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Research Solutions:

1. Abstract harmonic analysis:
   It is shown in that finite-codimensional ideals in group convolution algebras L1(G) factor in the sense that each element is the sum of two products. Partial results in this direction had been found earlier in [,,]. Positive structure theorems valid for all group algebras, such as this one, are rare.Banach spaces:

   It is shown in that there are Banach spaces that have the compact approximation property but do not have the approximation property. That had been an open question since the early 1970's, when it was shown by P. Enflo that not all spaces have the approximation property.Topological groups:

   A conjecture made in 1981 by K.H. Hofmann and A. Mukherjea concerning the structure of totally disconnected groups is proved in . The proof of the conjecture used ideas developed for that purpose in .Banach algebras:

   The question of whether Cohen's factorisation theorem has a converse is answered negatively in . The examples found suggest an intriguing, and largely unexplored, relationship between topological invariants for the carrier space and factorisation.

   The solutions to these and other problems have frequently involved the transfer of ideas between fields and my research interests have been continually widening in consequence. For example, a problem on automatic continuity of derivations from group algebras led to the question of factorisation in finite-codimensional ideals of group algebras eventually solved in . The solution to that problem involved the use of random walks on groups and led to the functional analytic definition of the boundary of a random walk given in . That paper also included a new proof of a conjecture of H. Furstenberg about amenable groups. Another question about the so-called `concentration functions' of a random walk on a group had been reduced to a question about the group structure by K.H. Hofmann and A. Mukherjea. They proved the conjecture for connected groups by using an `approximation by Lie groups' argument. I completed the proof of the conjecture in the general case by developing analogous techniques for totally disconnected groups (see the further discussion below). Thus from a Banach algebras question I was led to work on abstract harmonic analysis, then random walks on groups and then to group theory. It now seems likely that the circle will close and that the group theory techniques will be essential for proving further results about derivations on group algebras.

   Here is a description of the two principal research programs I am pursuing currently.

2. Locally Compact Groups:
   The study of general locally compact groups separates into the investigation of connected groups and of totally disconnected groups because the connected component of the identity in a locally compact group is a closed normal subgroup and the quotient by this subgroup is totally disconnected.

   Connected locally compact groups are well understood. With the solution of Hilbert's fifth problem it was seen that such groups can be approximated by Lie groups in the sense that they have arbitrarily small compact normal subgroups which, when factored out, leave a Lie group quotient. Lie groups are understood through their Lie algebra. That is an extremely powerful technique because it brings concepts from linear algebra such as eigenvalues and the Jordan canonical form to the study of groups.

Foundations for a structure theory of totally disconnected, or 0-dimensional, locally compact groups have been laid down in [,,,]. These papers develop a canonical form for automorphisms of 0-dimensional groups which is analogous to the Jordan canonical form of a linear transformation. Indeed, when the group is the additive group of a p-adic vector space, each automorphism is a linear transformation and the two canonical forms are essentially equivalent. The canonical form applies simultaneously to commuting automorphisms and includes analogues of eigenvectors and eigenvalues. Although as yet there is no analogue of the Lie algebra, the canonical form has been used to answer structural questions about totally disconnected groups by applying it to inner automorphisms, [,,,].

Many further applications and developments are the subject of current or proposed research.

* The canonical form is refined in the case when the group is abelian in and this refinement is being extended to nilpotent groups.
* A structure theory for general totally disconnected groups which replicates some features of Lie groups has been started in . The eventual aim will be to classify the compactly generated simple totally disconnected locally compact groups.
* Known results for connected groups which use approximation by Lie groups arguments may be extended to general locally compact groups. (A problem of this type was my reason for beginning work on totally disconnected groups.)
* Algebras associated with locally compact groups, including the measure algebra and the von Neumann algebra, may be investigated. In the case of totally disconnected groups there are also Hecke algebras associated and these are currently being independently investigated by C*-algebraists in connection with number theory and dynamical systems.
* Automorphism groups of locally finite graphs are totally disconnected and results about such groups may be applied to the study of graphs. On the other hand, totally disconnected groups might be studied by representing them on locally finite geometries.
* Each finitely generated discrete group may be embedded as a cocompact subgroup of the automorphism group of its Cayley graph. It is possible therefore that structural invariants for totally disconnected groups will provide a tool for studying finitely generated groups.

3. Banach Algebras

Banach algebras are the natural context for the spectral theory of linear operators and the Fourier and Laplace transforms. The Gel'fand theory of commutative algebras provides a unified description of these and other important ideas. However, the Gel'fand theory says nothing about radical algebras and these algebras are little understood. There are many examples of radical algebras, some arising in the theory of integral equations, but there are no general techniques for analysing them. I am undertaking research which aims to develop such general techniques. Observations first made in [,] led to the conjecture that there exist radical Banach algebras with the extreme and unlikely seeming property that they are singly-generated and that the sequence of normalised powers of the generator contains approximate units. Algebras having this property were constructed in . Current research is analysing the ideals and multipliers of these algebras. Future work will study the homomorphisms and modules of the algebras with a view to determining how typical their seemingly strange properties are of radical algebras, and to using them as models for quasinilpotent linear operators exhibiting novel features.

Another direction taken by my research on Banach algebras is to investigate the algebraic properties of algebras of operators on Banach spaces and relate them to the structure of the underlying Banach space. The papers [,,,,,] are in this direction. New tools for distinguishing between and classifying Banach spaces are needed following the construction of key examples by James, Tsirelson, Gowers, Maurey and others and algebraic properties of the operators supported by the Banach spaces are a promising source of such tools. I am particularly interested in relating cohomological properties of the operator algebras to the Banach space structure because these properties have proved to be important in many other contexts. Amenability is a cohomological property which has been seen to characterise an important class of groups in abstract harmonic analysis and an important class of C*-algebras in that theory. Work in investigated whether amenability of the algebra of compact operators on a space can be related to the structure of the space. It seems that it may be equivalent to a type of approximation property and current work is studying particular difficult examples. Amenability corresponds to the vanishing of certain cohomology groups and an even more interesting prospect is that in the non-amenable case cohomology groups may provide invariants for analysing Banach spaces.