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An automatic time increment selection scheme for simulation of elasto-viscoplastic consolidation of clayey soils

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An automatic time increment selection scheme for numerical analysis of long-term response of geomaterials is presented. The scheme is simple, rational and stable. Governed by a simple empirical expression, it can adaptively adjust the time increments depending on the strain rate-dependent temporal history of the material response. The proposed expression requires only a few parameters whose selection is a trivial task since they have a small effect on accuracy but have a significant effect on computational efficiency. This generalization has been made possible because of the enforcement of certain predefined control criteria to avoid extreme conditions. If any of the control criteria is satisfied, the computation is restarted by going a few time steps back to ensure the smoothness of the computed responses and time increments are again continuously adjusted through the governing equation provided. Performance of the automatic time increment selection scheme is investigated through finite element analyses of the long-term consolidation response of clay under different geotechnical profiles and loading conditions. Both elastic and elasto-viscoplastic constitutive relations are considered, including the consideration of the destructuration effects of geomaterials. Numerical results show that the performance of the automatic time increment selection scheme is reasonably excellent. While offering reasonable accuracy of the numerical solution, it can ensure temporal stability at optimal computational efficiency. In addition to the Euler implicit method, the automatic time increment selection scheme also performs well even when the explicit fourth-order Runge–Kutta method is employed for the integration of time derivatives.

Keywords: time increments; finite element methods; consolidation; elasto-viscoplasticity

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1. Introduction

During consolidation of geomaterials, pore water flow, deformation, volume change, strain softening or hardening, structural degradation, etc. can occur simultaneously, and consequently leads to time-dependent relations. Therefore, besides spatial discretization, the numerical analysis of the long-term behavior of geomaterials requires temporal discretization which can be solved by any of the numerical methods, such as the finite element method (FEM), even if the material is elastic or rate independent. For rational representation of the behavior of geomaterials, for instance, the elasto-viscoplastic constitutive model (Kimoto *et al*. 2004), with consideration of the structural degradation effect, can be considered. These advanced formulations can envisage localization phenomena as a result of strong concentration of viscoplastic strain in a distinct narrow zone. Again, inclusion of the soil-structure degradation effect may cause an increase to pore water pressure values even after removing the applied load. This leads to unstable soil behavior (Kabbaj *et al*. 1988; Oka *et al*. 1991) where a frequent change in the strain rate is expected. In all cases, the accuracy and stability of the numerical solution are dependent on the choice of time increments as well as on the mesh configuration if the finite element method is used as a numerical tool. However, it seems that only a little attention is paid to finding the proper method for selecting time increments, in contrast to the mesh adaptivity procedures. This is especially true for geomechanical problems involving the elasto-viscoplastic constitutive model.

Usually, the quality of the numerical simulation becomes better if the time domain is discretized through smoother time increment sequences with improved computational stability. To handle a great variety of situations in many complex problems, this can only be achieved through an adaptive or automatic selection approach. Transient computation of viscous flow through porous media provides an application where the adaptivity in time is crucial. The usual temporal error estimates may not be relevant and can be pessimistic when degradation of the structure of the porous medium is dominant. Since the degradation phenomenon is also a transient process, this could result in shorter time increments, thus lessening the computational efficiency. Therefore, the time increment scheme must reflect the numerical stability issues as a prime priority while maintaining reasonable accuracy with optimal computational efficiency. Also, the selection procedure must be able to adapt properly to a wide range of operating conditions including soil conditions.

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Time increments can be selected manually if they are relatively small and kept constant. However, a constant smaller time increment over the entire computation time would lead to high computational effort. One can also choose different time increments for different time spans by dividing the entire time domain into various subdomains. However, this selection criterion is strongly dependent on the user's own experience and prior knowledge about the problem. This is usually difficult to attain, especially in predicting long-term soil response where the time frame of seasonal change and structural degradation of the soil skeleton is not known beforehand. In fact, time increments should at least reflect the soil and model parameters, loading profile, strain rate sensitivity, strain softening or destructuration of the clay structure. As a consequence, proper selection of time increments is important to generate physically correct solutions and at the same time to ensure numerical stability and optimal computational efficiency.

Few adaptive time increment methods are available. Most of them are based on a posteriori temporal error estimation (Diebels *et al.* 1999; Ehlers *et al.* 2001). Several alternative approaches are also available such as the method of lines (Matthew *et al.* 2003), and Proportional-Integral (PI) controlled time-stepping (Soderlind 2002; Valli *et al.* 1999; Valli *et al.* 2002). Except for the a posteriori temporal error based methods, most of the other methods have not been applied to multiphase porous media. Recently, Sloan and Abbo (Sloan and Abbo 1999a; Sloan and Abbo 1999b) described an automatic time stepping approach for consolidation problems that subdivides the user specified coarse time increments, by controlling the local error tolerance in the nodal velocity obtained from the difference between the first-order and second-order accurate solutions. A few variants of this approach are also available (Sheng and Sloan 2003). A few empirical guidelines are also available for restricting the maximum time step size depending on the spatial discretization scheme, to minimize the possibility of numerical instability as well as temporal oscillations (Wang *et al.* 2001; LeVeque 1992).

In the numerical formulation of any time-dependent initial boundary value problem, the time domain can be discretized either by an implicit or explicit scheme or even a combination of both time-marching methods. In addition to the accuracy requirements, time increments are normally restricted by the stability condition of the adopted method. An explicit time integration method is usually more accurate but is stable only when the time increment is chosen sufficiently small, i.e., such methods are conditionally stable (Flanagan and Belytschko 1984). Implicit methods are unconditionally stable. The implicit time integration scheme is well suited to applications where moderate accuracy is needed, and it is the method of preference for most practical analyses. The explicit scheme is more suitable for problems where higher accuracy is required, and can be used to benchmark the accuracy of various integration schemes. Among various explicit schemes, the multistage Runge–Kutta method, especially the classical fourth-order Runge–Kutta scheme, is popular. Various implicit integration schemes are available, such as forward Euler, backward Euler, Crank–Nicolson, including an implicit version of the Runge–Kutta scheme (Brenan *et al.* 1989).

Because time increments play a crucial role for numerical implementation of any time-dependent boundary value problem, proper selection of time increments is essentially critical for numerical stability and accuracy as well as for computational efficiency. Therefore, a rational procedure for the selection of time increments is necessary. The procedure should possibly be adaptive in nature and capable of incorporating a wide range of material conditions and constitutive relations. It should be unconditionally stable even in simulating an unstable material response. It should also be equally applicable to implicit or explicit time integration schemes, and should not be dependent on the rigid temporal error tolerance criterion. Also, it should be capable of handling arbitrary finite element mesh configurations. In this paper, a robust automatic time increment selection scheme is presented that meets these requirements.

The paper is organized as follows. First, a detailed description of the automatic time increment selection scheme is presented. Later, its performance is investigated by analyzing long-term consolidation behavior of soils under different geotechnical profiles and loading conditions. Both elastic and elasto-viscoplastic constitutive models are considered under one- and two-dimensional plane strain conditions. The finite element method (FEM) is used as a numerical tool and the effectiveness of the proposed scheme is assessed by applying it to both the Euler implicit and Runge–Kutta explicit time integration methods.

2. The automatic time increment selection scheme

The proposed automatic time increment selection scheme is characterized by the change in time-dependent field variables for the given problem. It is capable of automatically adapting itself to the new conditions depending on the temporal material response. The scheme is dependent on the physical material conditions as well as on the limiting conditions of the constitutive model by which the material is described for numerical implementation. In the present time increment selection scheme, the time increment is assumed to be changing nonlinearly with time from an initial small input value. The time increment for the new time step is determined from the history of the strain rate up to the previous time step. Some control criteria are designed, which are defined by the physical conditions bounded by the constitutive model of the material in consideration. As long as the computed responses are below those bounded or constitutive model-specific failure criteria, time increments continue to grow. However, if any of the control criteria are satisfied, the computation is restarted by going a few time steps back to ensure smoothness of the computed responses, as well as a smooth transition in the change in time increments from large to small values. This procedure repeats itself and ensures temporal stability through continual adjustment of time increments. Even if numerical

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instability is observed during this process, it is assumed that this instability does not originate from the selected time increment profile. Instead, it is considered that the material has just reached its failure state as set by the working constitutive model, thus the computation has to be stopped unless the governing constitutive model or the mesh configuration is redefined. Note that this failure may not represent the true failure of the material because of the other concerns for numerical instability, including that arising from an improper mesh configuration. At any time step, the automatic time increment selection scheme assumes that the adopted FEM mesh configuration holds good for accuracy and spatial stability.

It is assumed that the strain rate is a good indicator of the current state of geomaterials including their structural degradation. Therefore, it is considered that the time increment profile should reflect the history of the strain rate. Based on this idea, an expression is fitted empirically evolving through the accumulated strain rate invariant by analyzing hundreds of cases of long-term consolidation problems in geomechanics under various soil and loading conditions, elastic and elasto-viscoplastic constitutive models, as well as various manual time increment selection schemes. In the proposed automatic time increment selection procedure, the time increment Δt is assumed to change nonlinearly and in a stepwise manner at H user-defined time step intervals (see Figure 1) from an initial user-defined small value Δt_c through the expression

$$\Delta t^{n+1} = (1-\Phi)\sqrt{\Upsilon\varsigma}\left(\mathrm{T}_b^{n+1}\right)^{\chi\varsigma}\Lambda^{n+1} + \Phi\Delta t_{cf} \qquad (1)$$

The governing expression 1 has two parts. The first part determines the time increment when no predefined control criteria are observed (i.e. when $\Phi = 0$). This is to avoid possible extreme conditions of the material. The second part yields the time increments during the occurrence of the control criteria (i.e. when $\Phi = 1$). The first part of Equation (1) is responsible for the increase of the time increment from the previous time step to the next step, which is reduced to a much lower value of Δt_{cf} if any of the predefined control criteria are observed presenting the occurrence of the $\Phi = 1$ condition. Note that the unit of time in the overall time increment selection procedure must be in seconds. The various parameters in Equation (1) are defined briefly as follows:

- Δt^{n+1} = computed time increment (in seconds) at the current time step $(n+1)$, where n is the previous time step;
- $\Delta t_{cf} = \Delta t_c$ or Δt_f, depending on the occurrence of the predefined control criteria;
- Δt_c = user-defined time increment (in seconds) at the first time step;
- Δt_f = user-defined reasonably smallest value of the time increment (in seconds) that is used only during the unstable temporal region or when failure is approaching;
- $\Phi = 1$ if any of the predefined control criteria are satisfied, otherwise always equals 0 from the first time step;
- Υ = parameter that influences the rate of increase of time increments only before approaching the unstable temporal region;
- T_b = main automatic time increment parameter by which the time increment is redefined depending on the past history of the strain rate invariant;
- χ = a user-defined dimensionless constant that controls the rate of increase of time increments, thereby directly affecting the computational efficiency and smoothness of the solution;
- $\varsigma = 2$, another dimensionless constant, which can be reduced to 1 for one-dimensional problems containing fewer elements in the adopted FEM discretization scheme;
- Λ = optional time increment parameter to promote the rate of increase of the time increment, mainly pronounced when the current time is far from the unstable temporal region.

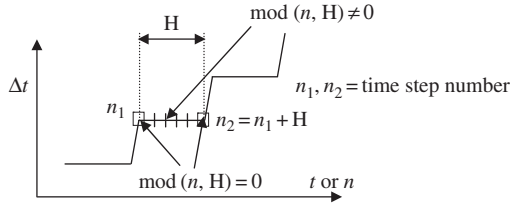

Figure 1. Parameter H: interval when Δt is redefined.

Through the enforcement of some predefined control criteria, mainly to identify the unstable temporal region, the automatic time increment selection approach can automatically take care of any possible numerical issue arising from the selected Δt. This eventually helps us to carry out the computation process successfully till the end of the physical computation time or the failure of the geomaterials, whichever is earlier. In fact, the automatic time increment parameters in Equation (1) can be adjusted manually to meet the user level requirements for accuracy and computational efficiency. As mentioned earlier, H denotes the time step interval when Δt is redefined, i.e. the time increment is changing in a stepwise way with respect to the physical time steps as shown in Figure 1. Therefore, the time increment Δt is redefined using Equation (1) if mod $(n, \mathrm{H}) = 0$, i.e. if the time step n becomes a multiple of H. The value of H can be chosen as $\mathrm{H} \geq 1$.

2.1 *Control parameter* Φ

Φ is a control parameter, which limits the time increment Δt^{n+1} when any of the predefined 'control' criteria are satisfied. From the beginning of the computation, it is assumed that only higher value of time increment Δt can cause numerical instability in the overall computational procedure. In the automatic time increment selection procedure governed by Equation (1), the time increment almost always tends to increase with time. The purpose of enforcing these controls is to ensure that the instability of the computation does not occur by the wrong selection of the time increment. Wrong selection of the time increment not only affects the accuracy, but can also lead to numerical instability at several stages of the computation. The error

caused by the wrong selection of Δt should be identified in proper time before it gets too late to avoid numerical instability. The purpose of enforcement of the $\Phi = 1$ control is to ensure that the error caused by Δt cannot be propagated further to cause the execution of the numerical code to bring it prematurely to a halt before the true material failure. The value of Φ is usually 0. But if any of the predefined control criteria are observed at any time step, the value of Φ is assigned to unity for that time step. As soon as the $\Phi = 1$ criterion is observed, Δt^{n+1} is reduced to the smaller value Δt_{cf} by restarting the computation H_2 time steps back. The value of H_2 is arbitrary provided that the computed material responses are reasonably accurate up to $(N - H_2)^{th}$ steps. N is the total number of time steps up to the current time step $(n + 1)$ whose time increments are accepted and can be expressed as $N = (n + 1)(n + 2)/2$. The value of H_2 can be simply taken as $H_2 = i\mathrm{H}$, where i is a scalar multiplier and $i \geq 2$. Note that if the value of H becomes as low as 1, a higher value of i is recommended for better accuracy and smoothness as well as for the reduction of the total number of $\Phi = 1$ occurrences.

Another control condition $\Xi = 1$ is also introduced that defines the unstable temporal region. The $\Xi = 1$ criterion exists only when the main control $\Phi = 1$ is repeated at frequent intervals, namely $H_3 = j\mathrm{H}$ time step intervals. Here j is a scalar and can be simply taken as $j \geq 3$. H_3 must be greater than H_2 ($H_3 > H_2$) to ensure that there are enough time steps behind if the computation requires restarting by going H_2 time steps back due to the occurrence of any control criteria. Both the $\Phi = 1$ and $\Xi = 1$ conditions bear the same meaning, implying that at least one of the predefined control criteria are satisfied due to the wrong selection of the time increment Δt. In addition, the $\Xi = 1$ condition indicates that the number of time steps between two consecutive $\Phi = 1$ impediments does not exceed H_3 intervals, i.e. the control criteria are observed frequently. The inclusion of the $\Xi = 1$ criterion is to minimize the possibility of frequently going back and forth in the computation process, which may arise from the larger time increments because of the nature of the parameter $\Upsilon = 1$ condition is observed, the rate of time increment is forced to be very low, starting from a much smaller value Δt_f. The value of Δt_f should be as small as possible, since the material is assumed to have failed if any of the control criteria are observed (i.e. $\Phi = 1$) during the time increment of $\Delta t = \Delta t_f$. If this situation is observed at any time step, the computation process is no longer carried out. This is because the resulting state will then be beyond the range of applicability of the constitutive model under consideration, unless the current state is redefined by another governing model. Because of this assumption, Δt_f should be considered as the lowest value of time increments and can be simply chosen as $\Delta t_f = 0.1$–10 seconds. It is worth noting that this failure may not represent the actual failure of the material, because the spatial discretization error is also involved in the numerical model among other possibilities and it is not considered here. Moreover, since the computation has to be restarted by going H_2 time steps back when any of the control criteria are observed, it is necessary to store all computational matrices at all steps or at least at an interval of H_2 time steps depending on the processor's memory; and for convenience, H_2 and H_3 can be chosen simply as a multiple of H.

When any of the control criteria are observed (i.e. either $\Phi = 1$ or $\Xi = 1$ or both), the time step parameters are reinitialized as

$$\Phi = 1, \Xi = 0, n < n_{\Xi} \rightarrow \begin{cases} \Delta t_{cf} = \Delta t_c \\ \Delta t^{n+1} = \Delta t_c \\ \mathrm{T}_b^{n+1} = \Delta t_c \\ \Lambda^{n+1} = 1 \\ \Upsilon = \mathrm{B} \\ n_{\Phi} = 0 \end{cases} \tag{2a}$$

$$\Phi = 1, \Xi = 0, n \geq n_{\Xi} \rightarrow \begin{cases} \Delta t_{cf} = \Delta t_f \\ \Delta t^{n+1} = \Delta t_f \\ \mathrm{T}_b^{n+1} = \Delta t_f \\ \Lambda^{n+1} = 1 \\ \Upsilon = 1 \\ n_{\Phi} = 0 \end{cases}$$

$$\Phi = 1, \Xi = 1 \rightarrow \begin{cases} \Delta t_{cf} = \Delta t_f \\ \Delta t^{n+1} = \Delta t_f \\ \mathrm{T}_b^{n+1} = \Delta t_f \\ \Lambda^{n+1} = 1 \\ \Upsilon = 1 \\ n_{\Phi} = 0 \end{cases} \tag{2b}$$

In Equations (2), n_{Φ} is the total number of time steps from the beginning or from that time step when the computation is restarted after going H_2 time steps back, and n_{Ξ} is the total number of time steps from the first time step till the first occurrence of $\Xi = 1$ criterion. The first part of Equation (2a) denotes that the $\Phi = 1$ criterion is observed but the $\Xi = 1$ criterion has never been satisfied in past time steps. The second part of Equation (2a) denotes that the $\Phi = 1$ criterion is observed but the $\Xi = 1$ criterion has been satisfied at least once in the past history. Equation (2b) determines that the $\Xi = 1$ criterion is satisfied, i.e., the occurrence of the $\Phi = 1$ condition has been repeated within H_3 time steps as discussed previously. Using the above redefined time step parameters, the time increment Δt^{n+1} is recalculated using Equation (1) and the whole process continues. Note that once the computation is restarted from H_2 time steps back, the conditions $\Phi = 1$ and $\Xi = 1$ have to be reset to $\Phi = 0$ and $\Xi = 0$, respectively, until another control criterion is observed. Overall the automatic time increment selection procedure is schematically presented in Figure 2. For ease of understanding of its implementation algorithm, a virtual example for the automatic time increment selection scheme is shown in Figure 3. Note that if $n \geq n_{\Xi}$ and $n_{\Phi} \leq H_3$, the automatic time increment selection scheme yields a constant time increment $\Delta t = \Delta t_f$ during which the occurrence of any control criteria leads to assumed material failure (Figures 2 and 3).

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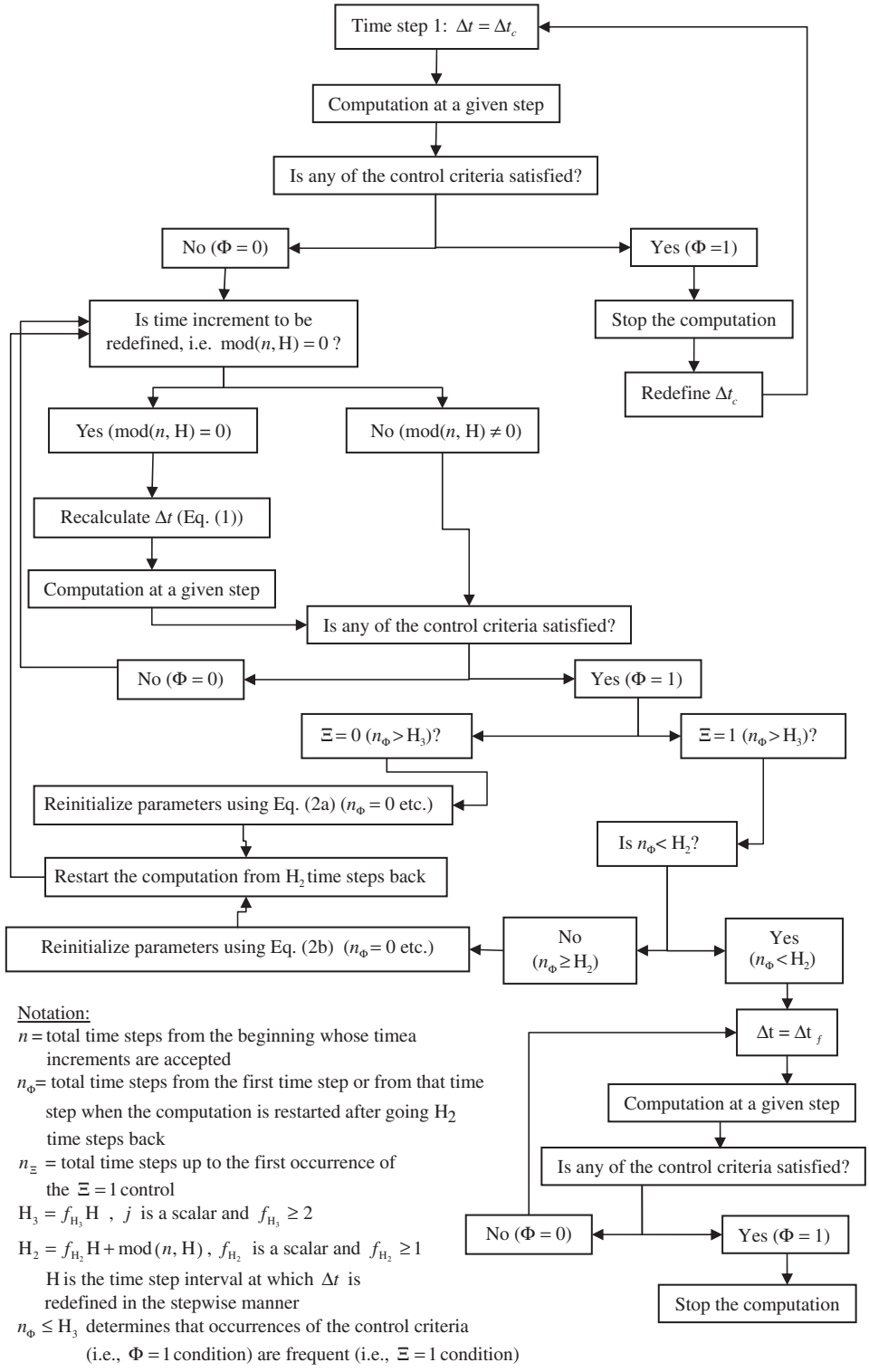

Figure 2. Schematic presentation of the automatic time increment selection scheme.

Control criteria are identified based on the limiting condition of the constitutive model of the material. Based on the series of numerical analyses for long-term consolidation response of normal to soft sensitive clay, any or more than one of the following control criteria can be considered:

(a) control for viscoplastic strain $\Phi_{\dot{\varepsilon}}$;
(b) control for mean effective stress Φ_σ;
(c) control for permeability Φ_k;
(d) control for pore water pressure Φ_{U_w};
(e) control for stress ratio Φ_{M_s};
(f) control for mesh overlap Φ_{DS}.

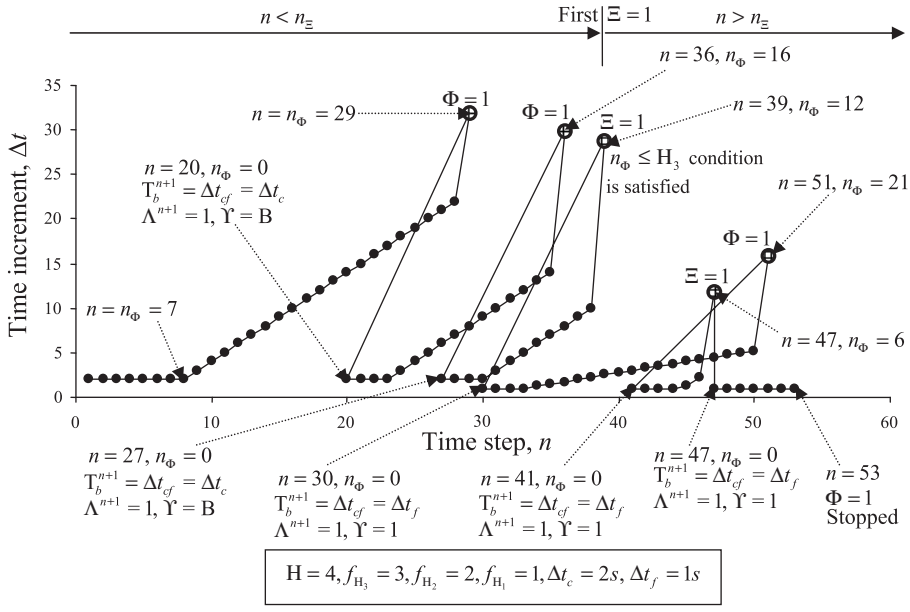

Figure 3. A virtual example for the automatic time increment selection scheme.

The purpose of enforcing these controls is to identify the period where the material behavior is unstable or when the material is near to or at the failure state. In other words, controls are designed to avoid the extreme case, for instance to avoid an extremely high strain rate. For simpler problems, such as for elastic analysis, enforcement of the control criteria can even become redundant. The above series of control criteria can be considered for highly non-linear rate sensitive materials involving an advanced constitutive model, for instance, the elasto-viscoplastic model with consideration of degradation of the soil structure (Kimoto *et al*. 2004). In fact, one control criterion could be adequate to control the rate of increase of time increments governed by Equation (1). For complex problems, more than one control criterion can be useful to identify the unstable temporal region in proper time to maintain the accuracy of the solution. Among these control criteria, control for the mean effective (or total) stress can be considered the most effective criterion since, in most cases, the minimum value of the mean effective stress is bounded by the constitutive law of the material. However, for consolidation analysis of water-filled saturated soil, it can be replaced by the control for pore water pressure. Note that no additional effort is necessary for implementing these controls, since no extra computation is involved.

2.1.1 *Control for viscoplastic strain* Φ_ε

The purpose of enforcing this control is to limit the increase in viscoplastic strain, which is usually the first victim of the wrong selection of time increment if an elasto-viscoplastic analysis is carried out. For example, a higher value of the time increment Δt^{n+1} can lead the viscoplastic volumetric strain $\varepsilon^{vp^{n+1}}$ from positive values (contraction) to negative values (dilation), and subsequently a few time steps later, this may end up close to infinite values if Δt^{n+1} is not restricted at proper time. Therefore, the control for viscoplastic strain Φ_ε can be expressed by either Equation (3a) or Equation (3b) as

$$\Phi = \Phi_\varepsilon = 1 \quad \text{if} \quad \left\{\varepsilon^{vp^{n+1}}\right\}^{\max} \le \varepsilon_c^{vp} \quad \text{when} \quad \varepsilon_c^{vp} \le 0 \tag{3a}$$

$$\Phi = \Phi_\varepsilon = 1 \quad \text{if} \quad \left\{\varepsilon^{vp^{n+1}}\right\}^{\max} \ge \varepsilon_c^{vp} \quad \text{when} \quad \varepsilon_c^{vp} > 0 \tag{3b}$$

where ε_c^{vp} is the tolerance limit of the viscoplastic volumetric strain.

2.1.2 *Control for mean effective stress* Φ_σ

This control immediately sets $\Phi = 1$ when the mean effective stress ${\sigma'_m}^{n+1}$ at the current time step $(n + 1)$ becomes greater than a specific given value σ_c. For soil, for example, this control can be used to enforce the requirement that the mean effective stress cannot be tensile. If compression is taken as positive, this can be written as

$$\Phi = \Phi_\sigma = 1 \quad \text{if} \quad \left\{{\sigma'_m}^{n+1}\right\}^{\max} \le \sigma_c \tag{4}$$

where σ_c can be given as any value and for soil $\sigma_c \le 0$. At time step $(n + 1)$, $\left\{{\sigma'_m}^{n+1}\right\}$ is the vector of mean effective stresses at the elements of the FEM discretization scheme.

2.1.3 *Control for pore water pressure* Φ_{U_w}

Taking compression as positive, this control can be written as

$$\Phi = \Phi_{U_w} = 1 \quad \text{if} \quad \left\{U_w^{n+1}\right\}^{\max} < U_{wc} \tag{5}$$

where U_{wc} is the critical value for pore water pressures which can be simply taken as $U_{wc} \le 0$ for saturated soils, and $\left\{U_w^{n+1}\right\}$ is the vector of elemental pore water pressures. This control is

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implemented to ensure that the computed pore water pressure at any time step U_w^{n+1} is always compressive. However, there is a possibility that the computed value of the pore water pressure could appear as tensile (of smaller value) during the initial period depending on the types of displacement boundary conditions used in the FEM analysis. This is an inherent numerical error in implementing initial conditions in many numerical methods, including FEM, which gradually diminishes towards the actual values as time passes. Because of this, the Φ_{U_w} control can abstain during some initial steps, namely up to 5H time steps where Δt usually has a lower value.

2.1.4 *Control for stress ratio* Φ_{M_s}

This control is invoked to ensure that possible early failure is not caused by the high value of the time increment chosen at any time step. The Φ_{M_s} control ensures that the computed stress ratio $\{M_s^{n+1}\}$ cannot be above the failure line (Adachi and Oka 1982). If M^* is the stress ratio at failure, this control can be expressed as

$$\Phi = \Phi_{M_s} = 1 \quad \text{if} \quad \{M_s{}^{n+1}\}^{\max} > M^* \tag{6}$$

For any element, the stress ratio M_s at any time step is computed as

$$M_s = \left|\frac{J_2}{\sigma'_m}\right| \tag{7}$$

Here J_2 is the second invariant of the deviatoric stress tensor given by

$$J_2 = \sqrt{(\sigma'_{xx} - \sigma'_m)^2 + (\sigma'_{zz} - \sigma'_m)^2 + \left(\sigma'_{yy} - \sigma'_m\right)^2 + 2\tau_{xz}^2} \tag{8}$$

where, at any step, σ'_{xx}, σ'_{zz}, σ'_{yy} and τ_{xz} are the components of the effective stresses at various planes.

2.1.5 *Control for permeability* Φ_k

Control for permeability indicates that calculation of the void ratio is wrong because a higher value of the time increment was chosen. This can be expressed as

$$\Phi = \Phi_k = 1 \quad \text{if} \quad \{k_1^{n+1}\}^{\max} \geq k_c \tag{9}$$

where k_c is a tolerance limit of k_1^{n+1} and can assume any value within $10 \leq k_c \leq 200$ to ensure that $exp(k_1^{n+1})$ does not lead to any value close to infinity. Note that the permeability is updated at each time step using k_1^{n+1} (Adachi and Oka 1982) as

$$k^{n+1} = k_0 \exp(k_1^{n+1}) \tag{10}$$

where

$$k_1^{n+1} = \left[\frac{(e^{n+1} - e_0)}{C_k^*}\right] \tag{11}$$

This is not an essential control and can be left out if the problem does not involve complexity. Also, if the constitutive model does not consider the change in material permeability, this control is not required.

2.1.6 *Control for mesh overlap* Φ_{DS}

This control can be useful when the FEM analysis is carried out on the fixed mesh configuration and localized phenomena are expected. If the computation has to be stopped due to the existence of this control criterion, one may consider changing the mesh configuration based on some adaptive algorithm. This control basically checks whether any of the FEM meshes overlap the neighboring meshes. If l and $l + 1$ are two consecutive nodes along the direction of loading, the control for mesh overlap Φ_{DS} can be expressed as

$$\Phi = \Phi_{DS} = 1 \quad \text{if} \quad u_l > u_{l+1} + S_{l,\,l+1} \tag{12}$$

where u_l and u_{l+1} are the displacements along the loading direction at nodes l and $l + 1$, respectively, and $u_l > u_{l+1}$. $S_{l,l+1}$ is the nodal spacing between these two nodes in the direction of loading. Displacements in other directions are not considered because they are assumed to be very small compared to that in the loading direction. This control can be regarded as the control for displacement.

2.2 *Time increment at first time step*

The time increment Δt at the first time step is defined by Δt_c, which can be of any reasonable value provided that it does not introduce any unrealistic or unstable results in the first few steps, namely H_2 steps. $\Delta t_c = \Delta t_f$ and Δt_c can be manually chosen as $\Delta t_c = 0.001$–100 seconds.

2.3 *Parameter* χ

χ has major influence on the rate of increase of the time increment governed by Equation (1). Although a higher value of χ significantly reduces the elapsed computational time, the value of χ can be assumed to be within $0 < \chi < 2$. If the computed responses are oscillatory with time under a specific combination of the automatic time increment parameters, reducing the value of χ can improve the quality or smoothness of the solution. For moderate rates of increase of time increments while maintaining reasonable accuracy, smoothness and efficiency of the computation, the parameter χ can be chosen as 0.75.

2.4 *Parameter* T_b

The parameter T_b is defined as

$$\begin{aligned} T_b^{n+1} &= T_b^n + (T_a^n)(T_a^{n+1}) \quad \text{if} \quad \Phi = 0 \\ &= \Delta t_{cf} \quad \text{if} \quad \Phi = 1 \end{aligned} \tag{13}$$

where

$$\begin{aligned} \mathrm{T}_a^{n+1} &= \mathrm{A}^n \mathrm{T}_a^n \quad \text{if} \quad \Phi = 0 \\ &= 1 \qquad \text{if} \quad \Phi = 1 \end{aligned} \tag{14}$$

The parameter A^n at time step n is defined as

$$\begin{aligned} \mathrm{A}^n &= \frac{\log \Psi^n}{\log \Psi^{n-1}} \quad \text{if} \quad \Phi = 0 \\ &= 1 \qquad \text{if} \quad \Phi = 1 \end{aligned} \tag{15}$$

where Ψ^n is the maximum of the invariant of the strain rate per second defined by

$$\Psi^n = \left\{ \Psi_1^n \quad \Psi_2^n \quad \cdots \quad \Psi_{N_e}^n \right\}^{\max} \tag{16}$$

where N_e is the total number of elements in the spatial discretization scheme and, for any element e, the invariant of the strain rate $\Psi_{N_e}^n$ is defined by

$$\Psi_{N_e}^n = \sqrt{\left(\frac{\varepsilon_{xx}^n - \varepsilon_{xx}^{n-1}}{\Delta t^n}\right)^2 + \left(\frac{\varepsilon_{zz}^n - \varepsilon_{zz}^{n-1}}{\Delta t^n}\right)^2 + \frac{1}{2}\left(\frac{\gamma_{xz}^n - \gamma_{xz}^{n-1}}{\Delta t^n}\right)^2} \tag{17}$$

where $\gamma_{ij} = 2\varepsilon_{ij}$. Moreover, at the first time step, T_a, T_b and A are initialized as

$$\mathrm{T}_a^1 = \mathrm{T}_b^1 = \Delta t_c \quad \text{and} \quad \mathrm{A}^1 = 1 \tag{18}$$

The above definition of the parameter T_b^{n+1} ensures smooth changes in time increments from one step to another. In fact, the parameter T_b is the accumulation of the strain rate dependent parameter Ψ^n that makes the time increment Δt^{n+1} almost always increase from the previous value Δt^n. Due to the dependency of T_b on the strain history of the material through the parameter Ψ, the current time increment selection approach is capable of holding the nonlinearity of the transient problem.

2.5 *Parameter* Υ

The parameter Υ is defined as

$$\begin{aligned} \Upsilon &= \mathrm{B} \quad \text{if} \quad n < n_\Xi \\ &= 1 \quad \text{if} \quad n \geq n_\Xi \end{aligned} \tag{19}$$

where n_Ξ is the total number of time steps when the first occurrence of the $\Xi = 1$ criterion is observed. This means that the value of Υ is equal to B from the first time step. But as soon as the $\Xi = 1$ criterion is observed, the value of the parameter Υ is reduced to unity thereafter. Therefore, the parameter Υ affects only the state before the initiation of structural degradation, or before approaching the unstable temporal region where the strain rate (alternatively Ψ^n) is highly fluctuating, i.e. before the occurrence of the $\Xi = 1$ control. The parameter B is given by

$$\mathrm{B} = \frac{\varsigma}{\chi}\left(\frac{Z_{\min}}{X_{\min}}\right)^{\frac{1}{\chi}}\left(\frac{\sigma'_{zz(0)}\big|_{\min}}{k_0|_{\max}\gamma_w}\right)^{\mathrm{M}} \tag{20}$$

where $\sigma'_{zz(0)}$ is the initial vertical effective stress, k_0 is the coefficient of permeability in per unit second, and γ_w is the unit weight of pore water. $|_{\min}$ and $|_{\max}$ denote the minimum and maximum values of the respective parameters, which are especially needed when the material is layered or inhomogeneous. $X_{\min}$ and $Z_{\min}$ are the minimum values of the nodal spacing of the spatial discretization scheme along the x- and z-directions, respectively, and M is an arbitrary constant having any value in the range $0 \leq \mathrm{M} < 1$. A very high value of M may result in a less smooth time history of material response. The best value of M is found to be equal to 1/3 in various case studies. In Equation (20), the aspect ratio of the FEM mesh is considered because a finer mesh generally requires finer time increments for the same degree of temporal accuracy of the coarser mesh (Hongyi 2001). Because of the inclusion of this property, the present automatic time increment selection scheme can be more effective when it is combined with space adaptivity.

For elastic geomaterials, $\sigma'_{zz(0)}$ can be replaced by the initial shear modulus G_0 which yields

$$\mathrm{B}_{elas} = \frac{\varsigma}{\chi}\left(\frac{Z_{\min}}{X_{\min}}\right)^{\frac{1}{\chi}}\left(\frac{G_0|_{\min}}{k_0|_{\max}\gamma_w}\right)^{\mathrm{M}} \tag{21}$$

2.6 *Parameter* Λ

The parameter Λ increases linearly with time step as defined by

$$\begin{aligned} \Lambda^{n+1} &= 1 + \frac{n_\Phi}{\mathrm{H}_1} \quad \text{if} \quad \Phi = 0 \\ &= 1 \qquad \text{if} \quad \Phi = 1 \end{aligned} \tag{22}$$

In Equation (22), H_1 is a constant and $\mathrm{H}_1 \geq 0$. A very high value of H_1 will make Λ^{n+1} produce a minimal effect on the calculation of the time increment. On the other hand, a much smaller value of H_1 can cause frequent occurrence of the $\Phi = 1$ condition because of the resulting higher value of Λ. H_1 can be taken as $\mathrm{H}_1 = f\mathrm{H}$, where f is a scalar multiplier and can be simply assumed to be within $2 \leq f \leq 10$. For a given value of M and χ, if a smoother soil response is required, the parameter H_1 can be increased to a relatively higher value. As mentioned earlier, the parameter n_Φ is the total number of time steps from the first time step or from that time step when the computation is restarted after going H_2 time steps back. This ensures a smaller rate of increase of the time increment just after the unstable temporal region. Inclusion of the parameter Λ is justified since the unstable temporal region can be carried off as the computation time progresses depending on the rate sensitivity or degree of structural collapse of the material. Moreover, the spatial mesh configuration can also be redefined after n_Ξ (time steps up to the first occurrence of the $\Xi = 1$ condition) time steps, when an adaptive mesh refinement approach is also combined. In view of

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these, the parameter Λ can further promote the rate of increase of the time increment, whose effect becomes prominent when the current time step is far from the unstable temporal region.

3. Performance of the automatic time increment selection scheme

The performance and validity of the automatic time increment selection scheme are investigated in detail through several test problems. The scheme is applied to the numerical analyses of the long-term consolidation response of clay under different geotechnical profiles and loading conditions. Both elastic and elasto-viscoplastic constitutive relations are considered, including consideration of the structural degradation of clay. The scheme is examined numerically within the framework of the finite element method (FEM). The automatic time increment selection scheme is applied mainly with Euler's implicit time integration method. However, its performance is also assessed against the fourth-order Runge–Kutta explicit time integration method. The computer run time or elapsed time for the numerical computation is denoted by E_t, which is dependent on the processor's specification, type of compiler and structure of the code. However, for comparison of computational efficiency, all analyses are carried out through the implementation of a two-dimensional code using MATLAB run on a Pentium IV 2.4 GHz CPU with 736 MB memory.

3.1 *Finite element formulation for elasto-viscoplastic consolidation analysis*

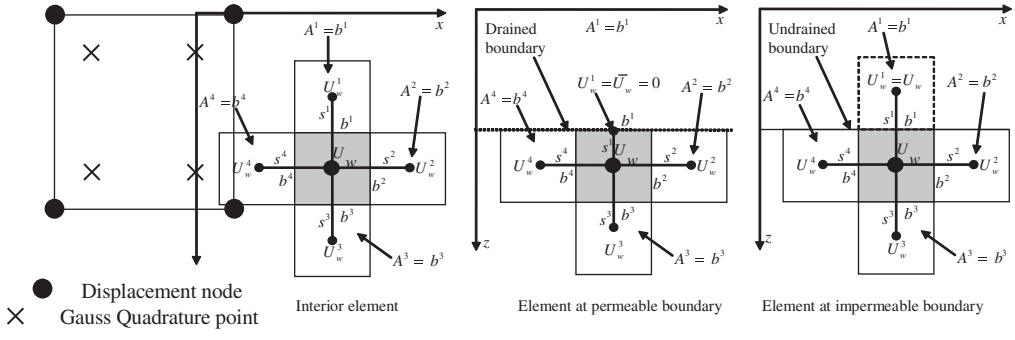
A finite element model for numerical simulation of the long-term consolidation response of clay is briefly presented. The model is based on the elasto-viscoplastic constitutive model (Adachi and Oka 1982) including an extension for the structural degradation of geomaterials (Kimoto *et al*. 2004; Kimoto and Oka 2005). This model can provide a reasonable description for some of the important features of natural overconsolidated soft clay highlighted mostly till date including the strain rate sensitivity, strain hardening, creep, secondary compression, stress relaxation, structural degradation or strain softening. The range of applicability of the present finite element formulation is limited by the infinitesimal strain theory under plane strain conditions. The basic properties of the soil are assumed to be: the soil is a two-phase porous medium in which pores are completely saturated with water; and both soil particles and pore water are incompressible. Under these assumptions, the governing equations of consolidation are those from the multidimensional equilibrium equation of a two-phase medium given by Biot (1941) and the storage equation given by Verruijt (1969). In the present displacement-based formulation, a four-node isoparametric element is considered for the approximation of the displacement u as shown in Figure 4(a), in which the pore water pressure U_w is taken into account through a special finite difference scheme (Oka *et al*. 1986) as shown in Figure 4(b). 2×2 Gauss quadrature points are used for evaluating the two-dimensional domain integrals and two points are used for evaluating the boundary or force integrals. Nevertheless, currently Euler's implicit time marching method is adopted for time integration. Under the above conditions, the final system of discrete equations for FEM analysis can be obtained (Oka *et al*. 1986; Karim 2006) as

$$\begin{bmatrix} [K] & [K_v] \\ [K_v]^T & \Delta t[\beta] \end{bmatrix} \begin{Bmatrix} \{\Delta u^*_{t+\Delta t}\} \\ \{U_{w,t+\Delta t}\} \end{Bmatrix} = \begin{bmatrix} [0] & [K_v] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\Delta u^*_t\} \\ \{U_{w,t}\} \end{Bmatrix} + \Delta t \begin{Bmatrix} \{Q\} \\ \{0\} \end{Bmatrix} \tag{23}$$

where

$$\{\beta\} = \left| -\beta + \sum_{i=1}^{4} \beta_i \right|_{\Omega} \quad , \quad \beta_i = \frac{k}{\gamma_w} \frac{b_i}{s_i} \quad , \qquad \beta = \sum_{i=1}^{4} \beta_i \tag{24a}$$

$$[K] = \int_{\Omega} [B]^T [D][B] d\Omega \quad , \quad \{K_v\} = \int_{\Omega} \{B_v\} d\Omega \tag{24b}$$



(a) u-interpolation (b) Inclusion of pore water pressure U_w in u-interpolation

Figure 4. Pore water pressure and displacement approximation.

$$\{Q\} = \{\dot{F}^*\} + \{\dot{F}^t\} + \{\dot{F}^b\} \tag{24c}$$

$$\begin{aligned} \{\dot{F}^t\} &= \int_\Gamma [N]^T \{\dot{\bar{T}}\} d\Gamma \quad , \\ \{\dot{F}^*\} &= \int_\Omega [B]^T \{\dot{\sigma}^*\} d\Omega \quad , \\ \{\dot{F}^b\} &= \int_\Omega [N]^T \{\dot{\bar{F}}^b\} d\Omega \end{aligned} \tag{24d}$$

in which $[N]$ is the FEM shape function, $[B]$ is the matrix which transforms the nodal displacement into strain, and $\{B_v\}$ is the vector which transforms the nodal displacement into the volumetric strain (Bathe 1996); furthermore, $\bar{F}_i^b$ is the body force tensor on the problem domain Ω, $[D]$ is the elastic modulus and $\{\dot{\sigma}^*\}$ is the relaxation stress vector expressed by (Oka *et al*. 1986)

$$\{\dot{\sigma}^*\} = [D]\{\dot{\varepsilon}^{vp}\} \tag{25}$$

Note that, in the adopted constitutive model, the total strain rate tensor $\dot{\varepsilon}_{ij}$ has been decomposed into two parts as

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^{vp} \tag{26}$$

where $\dot{\varepsilon}_{ij}^e$ is the elastic strain rate tensor and $\dot{\varepsilon}_{ij}^{vp}$ is the viscoplastic strain rate tensor. Based on the overstress type of viscoplastic theory (Perzyna 1963), the viscoplastic strain rate tensor is defined as

$$\dot{\varepsilon}_{ij}^{vp} = \gamma \langle \Phi_1(f_y) \rangle \frac{\partial f_p}{\partial \sigma'_{ij}} \tag{27}$$

where Φ_1 is the rate-sensitive material function for overconsolidated clay and

$$< \Phi_1(f_y) > = \begin{cases} \Phi_1(f_y) & : \quad f_y > 0 \\ 0 & : \quad f_y \leq 0 \end{cases} \tag{28}$$

where f_y is the static yield function and f_p is the viscoplastic potential function given by

$$f_y = \bar{\eta}^*_{(0)} + \tilde{M}^* \ln \frac{\sigma'_{\mathrm{m}}}{\sigma'(\mathrm{s})_{\mathrm{my}}} = 0 \tag{29}$$

$$f_p = \bar{\eta}^*_{(0)} + \tilde{M}^* \ln \frac{\sigma'_{\mathrm{m}}}{\sigma'_{\mathrm{mp}} = 0} \tag{30}$$

where $\sigma'_{my}(s)$ denotes the mean effective stress in the static equilibrium state and $\tilde{M}^*$ is defined as

$$\tilde{M}^* = \begin{cases} M^*_m & : \quad f_b \geq 0 \\ -\frac{\sqrt{\eta^*_{ij}\eta^*_{ij}}}{\ln(\sigma'_m/\sigma'_{mc})} & : \quad f_b < 0 \end{cases} \tag{31}$$

where σ'_{mc} is expressed as

$$\sigma'_{mc} = \sigma'_{mb} \exp \frac{\sqrt{\eta^*_{ij(0)}\eta^*_{ij(0)}}}{M^*_m} \tag{32}$$

The overconsolidation boundary surface f_b is defined as

$$f_b = \bar{\eta}^*_{(0)} + M^*_m \ln \frac{\sigma'_m}{\sigma'_{mb}} = 0 \tag{33}$$

where M^*_m is the value of $\sqrt{\eta^*_{ij}\eta^*_{ij}}$ at maximum compression and $\bar{\eta}^*_{(0)}$ is given by

$$\bar{\eta}^*_{(0)} = \sqrt{\left(\eta^*_{ij} - \eta^*_{ij(0)}\right)\left(\eta^*_{ij} - \eta^*_{ij(0)}\right)} \tag{34}$$

in which (0) denotes the state at the end of the consolidation and η^*_{ij} is the stress ratio tensor given by $\eta^*_{ij} = S_{ij}/\sigma'_m$ and S_{ij} is the deviatoric stress tensor. σ'_{mb} controls the size of the overconsolidation boundary surface given as

$$\sigma'_{mb} = \sigma'_{ma} \exp\left(\frac{1+e_0}{\lambda - \kappa}\varepsilon_v^{vp}\right) \tag{35}$$

where λ is the compression index, κ the swelling index, e_0 the initial void ratio and ε_v^{vp} is the viscoplastic volumetric strain. Equation (35) describes the destructuration effect of clay in terms of the softening of viscoplastic strain in addition to its hardening effect. Taking σ'_{mai} and σ'_{maf} as the initial and final values of σ'_{ma}, respectively, σ'_{ma} is defined by

$$\sigma'_{ma} = \sigma'_{maf} + (\sigma'_{mai} - \sigma'_{maf}) \exp(-\beta z) \tag{36}$$

where z is the accumulation of the second invariant of the viscoplastic strain rate given by

$$z = \int_0^t \dot{z} dt \quad \text{and} \quad \dot{z} = \sqrt{\dot{\varepsilon}_{ij}^{vp}\dot{\varepsilon}_{ij}^{vp}} \tag{37}$$

and β denotes the degree of possible collapse of soil structure or the degradation rate of clay.

Based on the experimental correlation proposed by Adachi and Oka (1984) for the rate-sensitive material function Φ_1, and incorporating Equations (27)–(35), the viscoplastic deviatoric strain rate $\dot{e}_{ij}^{vp}$ and viscoplastic volumetric strain rate $\dot{\varepsilon}_v^{vp}$ can be written as

$$\dot{e}_{ij}^{vp} = C\sigma'_m \exp\left\{m'\left(\bar{\eta}^*_{(0)} + \tilde{M}^* \ln \frac{\sigma'_m}{\sigma'_{mb}}\right)\right\}\left(\frac{\eta^*_{ij} - \eta^*_{ij(0)}}{\bar{\eta}^*_{(0)}}\right) \tag{38a}$$

$$\dot{\varepsilon}_v^{vp} = C\sigma'_m \exp\left\{m'\left(\bar{\eta}^*_{(0)} + \tilde{M}^* \ln \frac{\sigma'_m}{\sigma'_{mb}}\right)\right\}\left(\tilde{M}^* - \frac{\eta^*_{ij}(\eta^*_{ij} - \eta^*_{ij(0)})}{\bar{\eta}^*_{(0)}}\right) \tag{38b}$$

where C and m' are the viscoplastic parameters.

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Moreover, the following boundary conditions are used in deriving Equation (23):

$$\dot{\sigma}_{ji} n_j = \dot{\bar{T}}_i \quad \text{in} \quad \Gamma_\sigma \quad \text{and} \quad \dot{u}_i = \dot{\bar{u}}_i \quad \text{in} \quad \Gamma_u \qquad (39)$$

where Γ_σ and Γ_u are the parts of the total boundary Γ where the stress rate and displacement rate are prescribed as $\dot{\bar{T}}_i$ and $\dot{\bar{u}}_i$, respectively. Obviously, they satisfy the following relations:

$$\Gamma_u \cup \Gamma_\sigma = \Gamma \quad \text{and} \quad \Gamma_u \cap \Gamma_\sigma = \emptyset$$

3.2 *Application to elastic analysis*

Soil is assumed to be homogeneous and isotropic. The finite element formulation for the consolidation of soils under elastic conditions is not presented here; it can be found in many sources (Sandhu and Wilson 1969; Sandhu *et al*. 1977). Alternately, the elastic formulation can be derived from Equation (23) simply by setting the relaxation stress term equal to zero (i.e. $\{\dot{\sigma}^*\} = \{0\}$) since the viscoplastic strain rate tensor is now ignored (i.e. $\dot{\varepsilon}_{ij}^{vp} = 0$ in Equation (26)). In the current analyses, the soil is assumed to be fully saturated with water with the top surface fully permeable. All other boundaries are impermeable. The vertical boundaries are horizontally fixed and the bottom soil surface is assumed to be completely rigid in both the horizontal and vertical directions. The body force of the soil skeleton is not taken into consideration.

3.2.1 One-dimensional elastic consolidation response

A typical one-dimensional consolidation problem is considered as shown in Figure 5. Appropriate boundary conditions to represent the one-dimensional problem are used, i.e. a soil column of unit width is considered in the two-dimensional FEM code with vertical boundaries horizontally restrained. The 13.8 m soil layer is subjected to a static uniform load of 100 kN/m^2 on its top surface. The elastic soil parameters are presented in Table 1. Since the soil is elastic and its permeability is kept constant with time, controls for viscoplastic strain, stress ratio and permeability are not considered here. The automatic time increment parameters are given in Table 2.

Automatically computed time increments are presented in Figure 6(a). In this case, no control criterion is satisfied till after 100 days of computation time. This means that Equation (1) provides a good measure for the time increments. Moreover, the time increments Δt follow the observed strain history of the elastic medium as shown in Figure 6(b). Figures 7(a)-(b) illustrate the elastic soil responses under one-dimensional conditions. There is excellent agreement found in the soil displacements and pore water pressures between the FEM results and Terzaghi's close-form solutions (Terzaghi and Peck 1976). However, near the top permeable boundary and during the initial period, the FEM results deviate from the true solutions. This situation can be improved if a finer mesh or higher order shape functions are implemented in the FEM model and/or a much lower value of Δt_c is used. However, since the solution converges rapidly towards the analytical solution, this initial discrepancy is quite acceptable. It is also observed that all the results are oscillation-free. Therefore, it can be safely concluded that the present automatic time increment selection scheme provides a rational degree of accuracy. Moreover, it took only nine seconds to carry out the FEM analysis for 100 days.

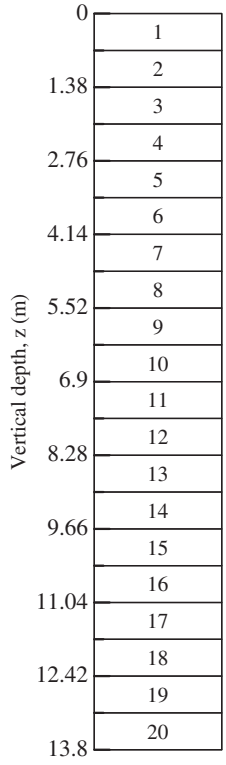


Figure 5. FEM mesh for one-dimensional elastic consolidation.

Table 1. Elastic soil properties

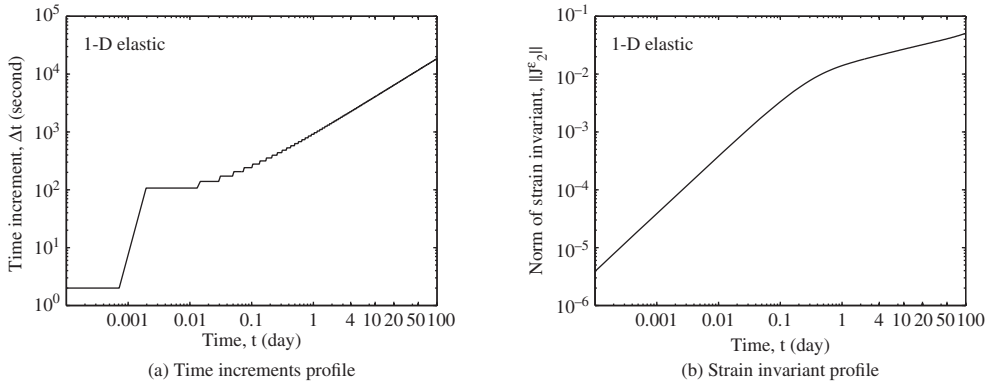
Shear modulus, G (kN/m^2)	1.5385×10^3
Poisson's ratio, ν	0.30
Permeability, k (m/s)	1.05×10^{-8}
Density of water, γ_w (kN/m^3)	9.81

Table 2. Time increment parameters for one-dimensional elastic consolidation

ς	Δt_c	Δt_f	H	H_1	H_2	M	χ	σ_c	U_{wc}	$Z_{\min}/X_{\min}$
1	5s	0.5s	10	20	10	0.333	0.75	0	0	0.69

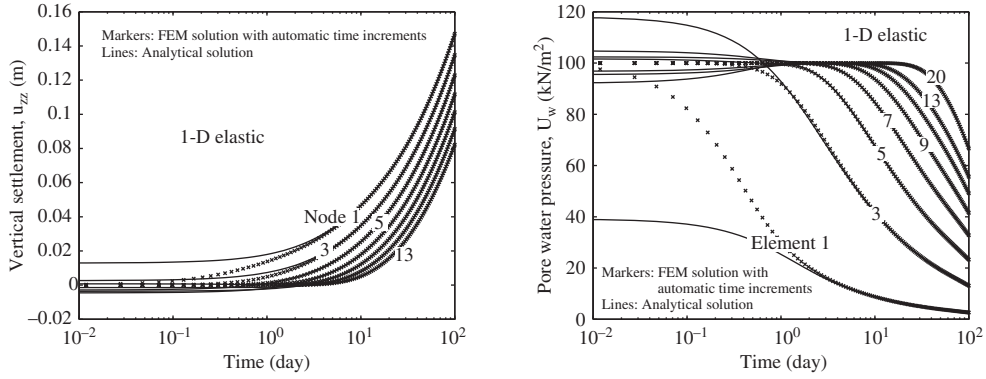
3.2.2 Two-dimensional elastic consolidation response

A typical two-dimensional consolidation problem is considered as shown in Figure 8. Soil properties and boundary conditions are kept the same as those with the elastic one-dimensional



(a) Time increments profile (b) Strain invariant profile

Figure 6. Time increments during one-dimensional elastic consolidation.



(a) Vertical displacements (b) Pore water pressures

Figure 7. Time history of soil response during one-dimensional elastic consolidation.

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analysis. The FEM mesh configuration is presented in Figure 9. The applied loading profile is shown in Figure 10(a), which is similar to the construction load of the test embankment D over the Champlain clay at St. Alban (Oka *et al*. 1991). Because of elastic considerations, only the controls for the effective stress and pore water pressure are implemented. The same set of time increment parameters is used as presented previously in Table 2, except that $Z_{\min}/X_{\min}$ is now 0.33 because of the two-dimensional discretization scheme as shown in Figure 9.

The time increments computed through the automatic selection scheme are presented in Figures 10(b) and 11(a). To ensure that the time increment profile reflects the loading history, the time increments Δt are always reassigned to $\Delta t = \Delta_{cf}$ when the construction loading is increased. This is a manual override, which introduces frequent increases and decreases in time increments during 5–18 days as shown clearly in Figure 10(b). The time increments between two consecutive construction loads or after the embankment construction is computed as usual using the current selection approach (i.e. using Equation 1). Note that the time increment profile also follows the strain history of the material as presented in Figures 11(a),(b). For this two-dimensional elastic consolidation analysis, it took only five minutes and three seconds to complete the computation up to 100 days. Temporal distributions of the two-dimensional elastic soil responses during and after the construction are presented in Figures 12(a),(b). It is found that the pore

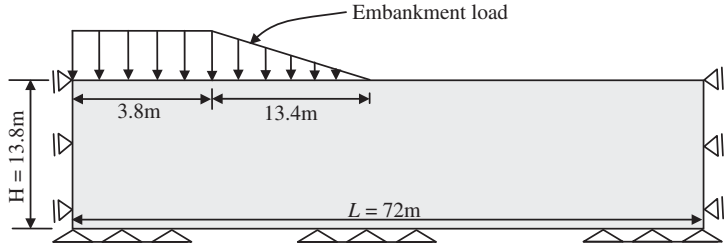


Figure 8. Typical two-dimensional consolidation problem.

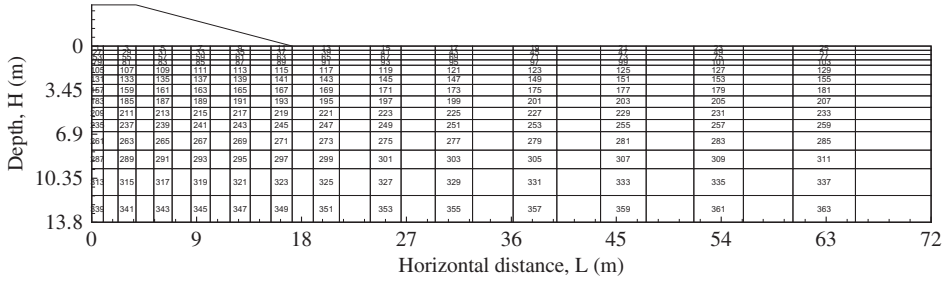
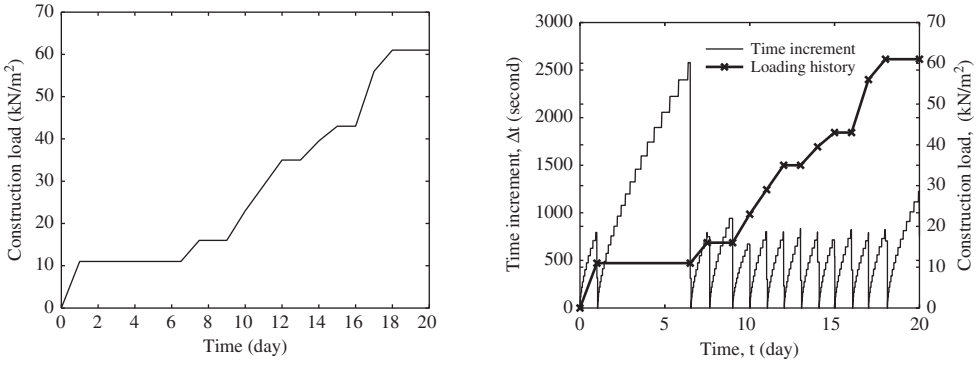
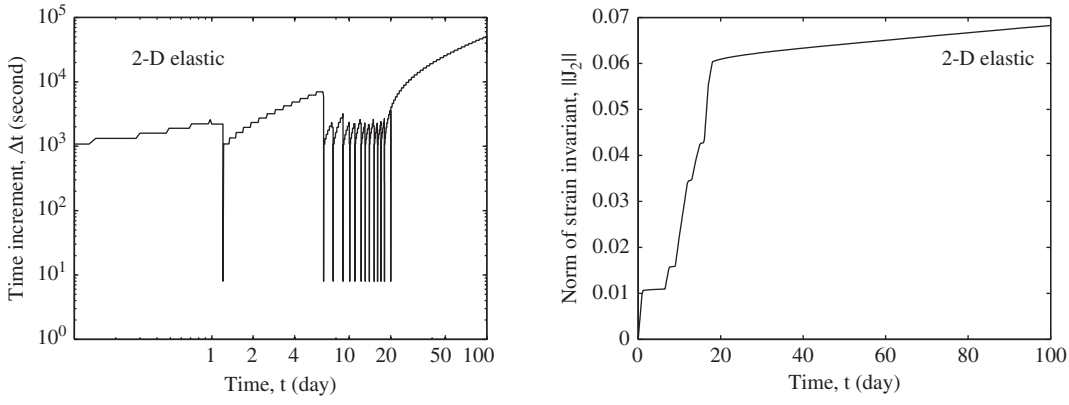


Figure 9. Discretization scheme for two-dimensional analysis.



(a) Construction loading (b) Time increments during construction

Figure 10. Loading profile and time increments during embankment construction (two-dimensional elastic).



(a) Overall time increment profile (b) Strain invariant profile

Figure 11. Time increments and observed strain history for two-dimensional elastic consolidation.

water pressure increases as the load increases. However, the pore water pressure starts to dissipate once the loading increment ceases (after 18 days). All the results are oscillation-free, including the computed stresses.

3.3 *Application to viscoplastic analysis*

For soil behavior beyond the elastic limit, an elasto-viscoplastic constitutive model with consideration for soil-structure degradation (Kimoto *et al*. 2004; Kimoto and Oka 2005) is now considered, for which the FEM formulation is presented in Section 3.1. Note that in addition to the time-dependent permeability (Equation 8), the shear modulus G of the soil skeleton is also varied with time from the initial value of G_0 as (Adachi and Oka 1982)

$$G = G_0 \sqrt{\frac{\sigma'_m}{\sigma'_{m(0)}}} \tag{40}$$

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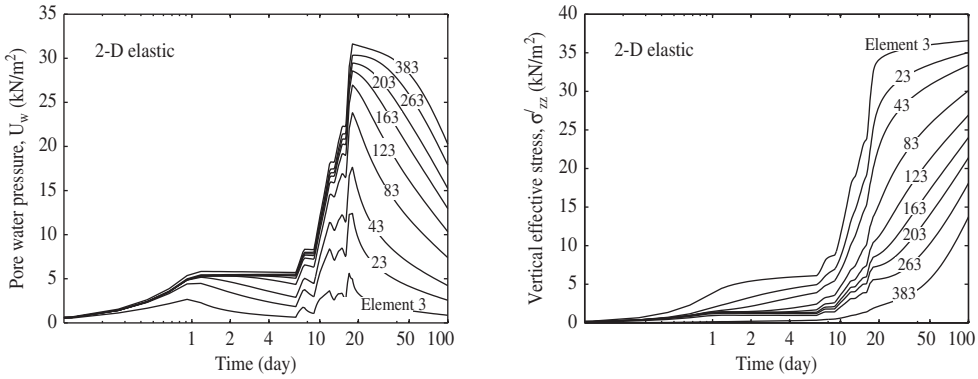

(a) Pore water pressures (b) Vertical effective stresses

Figure 12. Two-dimensional elastic consolidation response.

where σ'_m and $\sigma'_{m(0)}$ are the mean effective stresses at any time t and at $t = 0$, respectively.

3.3.1 *One-dimensional viscoplastic consolidation response*

The soil is assumed to be slightly overconsolidated with an overconsolidation ratio of 1.5. The soil is assumed to undergo an initial excess pore water pressure $U_{w(0)}$ of 1000 kN/m^2, which is equivalent to the application of an external load of the same amount. The physical characteristics of the soil skeleton, as well as the viscoplastic model parameters, are presented in Table 3. The thickness of the soil layer is taken as 20 m, which is discretized with 20 equally spaced elements similar to that shown in Figure 5. The automatic time increment parameters are shown in Table 4. The boundary conditions are kept the same as used previously in the elastic analysis.

The variation of the automatically computed time increments with time is shown in Figure 13(a). By measuring in terms of the strain invariant, the time increment profile shows that it follows the nonlinearity of the problem as presented in Figure 13(b). In fact, it is due to the strain-rate dependent parameter Ψ^n (Equation 16) that holds this important phenomenon. The temporal distributions of the pore water pressures are presented in Figure 14(a). The distribution of deviatoric stresses with mean effective stresses is presented in Figure 14(b). All these results demonstrate that the automatic time increment selection scheme works well. However, it produces smaller temporal oscillations in pore water pressures and thus in stresses, especially at the first element near the top permeable surface. This temporal oscillation can easily be eliminated if a lower value of χ or M and/or a higher value of H_1 is used. This situation could also be improved if a finer mesh near the top surface is used or a higher order of FEM shape function is considered. Note that the elapsed time for this computation was 15 minutes and 8 seconds for the total computation time of 100 years.

Now a particular one-dimensional case is analyzed, where the time increment is found to be very sensitive because of the pronounced structural degradation of the soil. The height of the soil column is now taken as 0.1 m. Discretizing the domain with the same number nodes, $Z_{\min}$ now becomes equal to 0.005 m. The soil parameters for this case are presented in Table 5. In fact, these are the parameters for the Tsurumi Pleistocene clay (Kimoto 2002). The time increment parameters are kept the same as shown previously in Table 4.

The time increment profile is shown in Figure 15(a). In this case, the time increments are found to be very sensitive between 10^3 s and 10^4 s during which structural changes of the soil are most prevalent. During this period, rapid fluctuation of the governing time increment parameter Ψ^n, i.e. the strain rate, is observed as shown in Figure 15(b). Therefore, the time increments are forced to keep as low as $\Delta t_f = 0.5$ s due to the automatic enforcement of the $\Phi = 1$ and $\Xi = 1$ controls. The time history of the pore water pressures and the viscoplastic strains are presented, respectively, in Figures 16(a),(b), which are in harmony with the results obtained by Kimoto (2002). As shown in Figure 16(a), an increase in pore water pressure was observed during the period 10^3–10^4 s as a result of the significant change in soil structure.

The effectiveness of the automatic time increment selection scheme can be explained as follows. Proper selection of the time increments for this case is not an easy task, unless the user has prior knowledge about the expected soil behavior. For

Table 3. Soil and viscoplastic parameters (case-1)

λ	κ	M^*	m'	C
0.508	0.0261	1.25	18.5	1×10^{-13} /s
C_k^*	β	σ'_{mai}	σ'_{maf}	e_0
0.1	20	300 kN/m^2	200 kN/m^2	1.7
K_0	$\sigma'_{zz(0)}$	$U_{w(0)}$	k_0	G_0
0.5	450 kN/m^2	1000 kN/m^2	8×10^{-10} m/s	3.61×10^3 kN/m^2

Table 4. Time increment parameters for one-dimensional viscoplastic consolidation

ς	Δt_c	Δt_f	H	H_1	H_2	M	χ	σ_c	ε_c^{vp}	k_c	U_{wc}	$Z_{\min}/X_{\min}$
1	5s	0.5s	10	20	10	0.333	0.75	0	$\pm e^{10}$	10	0	1

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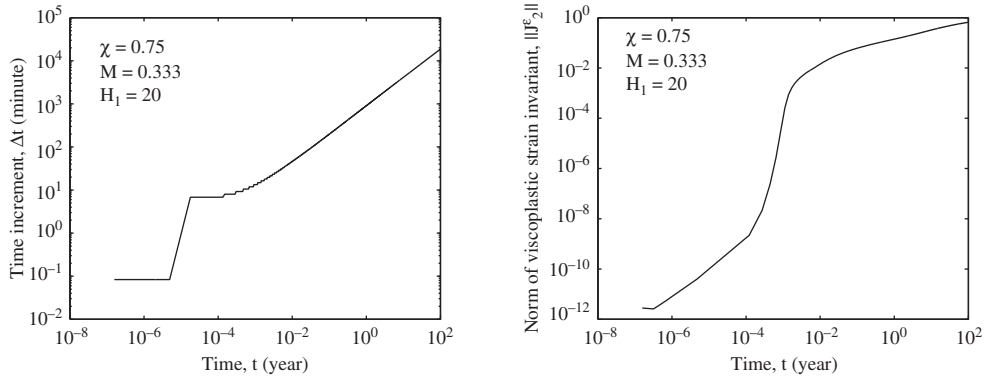
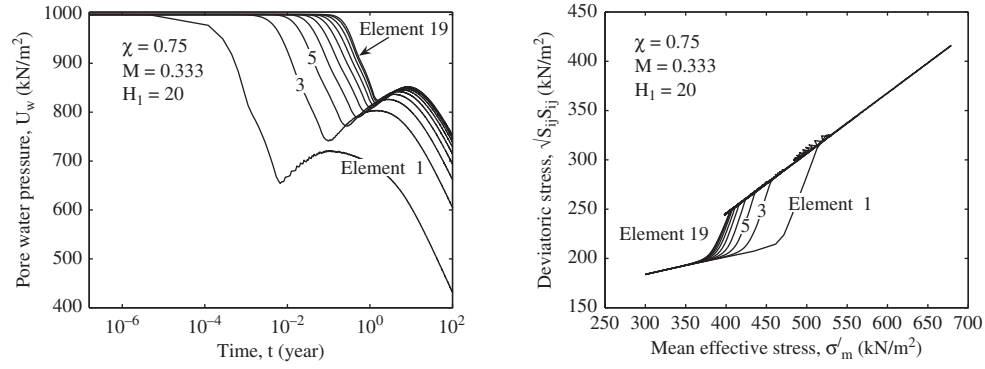


(a) Time increments profile (b) Strain invariant profile

Figure 13. Time increments in one-dimensional viscoplastic consolidation analysis (case-1).



(a) Pore water pressures (b) Stress paths

Figure 14. One-dimensional viscoplastic consolidation response (case-1).

Table 5. Soil and viscoplastic parameters (case-2)

λ	κ	M^*	m'	C
0.508	0.0261	1.09	18.5	1×10^{-13} /s
C_k^*	β	σ'_{mai}	σ'_{maf}	e_0
0.1	20	580 kN/m^2	300 kN/m^2	1.7
K_0	$\sigma'_{zz(0)}$	$U_{w(0)}$	k_0	G_0
0.5	870 kN/m^2	1160 kN/m^2	8×10^{-10} m/s	3.61×10^4 kN/m^2

example, even if a constant time increment as small as 2 seconds is used, numerical instability is observed in the computation. For constant time increments, it is only possible to complete this computation if the time increment is less than or equal to 1 second. However, this is not feasible at all for a long-term computation spanning up to 100 years. Figures 17(a),(b) show the results when constant time increments are used. In these figures, just after 2500 seconds, the viscoplastic strain becomes very high (close to infinity) when $\Delta t \geq 2$s. This difficulty, however, can be lessened if the manual time increment profile is chosen similar to that presented in Figure 18(a). Although the pore water pressure profiles are almost identical, this manual time increment selection is about 50 times slower than the automatic selection scheme, as shown in Figure 18(b). Compared to the performance of the automatic time increment selection scheme, this manual selection thus becomes too pessimistic. Above all, it is not always an easy task to select the time increments manually in that way. In the automatic time increment scheme, the user just needs to specify the minimum value of the time increment Δt_f that is used under the extreme case. The scheme always tries to increase the time increment for maximum computational efficiency, which is being monitored through the implementation of the control criteria. Since the control criteria ($\Phi = 1$) are based on the physical soil responses, they can catch the situation when the computation reaches the unstable or sensitive temporal region. If that happens, the time increment Δt automatically reduces to Δt_f as governed by Equation (1). In this way, the efficiency and stability of the computation is retained. The computational efficiency using the automatic time increment selection scheme is, however, dependent on the choice of discretization scheme, as shown in Figure 19(a). It is found that the computer run time increases as the nodal density increases. Usually, finer meshes represent the physical problem more clearly because of the closer mapping of the discontinuities of the field variables. This requires greater

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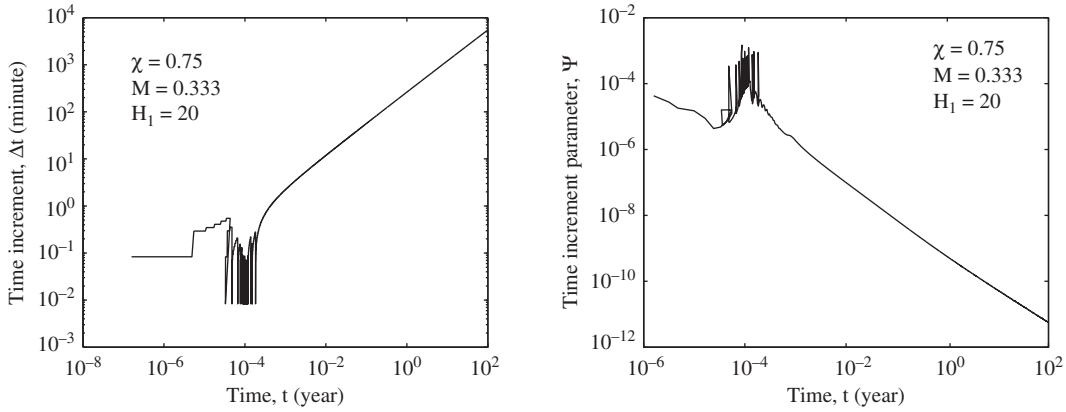
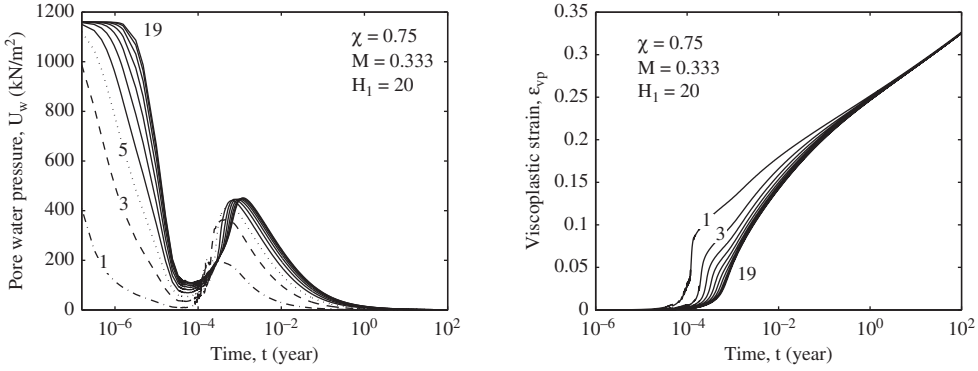


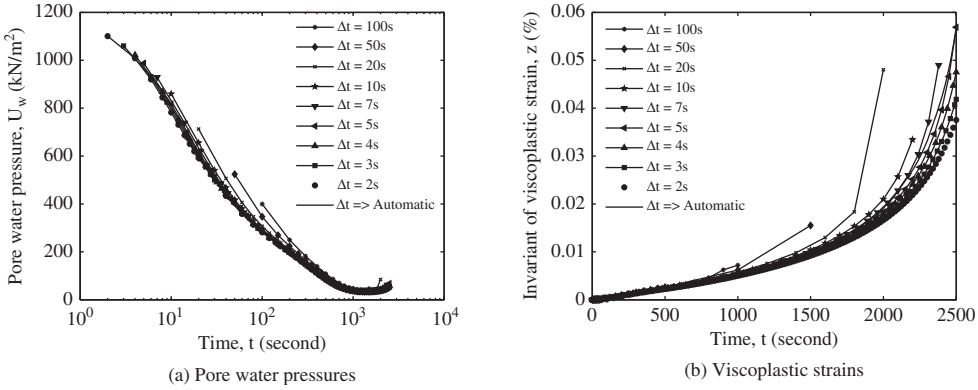
(a) Time increments (b) Parameter Ψ^n

Figure 15. Time increments in one-dimensional viscoplastic consolidation analysis (case-2).



(a) Pore water pressures (b) Viscoplastic strains

Figure 16. One-dimensional viscoplastic consolidation response (case-2).



(a) Pore water pressures (b) Viscoplastic strains

Figure 17. One-dimensional viscoplastic response under constant time increment.

temporal accuracy, which has been well reflected by the automatic time increment scheme. The pore water pressures for different nodal densities are shown in Figure 19(b) and are oscillation-free. These results show that the performance of the present automatic time increment selection scheme is not limited to the spatial mesh configuration; instead it provides optimal time increment selection for a given FEM discretization scheme.

3.3.2 Two-dimensional viscoplastic consolidation response

A similar two-dimensional consolidation problem as considered in the elastic case is now analyzed under the elasto-viscoplastic constitutive law (Kimoto *et al.* 2004). This example is similar to the conditions for the test embankment D on Champlain clay at Saint Alban (Oka *et al.* 1991). Although the Champlain clay appears as a layered medium (Oka *et al.* 1991), for simplicity, the soil parameters are homogenized as shown in Table 6. The

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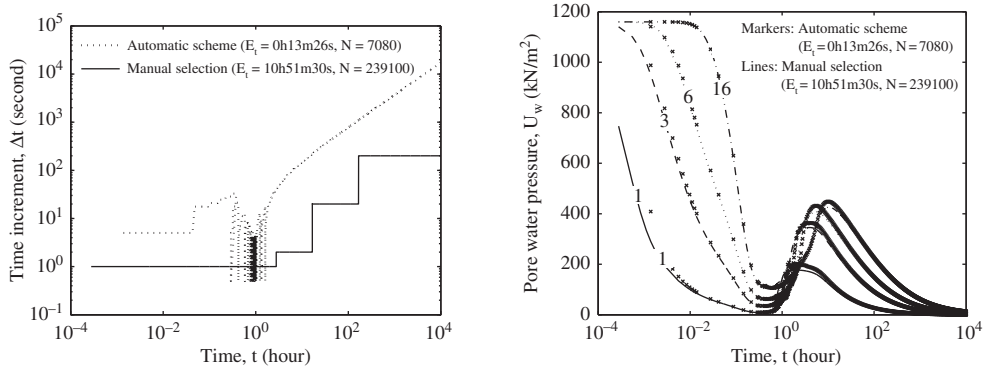
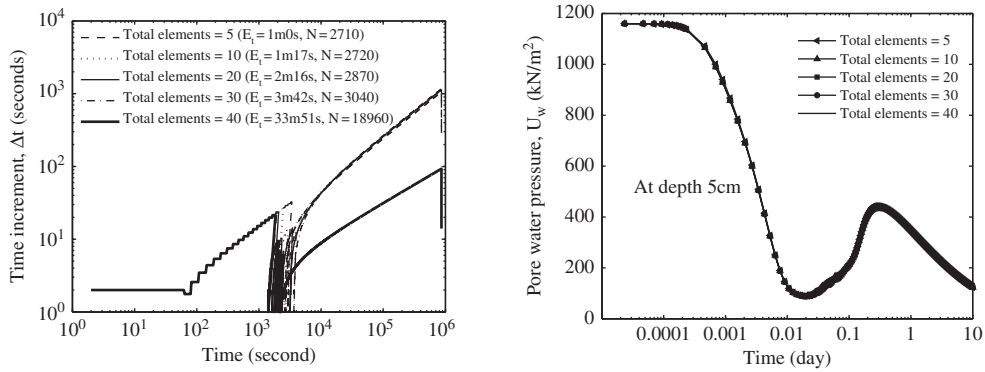


(a) Time increments (b) Pore water pressures

Figure 18. One-dimensional viscoplastic response under automatic and manual time increments.



(a) Time increments (b) Pore water pressures

Figure 19. One-dimensional viscoplastic response for various nodal densities.

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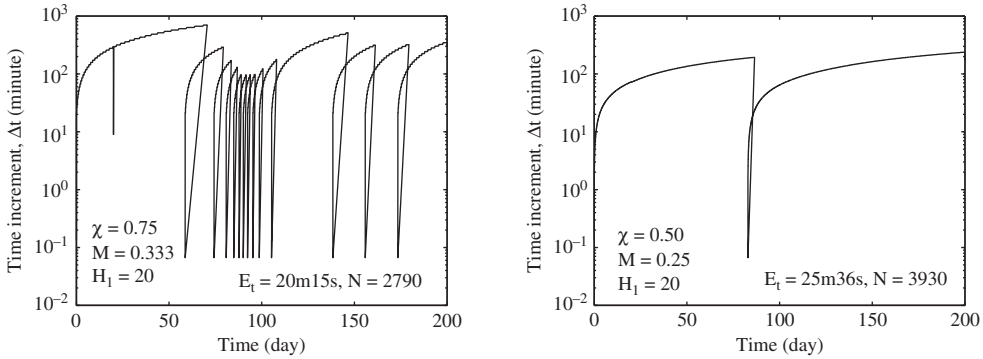
boundary conditions are kept the same as used in the elastic analysis (Figure 8). The loading history of the embankment, as well as the adopted FEM mesh configuration, are shown, respectively, in Figures 10 and 9. The responses of the soil foundation during and after the construction of the embankment are calculated throughout the time of 200 days. For this two-dimensional problem, the value of the parameter ς is taken as 2 and Δt_{cf} is defined as $\Delta t_c = 10$ s and $\Delta t_f = 1$ s. Other time step parameters remain the same, as presented in Table 4 for the one-dimensional analysis.

Automatically chosen time increments are plotted against time as shown in Figure 20(a). The time increment profile reflects the loading history, and the time increments are forced to be relatively smalle during the increase of construction loading as discussed previously in Section 3.2.2. Figure 20(a) also shows that the time increments are frequently fluctuating during 70–100 days which is even 50 days after the end of construction loading. This is because of the enforcement of the control criterion ($\Phi = 1$) to hinder the constantly upward nature of the rate of increase of the time increments. This ensures temporal stability of the computation and it almost does not affect the accuracy of the solution. This fluctuation, however, can be significantly eliminated if slightly lower values of χ and M are chosen. For example, for $\chi = 0.50$ and M = 0.25, there is no such frequent fluctuation in time increments, as shown in Figure 20(b). However, the elapsed run times for this numerical analysis was increased to 25 minutes and 36 seconds ($\chi = 0.50$ and M = 0.25, Figure 20(a)) from 20 minutes and 15 seconds ($\chi = 0.75$ and M = 0.333, Figure 20(b)). More importantly, both time increment profiles yield almost the same solution. The effects of various time increment parameters are discussed thoroughly in Section 4.

Table 6. Soil and viscoplastic parameters (two-dimensional)

λ	κ	M^*	m'	C
0.694	0.0045	0.98	27.245	5.9×10^{-11} /s
C_k^*	β	σ'_{mai}	σ'_{maf}	e_0
0.5	20	95.96 kN/m^2	26.89 kN/m^2	1.6429
K_0	$\sigma'_{zz(0)}$	OCR	k_0	G_0
0.8	31.03 kN/m^2	3.5684	1.05×10^{-8} m/s	4.015×10^3 kN/m^2

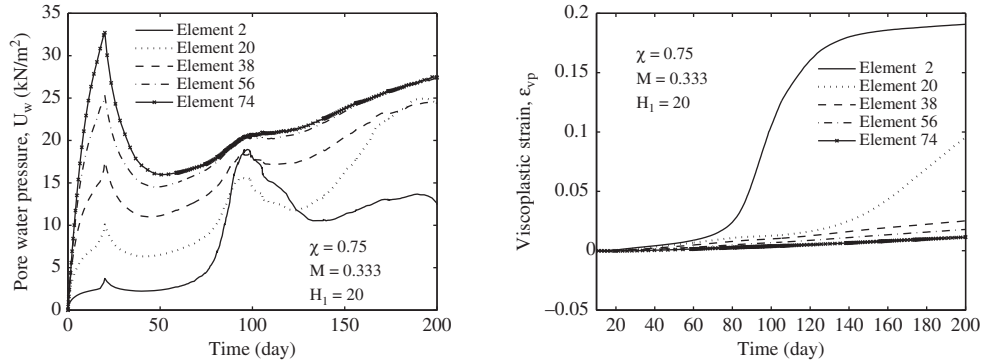
The temporal distribution of pore water pressure and viscoplastic volumetric strain are presented, respectively, in Figures 21(a),(b). From Figure 21(a), it is found that the pore water pressure increases noticeably some time after the end of the construction of the embankment. This anomalous pore water pressure generation has been frequently pointed out (Mesri and Choi 1985; Kabbaj *et al*. 1988; Oka *et al*. 1991). It has been possible to acquire this type of



(a) Time increments for $\chi = 0.75$ and $M = 0.333$

(b) Time increments for $\chi = 0.50$ and $M = 0.25$

Figure 20. Time increments in two-dimensional viscoplastic consolidation analysis.



(a) Pore water pressures

(b) Viscoplastic strains

Figure 21. Two-dimensional viscoplastic consolidation response.

behavior because of the inclusion of the degradation of soil structure into the governing constitutive model (Kimoto *et al.* 2004). However, this prohibits using larger time increments as shown in Figure 20. From the results presented, it can be concluded that the automatic time increment selection approach also works well for the two-dimensional elasto-viscoplastic analysis.

4. Guidelines for the selection of time increment parameters

The automatic time increment selection approach is a series of systematic steps that depends explicitly on the past history of the computed responses. Most of the time increment parameters depend on the material and the governing constitutive model parameters, as well as on the time-dependent material responses. The parameter H_2 depends on the parameter H which is basically the data storage interval. If the computed responses are stored at each time step, the value of H will then be equal to 1. However, since it is assumed that the time increments are varying in a stepwise manner, it is preferable to use $H \geq 1$. The parameter H_1 can also be related to H as discussed previously ($H_1 = fH$). However, the value of H_1 affects the rate of increase of the time increment, although its effect is minimum because of the nature of the parameter Λ as defined by Equation (22). Since the time increment selection scheme automatically adjusts the time increment at the H time step interval, the value of Δt_c can be any positive value (e.g. 100 s, 10 s, 1 s, 0.1 s, 0.001 s, etc.) provided that numerical stability and reasonable accuracy are in hand during the first H or H_2 time steps. Similarly, the minimum value of the time increment Δt_f can be chosen arbitrarily, but should be as small as possible ($\Delta t_f \leq \Delta t_c$) to avoid early termination of the computation ahead of the actual material failure. Therefore, the automatic time increment parameters are mainly χ, M and H_1. The effects of these parameters will now be discussed to establish the guidelines for their selection.

4.1 *One-dimensional analyses*

The same problem as described in Section 3.3.1 is considered. The soil parameters are listed in Table 3 (case-1). Unless otherwise specified, the time increment parameters are kept the same as presented previously in Table 4.

The effects of the time increment parameter χ on the pore water pressure calculation is presented in Figure 22(a). χ affects

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the computed time increments as shown in Figure 22(b). Although the smaller value of χ ($\chi = 0.25$) greatly increases the smoothness of the solution and results in oscillation-free soil responses, it requires significantly longer elapsed computation time (E_t). Although a higher value of χ ($\chi = 1.5$) produces smaller temporal oscillations near the top permeable surface, it requires much shorter elapsed time. Importantly, the gain in smoothness or accuracy is not significant compared to the cost of computer run times as shown in Figure 22(a). In general, the value of χ can be selected based on the accuracy or smoothness requirement. However, $0 < \chi < 1$ can be sufficient for achieving reasonable accuracy and $\chi = 0.75$ can be considered as the optimal value.

The effect of the parameter M is shown in Figures 23(a),(b). M also affects the accuracy of the solution as well as the elapsed computation time. But the value of M mainly affects the time increments before the first occurrence of the $\Xi = 1$ condition. Similar to the effect of χ, its effect is most prominent near the permeable boundary. For a smoother time history of the computed responses, a lower value of M can be used. If $M \leq 1$, the effects of the value of M on the overall accuracy and computational efficiency are not significant. Too high a value, i.e. $M > 1000$, is not recommended because it can yield too large time increments to skip that temporal region where strain localization initiates or degradation of soil structure is most prevalent. Figures 23(a),(b) also show that the inclusion of the parameter B (Equation 20) or B_{elas} (Equation 21) into the automatic time increment selection scheme (Equation 1) is rightly justified, which is assumed to be dependent on the initial vertical stress or shear modulus, the coefficient of permeability and the discretization scheme.

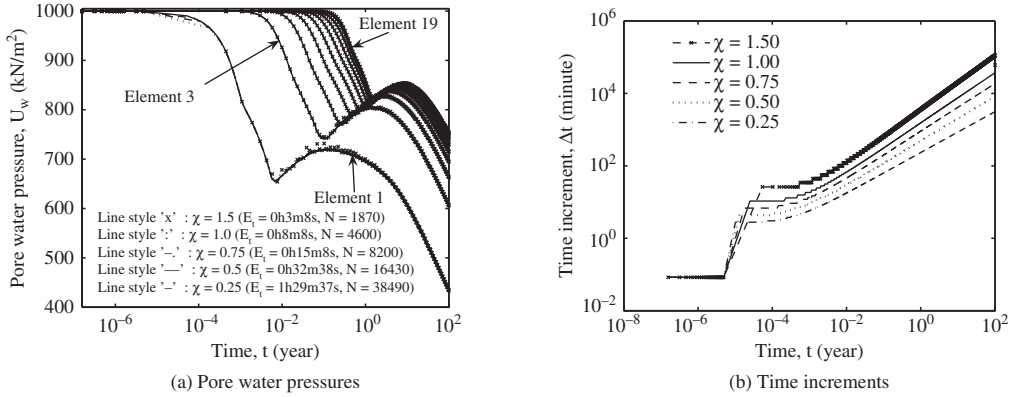
A higher value of χ or M does not cause any numerical instability in the computation process. This is because of the implementation of the $\Phi = 1$ control. However, frequent occurrence of the $\Phi = 1$ control leads to the existence of the $\Xi = 1$ control, which makes the parameter B inactive because of the inclusion of the parameter Υ. Therefore, the purpose of inclusion of the parameter Λ into Equation (1) is justified mainly when $B = \Upsilon$. However, the effect of the parameter Λ depends on the value of the parameter H_1 ($= fH$) as described in Section 2.6. The effect of H_1 on the pore water pressure calculation is shown in Figure 24(a). Time increments for various values of H_1 are presented in Figure 24(b). It is found that the value of H_1 does not significantly affect the accuracy of the solution, but it significantly affects the elapsed computational time (E_t). For example, the elapsed time is about 20 minutes for $H_1 = 40$ which is reduced to about five minutes when $H_1 = 1$. But $H_1 = 1$ introduces some small temporal oscillation. Therefore, the value of the parameter H_1 can be selected based on the individual's accuracy or smoothness requirement. However, $H_1 = 20$ can be regarded as an optimal value to achieve oscillation-free results for optimum elapsed computation time.

The above one-dimensional studies have shown that selection of the automatic time increment parameters is not rigid. They have a smaller effect on the accuracy but have a significant effect on the computational efficiency. Depending on the accuracy and efficiency requirements, these parameters can be easily selected for a specific problem. The numerical instability is not a concern because it is automatically ensured through the implementation of the control criteria.

4.2 *Two-dimensional analyses*

The effects of the automatic time increment parameters under two-dimensional boundary conditions are also studied. The same two-dimensional problem as described in Section 3.3.2 is considered. The soil and time increment parameters are kept the same as previously presented.

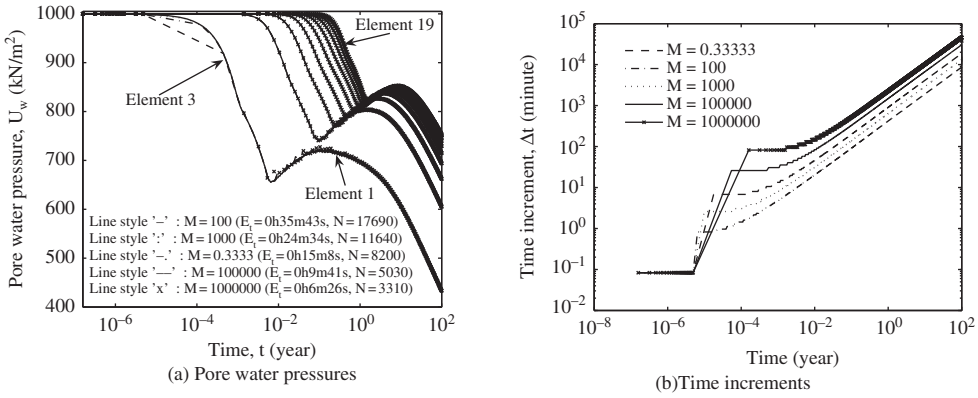
Keeping other parameters the same, the effect of χ on the time increments as well as on the computed responses are shown in Figures 25(a),(b). The effect of the parameter χ on the two-dimensional computed responses is also minimal and quite similar to that observed in one-dimensional analyses. However, it is found that the elapsed computation time for $\chi = 1.5$ is higher than that for $\chi = 1$, as seen from Figure 25(a). This is because of the frequent occurrence of $\Phi = 1$ caused by the higher value of χ. This leads to the existence of the $\Xi = 1$ control that reduces the B/Υ ratio to unity, thereby resulting in a lower rate of time increments as depicted in Figure 25(b).



(a) Pore water pressures

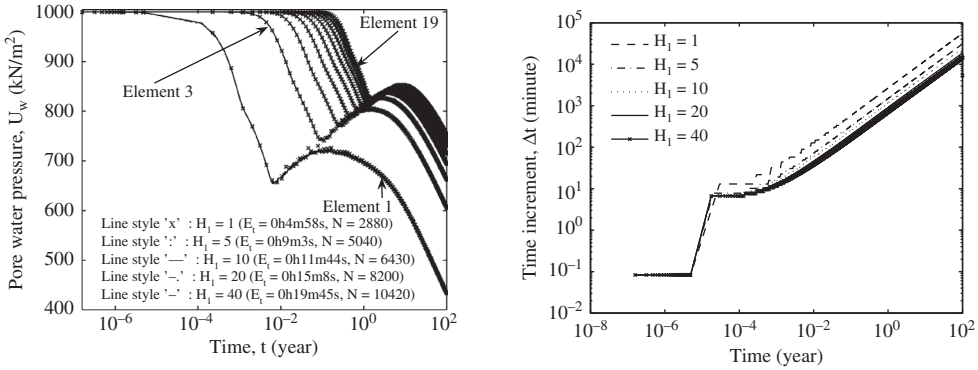
(b) Time increments

Figure 22. Effect of the parameter χ (one-dimensional).



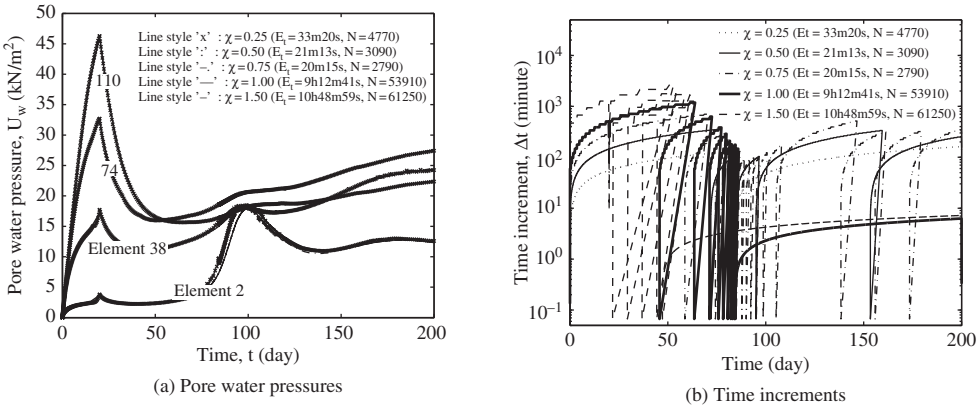
(a) Pore water pressures (b)Time increments

Figure 23. Effect of the parameter M (one-dimensional).



(a) Pore water pressures (b) Time increments

Figure 24. Effect of the parameter H_1 (one-dimensional).



(a) Pore water pressures (b) Time increments

Figure 25. Effect of the parameter χ (two-dimensional).

The effect of the parameter M is shown in Figures 26(a),(b). For the two-dimensional problem, it is found that the time increments become very large if $M > 1$. Therefore, the favorable value of M lies in the range $0 \leq M \leq 1$. As long as $M \leq 1$, the computed results are almost identical as presented in Figure 26(a). The optimal value of M can be taken as 1/3, since $M = 1/3$ gives the lowest computer run time. The accuracy or smoothness increases as the value of M decreases because of the resulting smaller time increments. But it decreases the computational efficiency. Up to a certain value of M, the computer run time decreases as the value of M increases. After that value, the computer run time again increases because of the enforcement of the control criteria arising from the resulting higher time increments. In this way, a problem can be easily calibrated to find the best value of M that will yield optimal computational efficiency.

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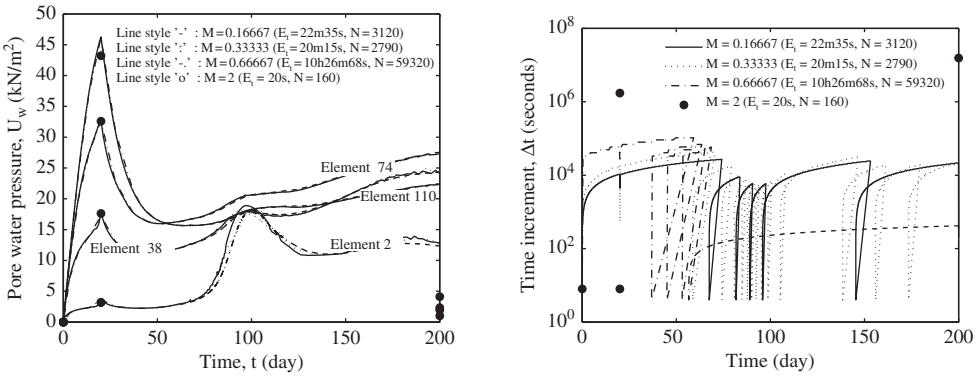


(a) Pore water pressures (b) Time increments

Figure 26. Effect of the parameter M (two-dimensional).

The effect of the parameter H_1 is shown in Figures 27(a),(b). It is also found that the effect of H_1 is minimal on the accuracy and efficiency of the computation. However, too small a value of H_1 (i.e. $H_1 = 1$) can require very long elapsed time. This is because of the resulting higher rate of increase of time increments that causes frequent occurrence of the $\Phi = 1$ condition and thus frequently going back and forth in the computation.

The above studies reveal that selection of the automatic time increment parameters is quite simple and does not require any prior knowledge. In fact, the proposed automatic time increment selection approach provides an effective means for the numerical analysis of transient soil response governed by an advanced constitutive law such as the elasto-viscoplastic constitutive model considering structural changes, (Kimoto *et al.* 2004). The approach is also equally applicable for the simpler soil models such as for the elastic model as presented.

5. Application of the automatic time increment selection scheme to an explicit time integration method

In all the above analyses, Euler's implicit time integration method is implemented. Now, the automatic time integration scheme is applied to the explicit fourth-order Runge–Kutta integration method. The explicit Runge–Kutta method treats every time step in a sequence of steps in an identical manner. In each step, the time derivative is evaluated four times and the final function value is calculated from these derivatives. Although various forms of Runge–Kutta schemes are available (Butcher 1987), the fourth-order Runge–Kutta integration method is a classic time integration approach. Detailed descriptions of this method can be found, e.g. in (Gear 1971), (Butcher 1987).

In the Eulerian finite element framework, the final system of equations governing the soil consolidation (Equation 23) can be written as

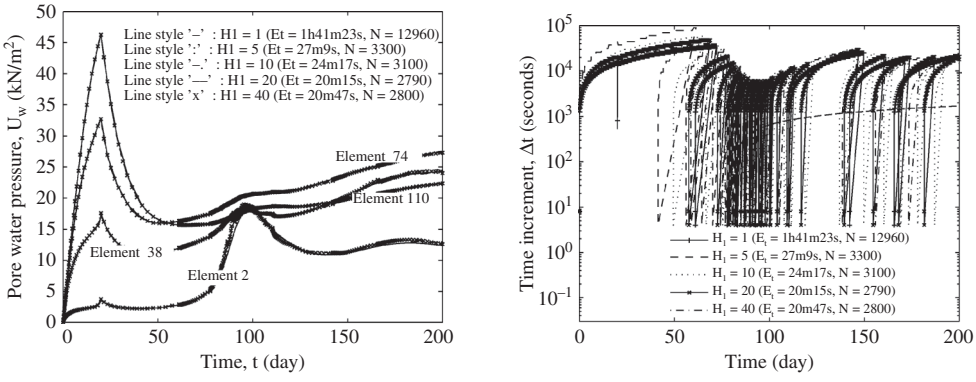
$$[K]\{\dot{u}\} + [K_v]^T\{\dot{U}_w\} = \{\dot{F}\} \tag{41a}$$

$$[K_v]\{\dot{u}\} - [K_h]\{U_w\} = \{\dot{Q}\} \tag{41b}$$

To represent the fourth-order Runge–Kutta scheme, Equations (41a) and (41b) are written as

$$\dot{U}_w = [A]\{\dot{F}\} - [B]\{\dot{u}\} \tag{42a}$$

$$\dot{u} = [C]\{\dot{Q}\} + [D]\{U_w\} \tag{42b}$$



(a) Pore water pressures (b) Time increments

Figure 27. Effect of the parameter H_1 (two-dimensional).

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where $[A] = \left[[K_v]^T\right]^{-1}$, $[B] = [A][K]$, $[C] = [K_v]^{-1}$, $[D] = [C][K_h]$.

Numerical solution based on the fourth-order Runge–Kutta algorithm will then be given by

$$\{U_w\}_{n+1} = \{U_w\}_n + \Delta t(\{L_1\} + 2\{L_2\} + 2\{L_3\} + \{L_4\})/6 \quad (43a)$$

$$\{u\}_{n+1} = \{u\}_n + \Delta t(\{M_1\} + 2\{M_2\} + 2\{M_3\} + \{M_4\})/6 \quad (43b)$$

The elements of the vectors $\{L_1\}$, $\{L_2\}$, $\{L_3\}$, $\{L_4\}$, $\{M_1\}$, $\{M_2\}$, $\{M_3\}$, $\{M_4\}$ are given as

$$\{L_1\} = [A(t_n)]\{\dot{F}(t_n)\} - [B(t_n)]\{M_1\} \quad (44a)$$

$$\{M_1\} = [C(t_n)]\{\dot{Q}(t_n)\} + [D(t_n)]\{U_w(t_n)\} \quad (44b)$$

$$\{L_2\} = \left[A\left(t_n + \frac{\Delta t}{2}\right)\right]\left\{\dot{F}\left(t_n + \frac{\Delta t}{2}\right)\right\} - \left[B\left(t_n + \frac{\Delta t}{2}\right)\right]\{M_2\} \quad (45a)$$

$$\{M_2\} = \left[C\left(t_n + \frac{\Delta t}{2}\right)\right]\left\{\dot{Q}\left(t_n + \frac{\Delta t}{2}\right)\right\} + \left[D\left(t_n + \frac{\Delta t}{2}\right)\right]\left\{U_w(t_n) + \frac{\Delta t}{2}\{L_1\}\right\} \quad (45b)$$

$$\{L_3\} = \left[A\left(t_n + \frac{\Delta t}{2}\right)\right]\left\{\dot{F}\left(t_n + \frac{\Delta t}{2}\right)\right\} - \left[B\left(t_n + \frac{\Delta t}{2}\right)\right]\{M_3\} \quad (46a)$$

$$\{M_3\} = \left[C\left(t_n + \frac{\Delta t}{2}\right)\right]\left\{\dot{Q}\left(t_n + \frac{\Delta t}{2}\right)\right\} + \left[D\left(t_n + \frac{\Delta t}{2}\right)\right]\left\{U_w(t_n) + \frac{\Delta t}{2}\{L_2\}\right\} \quad (46b)$$

$$\{L_4\} = [A(t_n + \Delta t)]\{\dot{F}(t_n + \Delta t)\} - [B(t_n + \Delta t)]\{\dot{u}(t_n) + \{M_4\}\} \quad (47a)$$

$$\{M_4\} = [C(t_n + \Delta t)]\{\dot{Q}(t_n + \Delta t)\} + [D(t_n + \Delta t)]\left\{U_w(t_n) + \frac{\Delta t}{2}\{L_3\}\right\} \quad (47b)$$

Equations (43a)–(47b) are computed step by step and are repeated up to the end of the computation time. The value of the time increment Δt is computed automatically through the present time increment selection scheme.

Figures 28(a),(b) show the comparison of pore water pressures for elastic consolidation, when the automatic time increment selection scheme is applied to the explicit as well as implicit time integration methods. For one-dimensional elastic analysis, the same problem as described in Section 3.2.1 is considered and the adopted two-dimensional elastic consolidation example is presented in Section 3.2.2. For both one- and two-dimensional elastic consolidation analyses, excellent agreement between the implicit and explicit time integration methods is observed.

One-dimensional elasto-viscoplastic consolidation responses for the explicit time integration method are shown in Figure 29(a). The boundary conditions, soil and model parameters are the same as those given in Section 3.3.1 for the case-2 example problem. Figure 29(a) show that the explicit Runge–Kutta integration method produces a smoother response especially near the top permeable boundary than that found using the implicit scheme. This is because of the lower values of the time increment associated with the explicit time integration method as shown in Figure 29(b). But this improved performance needed a computer run time of around 100 times longer. Note that a similar improvement can also be obtained using the implicit method if smaller time increments, i.e. if smaller values of χ (< 0.75) or M (< 1/3) and/or higher values of H_1 (> 20) are used in the automatic time increment selection scheme.

Two-dimensional viscoplastic consolidation responses under different integration methods are now shown. The same problem as described in Section 3.3.2 is considered. The time increment parameters are also taken as the same, i.e., $\chi = 0.75$, $M = 1/3$ and $H_1 = 20$. Temporal distributions for lateral displacements and pore water pressures are shown, respectively, in Figures 30(a),(b). Both explicit and implicit methods produce almost the same results. However, the maximum value of the time increments for the explicit Runge–Kutta method is much smaller compared to Euler's implicit method which reduces the efficiency of the computation for the former as shown in Figure 31(a). Frequent ups and downs in the time increments are observed when the automatic selection scheme is implemented in the Runge–Kutta integration system. This means that the Runge–Kutta method does not allow larger time increments to preserve the stability of the solution. Therefore, it is preferable to use a relatively lower value of χ and M or a relatively higher value of H_1 to recover the loss in computational efficiency if the Runge–Kutta explicit integration method is invoked. Figure 31(b) shows that computational efficiency is significantly gained with the Runge–Kutta method if the time increment parameters are chosen as: $\chi = 0.125$, $M = 0.125$ and $H_1 = 50$. Importantly, the accuracy is not affected by the choice of the time increment parameters as shown in Figure 32. It is also confirmed that the accuracy of the solution by the proposed time increment selection scheme is at the same level as that obtained by the Runge–Kutta method.

The above studies show that the automatic time increment selection scheme is independent of the time integration method. It can automatically adjust the time increments depending on the stability requirement of the time integration method. The fourth-order Runge–Kutta integration method can increase the accuracy of the solution over that obtained from Euler's implicit scheme, but it reduces the computational efficiency, although the amount of efficiency loss is problem-specific and depends on the choice of the automatic time increment parameters. This is mainly because the fourth-order Runge–Kutta integration scheme requires four evaluations of the field variables per time step and it may not be unconditionally stable to tolerate larger time increments. Euler's implicit

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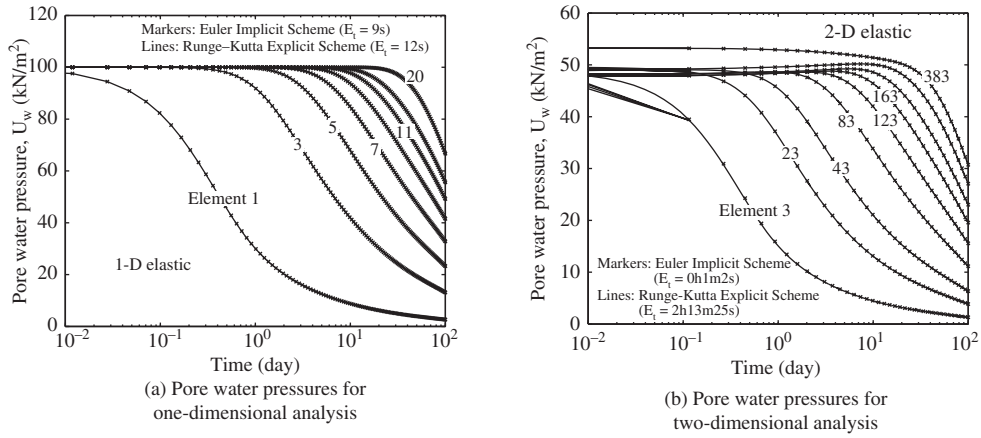
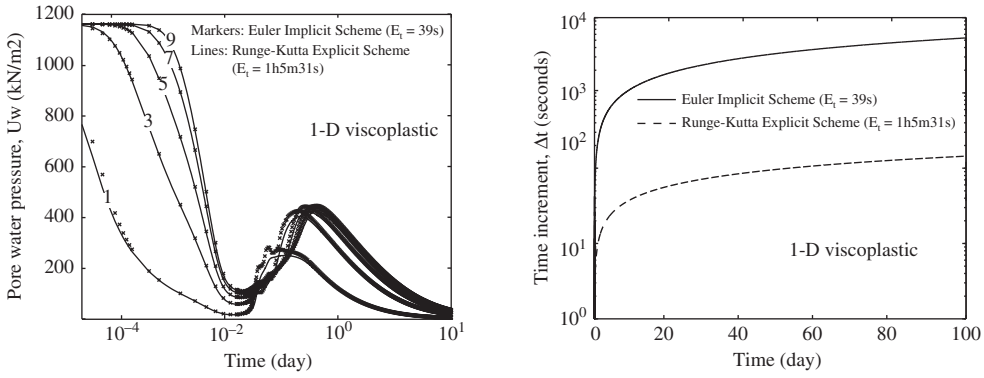


(a) Pore water pressures for one-dimensional analysis

(b) Pore water pressures for two-dimensional analysis

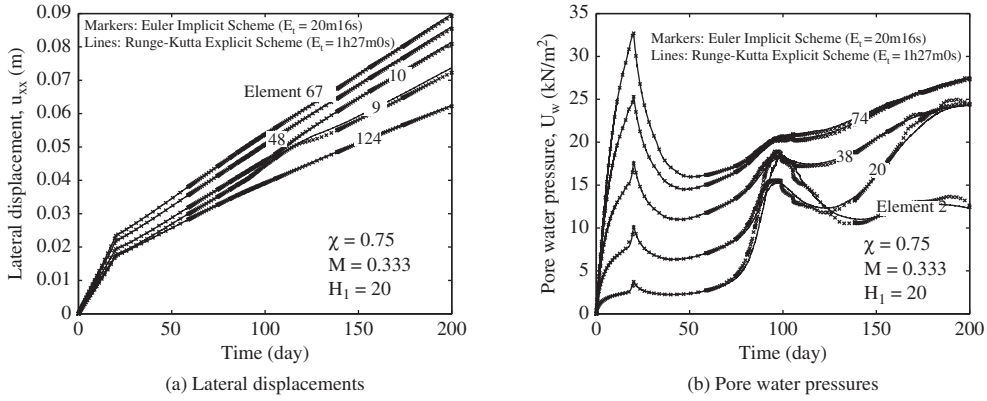
Figure 28. Elastic soil response for different time integration methods.



(a) Pore water pressures

(b) Time increments

Figure 29. Viscoplastic soil response and time increments for different time integration methods (one-dimensional).



(a) Lateral displacements

(b) Pore water pressures

Figure 30. Viscoplastic soil response for different time integration methods (two-dimensional).

method can unconditionally allow larger time increments and can be one of the handy choices for the time integration approach for the long-term consolidation analysis of geomaterials that sometimes spans long durations.

6. Conclusions

An automatic time increment selection scheme for the numerical analysis of the long-term response of geomaterials is

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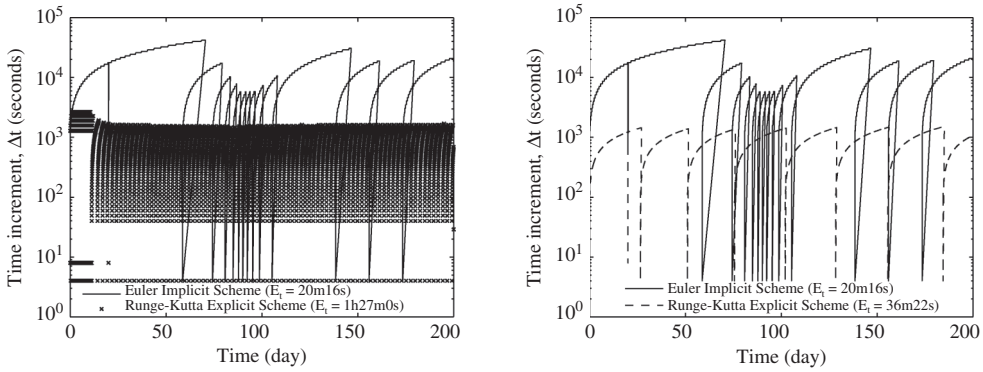

(a) For same time increment parameters

(b) For different time increment parameters

Figure 31. Time increments for different time integration methods (two-dimensional).

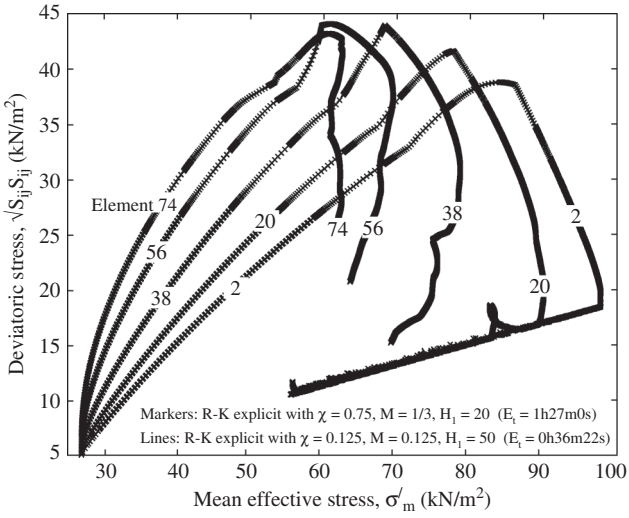

Figure 32. Stress paths using explicit time integration methods (two-dimensional).

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presented. It considers the temporal stability issue as the key priority while offering reasonable accuracy of the numerical solution. The approach is simple and robust. It can automatically adapt itself to the new conditions depending on the rate-dependent temporal history of material response.

The strain rate is a good indicator for describing the current state of geomaterials upon which the automatic time increment selection scheme is evolved through the proposed simple empirical equation that requires only a few extra parameters. These parameters are flexible and easy to choose. The selection of these parameters is a trivial task and does not require prior knowledge about the physical problem, since they have a small effect on the accuracy but have a significant effect on the computational efficiency. The approach uses a few control criteria depending on the physical conditions as well as on the governing constitutive model of the material. The temporal stability of the solution is automatically ensured through the implementation of these control criteria.

Based on the framework of the finite element method (FEM), the performance of the automatic time increment selection scheme is investigated through various numerical analyses of long-term consolidation response of clay under different geotechnical profiles and loading conditions. Both elastic and viscoplastic constitutive relations are considered together with the consideration for the destructuration effect of clay. Numerical results show that the performance of the automatic time increment selection scheme is fairly excellent. It is also effective in simulating unstable material responses, such as the anomalous generation of pore water pressure after the construction of an embankment on soft sensitive clay. The accuracy of the solution obtained by the proposed method is at the same level as that by the Runge–Kutta method.

The present automatic time increment selection scheme is applied to both Euler's implicit time integration method and the fourth-order Runge–Kutta explicit method. The approach can automatically adjust the time increments depending on the stability requirement of the governing time integration method. Compared

to Euler's implicit method, the fourth-order Runge–Kutta explicit integration scheme can further increase the accuracy of the solution, but it generally reduces the computational efficiency.

The automatic time increment selection scheme is equally applicable to any spatial mesh configuration. Since it considers the quality of the FEM mesh configuration, the present approach could be effective in designing a robust space-time adaptive numerical tool for long-term analysis of geomaterials. The proposed automatic time increment selection approach is intelligent, adaptive and unconditionally stable. Because of its robustness, the present automatic time increment selection scheme could be an competitive choice over the fixed time increment or temporal error-based or other empirical approaches commonly used until today for long-term numerical analysis of geomaterials.

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