multiply \( c^s \) by -1 if a compressive load is to be maximized.

### 3.6 ASSEMBLY OF CONSTRAINT EQUATIONS

The various constraints are assembled to give the overall constraint matrix according to

\[
A = \sum_{i=1}^{E} A_{\text{eqil}}^i + \sum_{d=1}^{D} A_{\text{eqil}}^d + \sum_{l=1}^{L} A_{\text{bound}}^l
\]

where the coefficients are inserted into the appropriate rows and columns and \( E \) is the total number of elements, \( D \) is the total number of discontinuities and \( L \) is the total number of boundary edges with prescribed tractions.

Similarly, the vector \( b \) and objective function coefficients \( c \) are assembled according to

\[
b = \sum_{i=1}^{E} b_{\text{eqil}}^i + \sum_{d=1}^{D} b_{\text{eqil}}^d + \sum_{l=1}^{L} b_{\text{bound}}^l
\]

\[
c = \sum_{i=1}^{S} c^s
\]

where \( S \) is the total number of boundary edges over which the stresses are to be optimized.

### 4 ACTIVE SET ALGORITHM WITH DYNAMIC LINEARIZATION

A comprehensive description of the active set algorithm may be found in Best & Ritter (1985) and a detailed description of the method is beyond the scope of this paper. Each iteration of the algorithm is comprised of three distinct steps. These are:

1. the determination of the search direction and test for optimality;
2. the determination of the maximum feasible step size and new active constraint and
3. updating the solution and active constraint matrix.

At each iteration the objective function decreases (or in the case of zero maximum feasible step size remains the same) and the algorithm moves along an edge of the feasible region from one vertex to another. The iteration is halted once it is no longer possible to decrease the objective function further by moving to an adjacent vertex. The canonical form required by the proposed modification of active set algorithm is

Maximize \( c^T x \)

Subject to \( a_i^T x = b_i ; i \in R = \{1, 2, ...., r\} \)
feasible point, it is also convenient to introduce artificial rows which are denoted by \( I_j = \{ 1, 2, \ldots, r+m+q \} \), where each of the numbers \( I_j \) is an element of the set \( \{ 1, 2, \ldots, r+m \} \). The index set \( I_j \) defines the constraints which are active (or being) at iteration \( j \) and hence defines the active constraint matrix \( D^j \). If \( I_j = k \), this implies that the \( ith \) row of the \( n \times n \) matrix \( D^j \) is given by \( a_k^j \) (the \( kth \) row of the constraint matrix). In order to start the algorithm with an arbitrary feasible point, it is also convenient to introduce artificial rows which are denoted by \( I_j = 0 \). If \( I_j = 0 \), this implies that the \( ith \) row of \( D^j \) is equal to the \( ith \) row of the identity matrix. Inequalities \( (i \in M) \) can be linearized in advance, and others \( (i \in P) \) are linearized during the optimization process. It is unknown how many such dynamic constraints will be involved in the active set, but their maximal possible number is

\[
q = \sum_{i=1}^{n} DOF_i,
\]

where \( DOF_i \) is the number of variables which each function \( f_i \) depends on. Thus, we can reserve sufficient computing memory for these constraints in beforehand. All the non-linear constraints will give not more than \( q \) linear constraints and they will be dynamically numbered by any of nonallocated numbers (i.e. numbers which are not in active set) from the range \( N = \{ r+m+1, \ldots, r+m+q \} \). In such a manner an ordered index set \( I_j \) is extended and each of the numbers \( I_j \) is now an element of the set \( \{ 1, 2, \ldots, r+m+q \} \). Employing extended definition of the index set \( I_j \) we can write the active set algorithm with the dynamic linearization.

**ALGORITHM** [active set algorithm (Sloan 1988) with dynamic linearization]:

**Step 1.0** - (Initialisation)

Set \( j=0 \). Start with an initial feasible point \( x^0 \), an initial index set \( I^0 = \{ I^0_1, I^0_2, \ldots, I^0_s \} \) and \( [D^0]^{-1} \), where \( [D^0]^T = [d_1^0, d_2^0, \ldots, d_s^0] \) and \( D^0 \) is non-singular. Each column \( d_i^0 \) corresponds to a row of the constraint matrix such that \( d_i^0 = a_{ij} \) (or an artificial row in which case \( I_i^0 = 0 \) and \( d_i^0 \) is the \( ith \) column of the identity matrix) and \( 0 \leq I_i^0 \leq (r + m + q) \). Each equality constraint \( (i \in R) \) must be in \( I^0 \) (provided it is linearly independent of all other equality constraints).

**Step 1.1** - (Computation of search direction)

1.1.0 Set \( \lambda_i^j = [D^j]^T c \)

1.1.1 If \( D^j \) contains no artificial rows (i.e. \( 0 \notin I^j \)) go to step 1.1.4

1.1.2 If \( \lambda_i^j = 0 \) for all \( i \) with \( I_i^j = 0 \) go to step 1.1.4. Else determine the smallest index \( k \) such that

\[
|\lambda_i^j| = \max_{i: I_i^j=0} \left( |\lambda_i^j| \right)
\]

1.1.3 Set \( s_k^j = [D^j]^{-1} e_j \) if \( \lambda_k^j > 0 \), or \( s_k^j = -[D^j]^{-1} e_j \) if \( \lambda_k^j < 0 \), and omit steps 1.1.4-1.1.5.

1.1.4 If \( \lambda_i^j \leq 0 \) for all \( i \) with \( I_i^j > r \), exit with optimal solution \( x^j \). Else determine the smallest index \( k \) such that
1.1.5 Set $s_i' = [D^i]^{-1} e_i$

Step 1.2 - (Computation of maximum feasible step for linear inequalities)

Compute the smallest index $l$ and $\delta_j'$ such that

$$\delta_j' = \frac{a_j^s x - b_j}{a_j^s s_j'} = \min_{i \in M, i \notin I'} \left\{ \frac{a_j^s x - b_j}{a_j^s s_j'} \right\}$$

Step 1.3 - (Computation of overall maximum feasible step)

1.3.0 Compute (analytically, if possible, or numerically) feasible steps for all $f_i(x) \leq 0; i \in P$ and store them in $w_j$. Set $Q = 0$.

1.3.1 Compute the smallest index $nl$ and $\delta_{nl}'$ such that

$$\delta_{nl}' = \min_{i \in P, i \notin Q} w_i$$

1.3.2 If $\delta_{nl}' = 0 (i.e. a_j^s s_j' \geq 0$ for all $i \in M$ and $i \notin I')$, print message that problem is unbounded from below and stop, else go to step 1.4 with unchanged $l$ and $\delta_j' = \delta_j'$.

1.3.3 If $\delta_{nl}' < 0$ and $\delta_{nl}' > \delta_j'$ go to step 1.4 with unchanged $l$ and $\delta_j' = \delta_{nl}'$, else compute the gradient $\nabla f_{nl}$ of scalar function $f_{nl}$ at the point $y = x' - \delta_{nl}' s_j'$. Vector $\nabla f_{nl}(y)$ defines the tangential hyperplane to surface $f_{nl}$ at the point $y$.

1.3.4 Deflect vector $\nabla f_{nl}(y)$ in such a manner that the resulting hyperplane $v_{nl}$ would cut the constraint $f_{nl}$ with a needed accuracy of linearization $\mu$ in the vicinity of the point $y$.

1.3.5 If $v_{nl}' s_k' \geq 0$ put index $nl$ in $Q$ and go to step 1.3.1. Else compute the smallest index $l$ such that

$$l = \min_{i \notin M, i \notin I'} i$$

and set $\delta_j' = \delta_{nl}'$, and $a_j = v_{nl}$.

Step 1.4 - (Update solution and active set data)

1.4.0 Set $x^{i+1} = x' - \delta_j' s_j$

1.4.1 Obtain $I^{i+1}$ from $I'$ by setting $I_k^{i+1} = I$ and $I_j^{i+1} = I'$ for $i = 1, 2, ..., n$ with $i \neq k$

1.4.2 Obtain $[D^{i+1}]^{-1}$ from $[D^i]^{-1}$ by replacing row $k$ of $D^i$ with $a_j^s$ or compute $[D^{i+1}]^{-1}$ afresh using $I^{i+1}$ and the constraint matrix coefficients.

1.4.3 Replace $j$ with $j+1$ and go to step 1.1.

The computation of the feasible step size (1.3.0) and determination (1.3.3) and deflection (1.3.4) of the gradient $\nabla f_{nl}$ need individual adjustment for different types of non-linear constraints. As the algorithm is problem dependent, detailed description of each of the above mentioned steps is presented in the following section for a particular type of function $f$. 
5 DYNAMIC LINEARIZATION FOR HOEK-BROWN CRITERION

To ensure that the stresses do not violate the yield condition and hence satisfy the requirements of the lower bound theorem, it is necessary to insist \( f \leq 0 \) throughout each triangle. Therefore, the non-linear inequality constraints in this case have the following form

\[
f = (\sigma_x - \sigma_y)^2 + 4\sigma_y^2 + \frac{m\sigma_y}{2} \left[ (\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\sigma_y^2} \right] - s\sigma_x^2 \leq 0
\]  

(13)

This constraint needs to be satisfied at each node of every element. Assuming that the parameters \( m, \sigma \) and \( \sigma_c \) are constant throughout each element and reminding that the components of the stress tensor inside a triangle are in linear combinations of the nodal stresses, it is straightforward that the equation (13) is satisfied at any point inside a triangular element if it is satisfied by each set of the nodal stresses (Figure 6).

\[
f(a) = 0, \text{ where } a = [\sigma_x, \sigma_y, \tau_{xy}]^T
\]

\[
a^* = \frac{1}{N} \sum_{i=1}^{N} N_i a_i \\
\Rightarrow f(a^*) \leq 0
\]

\[
f(a) - \text{ convex}
\]

Figure 6. Geometrical illustration showing the sufficiency of the yield constraints application

5.1 COMPUTATION OF FEASIBLE STEP SIZE

In the dynamic linearization process, the feasible step size from an initial stress point towards the yield surface is computed using the search vector at each iteration. Therefore, the intersection points between the search vector and the yield surface can be expressed in terms of a single variable step size \( \delta \). For any point along the search vector, \( s \), we have

\[
\begin{align*}
\sigma_x &= \sigma_{x0} + s_x \delta; \\
\sigma_y &= \sigma_{y0} + s_y \delta; \\
\tau_{xy} &= \tau_{xy0} + s_{xy} \delta
\end{align*}
\]

(14)

Substitution of equation (14) into equation (13) produces a non-linear equation for \( \delta \). The feasible \( \delta \) is computed using well-known Brent’s (1973) method.

5.2 COMPUTATION OF GRADIENT

Equation (3) is a smooth function whose gradients are given by

\[
\begin{align*}
\frac{\partial f}{\partial \sigma_x} &= 2(\sigma_x - \sigma_y) + m\sigma_y \left( 1 + \frac{\sigma_x - \sigma_y}{R} \right) \\
\frac{\partial f}{\partial \sigma_y} &= -2(\sigma_x - \sigma_y) + m\sigma_y \left( 1 - \frac{\sigma_x - \sigma_y}{R} \right) \\
\frac{\partial f}{\partial \tau_{xy}} &= 8\tau_{xy} + \frac{m\sigma_y}{2} \left( \frac{4\tau_{xy}}{R} \right)
\end{align*}
\]

(15)

where
To ensure the lower bound analysis, these gradients are deflected to keep the search direction inside the yield surface at the next iteration. Then the new search direction becomes \( \mathbf{v} = \nabla f(\mathbf{y}) + \mathbf{w} \), where \( \mathbf{w} \) is the direction and magnitude of the deflected vector.

5.3 Computing the deflection of the gradient \( \nabla f \)

In the dynamic linearization scheme, only the inequalities with the smallest step in the search direction are locally linearized at each iteration. Consequently we must generate only the sides of the inscribed polyhedron which is formed at the intersection between the search vector and the yield surface. For two dimensional stability problems, yield function depends on three unknown stresses and at least three planes are needed to perform the linearization. In practice it is more convenient to employ four planes rather than three, as this permits partially symmetric linearizations to be constructed in a straightforward manner. Let us define the intersection point as the origin of an orthogonal coordinate system which spans the space of the tangential plane. The process of constructing the relevant part of the polyhedron can be described in terms of a deflection of the gradient of the yield function, where the deflection vectors lie in the directions of the axes of such a system. Figure 7 illustrates the local linearization of a spherical yield surface.

![Figure 7. Local linearization of spherical surface](image)

Here \( \mathbf{x}' \) is an intersection point, \( \nabla f \) is a gradient of \( f \) at this point, \( \mathbf{w}_i \) is a unit vector in the \( i \)th deflection direction, \( d_i \) is a deflection factor for the \( i \)th deflection vector and \( \mathbf{n}_i = \nabla f(\mathbf{x}') + d_i \mathbf{w}_i \) is a normal vector to the \( i \)th side of the polyhedron. Once the components of \( \mathbf{n}_i \) are known, the equation of the corresponding \( i \)th linear constraint is

\[
 n_{i_1} \sigma_{x} + n_{i_2} \sigma_{y} + n_{i_3} \tau_{xy} = b
\]

where \( b \) is a constant which can be determined by substituting the known values of \( \mathbf{x}' = (\sigma_x, \sigma_y, \tau_{xy})' \) at the intersection point. This type of procedure is quite general and can be used to linearize any convex surface in \( n \)-dimensional space. For a non-spherical yield function, the deflection coordinate system should be oriented to maximise the symmetry of the linearization.

The accuracy of the linearization for a general yield surface can be controlled by comparing the curvature of the yield surface in the deflection direction with the curvature of a circle which passes through \( \mathbf{x}' \) and whose centre is located on the hydrostatic axis. Let \( \mu \in (0, 1) \) define a measure of the accuracy of the deflection factor \( d_i \) and consider the linearization of a circle given by equation: \( f_c(\mathbf{x}') = \sqrt{x_1^2 + x_2^2} = R \) at the point \( \mathbf{x}' \) (Figure 8). Let \( e \) and
Figure 8. Deflection of gradient of circle with radius $R$

$$
\mathbf{H}_f = \begin{bmatrix}
\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\
\frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2}
\end{bmatrix}
$$
denote the unit vector tangential to $f$, and the Hessian of $f$, at $x'$ respectively. Then the directional derivative of $\nabla f$, at $x'$, in the direction of $e$, is given by

$$
\mathbf{q}_c = d_e \mathbf{e} = \mathbf{H}_f \mathbf{e}
$$

with

$$
\|\mathbf{q}_c\| = 1/R
$$

Once $\nabla f$, and $\mathbf{q}_c$, have been found, the vector normal to the linearized part of the circle may be computed as $\mathbf{n}_c = \nabla f + d_e \mathbf{e}$. The accuracy of such a unit linearization can be assessed by solving the following quadratic equation for $\mu$.

$$
\|\mathbf{q}_c\| = \frac{1}{R} = \tan \alpha = \frac{\sqrt{\mu(2-\mu)}}{1-\mu}
$$

To control the accuracy of the dynamic linearization for an arbitrary convex yield function, we assume that

$$
\frac{d_e}{d_e} = \frac{\|\mathbf{q}_c\|}{\|\mathbf{H}_f \mathbf{w}_i\|} = \frac{\|\mathbf{H}_f \mathbf{w}_i\|}{\|\mathbf{H}_f \mathbf{e}\|}
$$

where $\mathbf{H}_f$ is the Hessian of $f$ at the intersection point $x'$. This implies that the ratios of the deflection factors for the arbitrary and circular yield surfaces are set equal to the ratios of their corresponding directional derivative norms. Substituting (16) in (17) and noting that

$$
d_e = \frac{\sqrt{2\mu - \mu^2}}{1-\mu}
$$

furnishes the result that

$$
d_e = \|\mathbf{H}_f \mathbf{w}_i\| \frac{\sqrt{2\mu - \mu^2}}{1-\mu}
$$

where

$$
R = \sqrt{(\sigma_x - s)^2 + (\sigma_y - s)^2 + \tau_{xy}^2}
$$

and

$$
s = \frac{1}{2}(\sigma_x + \sigma_y)$$
Given a specified tolerance $\mu$, equation (18) provides a means for controlling the accuracy of the dynamic linearization. Typical values for $\mu$ are around 0.01. Figure 9 shows a typical example of dynamic linearization of the Hoek-Brown yield function.

6 NUMERICAL RESULTS

Results of a plane strain footing problem is briefly presented to demonstrate the performance of the present algorithm. The finite element mesh used for all of the analysis is shown in Figure 10. It contains 813 nodes, 271 triangular elements and 389 stress discontinuities. Due to symmetry only half of the problem is considered, where $B$ is the footing width. The stress boundary conditions are also shown in the Figure 10.

To assess the performance of the proposed formulation the lower bound collapse pressure of a rigid strip footing analysis is considered on heavily jointed Panguna andesite from Bougainville, Papua New Guinea. Comprehensive test data are available for this material and Figure 11 shows the triaxial test results with further extension of the yield criterion of an undisturbed sample (Hoek and Brown, 1980). The dynamically linearized model of the Hoek-Brown criterion produces a lower bound limit pressure 48.7 MPa using the empirical constants $m$, $s$ and the uniaxial compressive strength $c$ of this sample as shown in Table 1. The non-linear Hoek-Brown failure envelope is then piecewise linearized to eight linear failure envelopes of $c$-parameters within the loading range. A rigorous analysis has been carried out at this stage to confirm that the approximated yield envelopes are not violating the original Hoek-Brown failure envelope within the loading range. These piecewise linear envelopes individually represent a Mohr failure envelope for the Mohr-Coulomb criterion and they are used together in the lower bound linear programming formulation of Sloan (1988) to compute another collapse pressure. These approximated envelopes of $c$-$\phi$ parameters produce a lower bound limit pressure 47.0 MPa (Table 1), which is approximately 3% lower than that of the result produced by the dynamically linearized Hoek-Brown criterion. This expected lower result is due to the fact that the manually linearized failure envelopes always lie inside the Hoek-Brown criterion with a limited number of piecewise linearization. Therefore, the authors wish to comment that the dynamically linearized Hoek-Brown criterion produces a more accurate lower bound limit pressure than that of the approximated Mohr-Coulomb criteria. The advantage of the proposed formulation is that it includes the empirical constants $m$ and $s$ as a direct input and it doesn’t need any manual linearization.

Table 1. Comparison of lower bound limit pressure between Mohr-Coulomb and Hoek-Brown criterion
Figure 10. Mesh used for lower bound limit analysis

<table>
<thead>
<tr>
<th>Yield function</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>q (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mohr-Coulomb</td>
<td>c (MPa)</td>
<td>0.51</td>
<td>0.72</td>
<td>0.9</td>
<td>1.4</td>
<td>2.8</td>
<td>5.0</td>
<td>7.0</td>
<td>9.5</td>
</tr>
<tr>
<td></td>
<td>( \phi^o )</td>
<td>65.0</td>
<td>57.0</td>
<td>48.0</td>
<td>37.3</td>
<td>27.9</td>
<td>22.0</td>
<td>18.5</td>
<td>15.4</td>
</tr>
<tr>
<td>Hoek-Brown</td>
<td>( m=0.278 ) ; ( s=0.0002 ) ; ( \sigma_c=265.0 \text{ MPa} )</td>
<td>48.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The performance assessment of this new procedure is extended for other types of jointed rocks as well. Jointed carbonate rocks with well developed crystal cleavage is considered in this case. Figure 12 shows the approximate normalised shear strength-normal stress relationship for different degrees of joints of carbonate rock. For the sake of conciseness, only a single type of jointed rock namely “very good quality rock mass” is considered in this test. The yield criterion of this rock mass with the empirical parameters \( m \) and \( s \) as 3.5 and 0.1 is shown in Figure 13. The dynamically linearized model produces a lower bound collapse pressure 2.86 \( \sigma_c \) for this type of rock. Similar to the previous calculation, the Hoek-Brown criterion of this rock is piecewise linearized to eight set of Mohr failure envelope. These piecewise linear approximations produce a lower bound collapse pressure 2.69 \( \sigma_c \), which is approximately 6% lower than the factor computed by the dynamically linearized Hoek-Brown model (Table 2). Again we find that the lower bound collapse pressures computed by these two methods are in a very good agreement and the Mohr-Coulomb criteria with multi-linear yield surface produce relatively a lower value than that of the dynamically linearized Hoek-Brown criterion. These findings confirm that the proposed formulation of the Hoek-Brown criterion can be used to obtain the lower bound limit pressure for any other type of jointed rock, where the empirical parameters \( m \) and \( s \) and the uniaxial compressive strength \( \sigma_c \) are known. Table 3 shows the lower bound collapse pressures for other types of jointed carbonate rock computed using the dynamically linearized Hoek-Brown criterion.

Table 2. Comparison of lower bound limit pressure between Mohr-Coulomb and Hoek-Brown criterion
Figure 11. Hoek-Brown envelope for jointed Panguna andesite from Bougainville, PNG. (after Hoek-Brown, 1980) with c-f approximation.

Table 3. Lower bound collapse pressure for carbonate rocks

<table>
<thead>
<tr>
<th>Yield function</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>$(q/\sigma_c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mohr-Coulomb</td>
<td>c (MPa)</td>
<td>0.057</td>
<td>0.072</td>
<td>0.10</td>
<td>0.14</td>
<td>0.21</td>
<td>0.28</td>
<td>0.40</td>
<td>0.60</td>
</tr>
<tr>
<td></td>
<td>$\phi^\circ$</td>
<td>49.0</td>
<td>40.0</td>
<td>34.0</td>
<td>29.0</td>
<td>24.0</td>
<td>21.0</td>
<td>17.5</td>
<td>15.0</td>
</tr>
<tr>
<td>Hoek-Brown</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Estimated lower bound collapse pressure by dynamic linearization of Hoek-Brown yield criterion. Carbonate rocks with well developed crystal cleavage.

The table shows that for carbonate rocks with well developed crystal cleavage, the empirical parameters $m$ and $s$ for jointed rocks can be estimated as follows:

- **Very good quality rock mass**: $m=3.5$; $s=0.1$, giving a lower bound collapse pressure of 2.86.
- **Good quality rock mass**: $m=0.7$; $s=0.004$, giving a lower bound collapse pressure of 0.57.
Figure 12. Mohr failure envelope of carbonate rocks for different joint quality

<table>
<thead>
<tr>
<th></th>
<th>(m)</th>
<th>(s)</th>
<th>(q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fair</td>
<td>0.14</td>
<td>0.0001</td>
<td>0.10</td>
</tr>
<tr>
<td>Poor</td>
<td>0.04</td>
<td>0.00001</td>
<td>0.031</td>
</tr>
</tbody>
</table>

where \(q\) is lower bound collapse pressure and \(\sigma_c\) is uniaxial compressive strength of intact rock.

7 CONCLUSIONS AND RECOMMENDATIONS

A new method has been presented to compute the lower bound limit pressure of jointed rocks using the Hoek-Brown yield criterion. Inclusion of the dynamic linearization in the optimisation procedure makes the parameter selection simple for the user and the manual linearization becomes unnecessary. The method shows a good potential to apply on different types of jointed rocks. The technique can be upgraded to 3-D stress state, especially where the intermediate principal stress is of great concern.

REFERENCES


Figure 13. Mohr failure envelope of carbonate rock with $c-f$ approximation


